M. KELEMEN
Central Institute
of Informatics

Fuzzyness and Structural Stability

The paper suggests a connexion between fuzzyness and structural stability. This is illustrated by a classical decision model, viewed from the stand-point of Catastrophe Theory.

I. FUZZYNESS AND STRUCTURAL STABILITY

The fuzzy system theory was created in order to improve the quality of the models for decision-making processes subject to uncertainty.

The intuitive and philosophycal significance of fuzzyness was generally concentrated only on the membership function ($f : X \to [0, 1]$, which "softens" the crisp determination of the classical characteristic function ($g : X \to \{0, 1\}$).

In this way, most of the proposed models (which are essentially classic, [1], are dealing with these "f"'s only as data (sometimes even unvalidated data). And so, the results and the manner of obtaining them miss a deeper justification and insight which would be necessary according to the purposes of the fuzzy-set theory.

In order to bypass this shortcoming we suggest the following idea:

We claim that behind every *correct fuzzy decision model* (i.e. a model that does not fail in reaching the purpose for which it was developed) stands a *structurally stable model.*

In other words, such a decision-making model should have two descriptions:

d_i — an internal one, which-by structural stability-explains *why* the proposed model can be used;

d_e — an external one, which — by fuzzyness — shows *how* the proposed model can be easier used.

In this way we suggest to analyse some fuzzy results, using the powerfull notion of classical mathematics, structural stability.

(The discussion on structural stability can be embedded in the larger problem of the dogma of stability [2] and the axiom of structural stability [3], while the agreement of d_i and d_e is a kind of adequacy [2]).

II. AN ILLUSTRATIVE EXAMPLE

In this section we shall give an example for what we claimed in the previous section.

We are dealing with a process which must be controlled by a skilled human operator, in order to stabilize its state at the equilibrium $e = 0$, $\Delta e = 0$, where e and Δe are the steady-state error and its variation.

The decisions he makes regarding the controls Δu depend on e and Δe, and are established in an empiric way by means of fuzzy variables [4]; they are given in the following table.

where *PB* is positive big; *PM* is positive medium; *PS* is positive small; *PO* is positive zero and similarly for the negative ones. (The entries in the matrix are the controls Δu)

The effective controls are obtained in [4] by converting the fuzzy variables into numbers, using fuzzy membership functions; this description o Δu is an external one, d_e.

We shall now obtain an internal one, d_i, using some results from the Catastrophe Theory, [3]. For this we assume that the following (we hope natural) requirements are fulfilled

r_1 — Applying (T), the human operator has to face two situations
1° When Δe i large, the decisions of the human operator are affected by *hysterezis* at $e = 0$.
2° When Δe is small, the *hysterezis disappears* (The hysteresis is a delay in the reactions of the human operator, who cannot follow rapid variations of the error).

(T)

error variation ↑									
	PB	PB			PM	NM			NB
	PM	PB			PM	NM			NB
	PS	PB		PM	PS	NS	NM		NB
	PO	PB		PM			NM		NB
	NO	PB			NO	PO			NB
	NS	PB PM	PM		NS	PS		NM	NB NM
	NM				NM	PM			
	NB				NM	PM			
		NB	NM	NS	NO	PO	PS	PM	PB

error →

Fig. 1

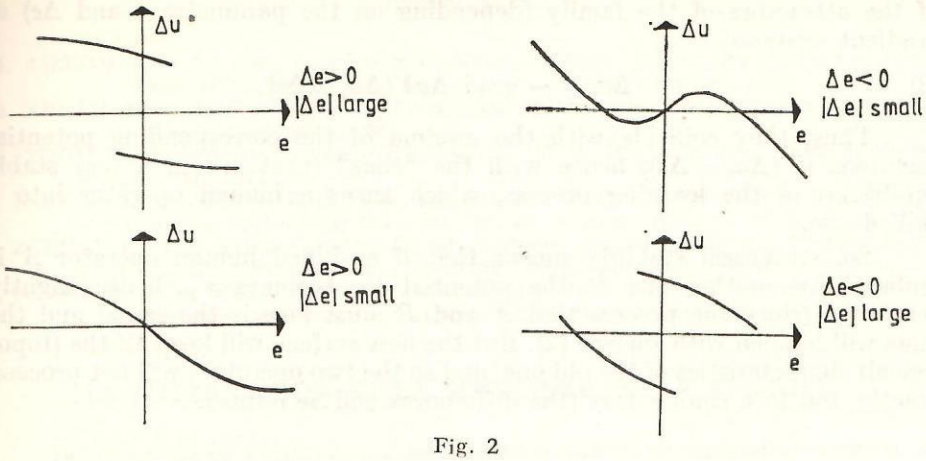


Fig. 2

r_2 — The equilibrium (0, 0) is stable.

r_3 — The effect of Δu is "smooth" in some neighbourhood of the origin.

Now, from (T) and r_1 we can obtain a "qualitative" dependence Δu versus e, for different Δe, which we represent graphically as follows:

Then, according to [5], the first two and the last two families of curves can be "glued" into two functions of the "Riemann-Hugoniot"-type (characterized by cusp singularities).

Following the modelling technique in [6], we can find the local model (for a neighbourhood of the equilibrium) of Δu, for a skilled human operator, having the expression:

$$(1)\quad \begin{cases} a_1(\Delta u)^3 + b_1(\Delta e)\,(\Delta u) + c_1 e = 0, & \Delta e > 0 \\ a_2(\Delta u)^3 + b_2(\Delta e)\,(\Delta u) + c_2 e = 0, & \Delta e < 0 \end{cases}$$

(the coordinate axes are shifted).

Here $a_1, \ldots, c_2$ can be determined by means of the least-square method knowing the controls given by the specified human operator. The values Δu are optimal, since they were adopted by the human operator as a result of a *learning* process (that is why the human operator is called "skilled").

The main quality of the above model is its *structural stability*. Indeed, the two submodels, (for $\Delta e > 0$ and $\Delta e < 0$) are structurally stable by the main theorem of the Catastrophe Theory, and we can consider that they are "glued" together into a structurally stable whole, since small perturbations in Δu (see r_3) do not affect the control (see r_2).

Here, the significance of structural stability is the following:

We can write (1) in an equivalent form

$$(2)\quad \text{grad } V\ (\Delta u, e, \Delta e) = 0$$

where the function V could be considered a "control-error potential" of the operator. In this way, the exterior sheets of the surface (1), or (2), consist

of the attractors of the family (depending on the parametra e and Δe) of gradient systems.

$$\Delta u = -\operatorname{grad}_{\Delta u} V(\Delta u, e, \Delta e). \tag{3}$$

Thus, they coincide with the minima of the corresponding potential functions $V(\Delta u, e, \Delta e)$, hence with the "ends" (that are in a way stable equilibria) of the learning process, which turns a human operator into a skilled one.

So, structural stability means that if a skilled human operator A is replaced by another one B, the potential V_A becomes V_B, hence slightly perturbed (since the process that A and B must face is the same) and the same will happen with surface (2). But the new surface will keep all the (topological) characteristics of the old one, and so the two operators will not proceed exactly, but in a similar way (the differences will be minor).

III. WHAT DO WE HOPE FROM THIS IDEA?

We shall try to obtain mathematical results of the following type:

— We want to see if a structural stable model of the behaviour of the human operator does exist near the "stable equilibrium of the learning process" for this operator.
(The notions above could make sense using an analogy with the qualitative theory of dynamical systems).

— If such a model exists (d_i), but cannot be written explicitely, to what extent is the fuzzy description (d_e) related to it?
So d_i should be a criterion for validating d_e.

CONCLUSIONS

— This paper tries to suggest a connexion between fuzzyness and structural stability.

— The use of Catastrophe Theory was not essential; here it helped to find in an easier way a structurally stable model.

— Not any fuzzy description should be adopted, but only those which are in accordance with the requirement of structural stability; we call those "coherent" fuzzy descriptions.

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LA FUZZIFICATION ET LA STRUCTURE DE LA STABILITÉ

Dans l'ouvrage on présente une connexion entre la stabilité de fuzzification et celle structurelle. Cette connexion est illustrée par un modèle de décision classique envisagé du point de vue de la théorie des catastrophes.

ФАЗЗИФИКАЦИЯ И СТРУКТУРА УСТОЙЧИВОСТИ

В работе указано на взаимосвязь между устойчивостью фаззификации и структуральной устойчивостью. Это проиллюстрировано классической моделью принятия решений, рассматриваемой с точки зрения теории катастроф.

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