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On Disjunctive Linear Fractional Programming

This paper deals with a class of linear fractional programming problems where the constraints are of "or" type. The paper is divided into five sections. The first section has an introductory character. In the second section there are given a few situations where such problems arise. The third section contains several variants of disjunctive linear fractional programming problem and each variant is transformed in a mixed integer linear fractional programming problem. The fourth section presents for such one variant a solution method which is illustrated by an example. The final section contains several conclusions.

1. INTRODUCTION

The investigations in the field of fractional programming followed closely on the lines of development in linear and nonlinear programming. Different aspects of fractional programming have been studied and the recent bibliography prepared by I. M. Stancu-Minasian [11] covers most of the important papers. The main advantage in fractional programming is that two objectives (may be going in opposite direction) can be treated simultaneously by defining them as a ratio. In general, fractional programming problems are met with in problems with several quantitative criteria and for which the optimizing of a qualitative criterion is looked for, a criterion that appears like a ratio of quantitative criteria. Thus prior to the development of multiobjective programming algorithms, fractional programming proved as a suitable tool.

In this paper we study a class of linear fractional programming problems where the constraints are of "or" type i.e. defined by the union of set of linear constraints so that the resulting feasible region is *nonconvex* in nature. The plan of the paper is as follows. In the second section a few situations are des-

cribed where such problems arise. The third section contains several variants of disjunctive fractional programming problem while in the fourth section one such variant is discussed in details with a numerical example. The final section gives several conclusions.

2. PRACTICAL SITUATIONS

The disjunctive fractional programming problems appear in various contexts, e.g. in scheduling problems [8] where the objective is of finding minimum processing-time of a product in a machine flow-shop.

Recently, Golovanov and Sarkisyan [3] considered a problem of forecasting of vehicle system for future where the use of fractional criterion, namely minimization of ratio of expenses to effect, is discussed. Here besides the general constraints on resources, those associated with each particular vehicle system are also to be included. Thus this is a situation where disjunctive fractional programming would be applicable because generally the choice of one system excludes the others. This is particularly true in choosing the defence equipment for future where the choice of one technology rules out the other.

In regional planning, especially in deciding the relocation of certain activities like industries, markets etc., the objective of decongestion is to be achieved through maximization of the benefit to the cost. However, the linkages between various activities often rule out shifting of certain activities which could be expressed by disjunctive constraints.

Similar situations arising in portfolio management, media selection in advertising, selection of appropriate technology could also be treated by the development presented in this paper.

3. VARIANTS OF DISJUNCTIVE FRACTIONAL PROGRAMMING PROBLEM

Generally in programming problems, the constraints are of "and" type. But in many practical problems, as we have seen in the previous section, there are situations where the constraints are disjunctive i.e. of "or" type. The modality for appearance of these constraints vary from problem to problem, depending on the situation. For solving such problems, a use can be made of branch and bound method (see F. Radó [10] for the linear case) or their transformation to mixed integer nonlinear programming problems. The two methods are similar and have same degree of difficulty. Here we shall concentrate on the second approach. Of course, we are aware of many interesting papers on disjunctive linear programming, specially those of E. Balas, R. G. Jeroslow, F. Glover and so on, giving the cutting — Plane methods and their different extensions. However, we are not exploring those possibilities in this paper.

Let us consider the following linear fractional programming problem which would be later considered in various disjunctive forms:

$$\text{Minimize (Maximize)}\, F(X) = \frac{f_1(X)}{f_2(X)} = \frac{C^t X + \alpha}{D^t X + \beta} \tag{1}$$

Subject to

$$\begin{aligned} g_1(X) &= A_1X - b_1 \leqslant 0 \\ g_2(X) &= A_2X - b_2 \leqslant 0 \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ g_p(X) &= A_pX - b_p \leqslant 0 \\ X &\geqslant 0 \end{aligned} \tag{2}$$

where X, C, D are $n\times 1$ vectors, $A_i (i = 1,2, ..., p)$ is $m_i \times n$ matrix, b_i $(i = 1,2, ..., p)$ is an $m_i \times 1$ vector, $f_i: R^n \to R$, $(i = 1, 2)$, $g_i: R^n \to R^{m_i}$ $(i = 1,2, ..., p)$ are linear functions of variable $X \in R^n$, α and β are scalar constants and the superscript "t" denotes the transpose.

It is also assumed that the denominator $(D^tX + \beta)$ is strictly positive over the feasible region.

Problem I. Let i_0, $j_0 \in I = \{1, 2, ..., p\}$ and define $I^* = I \setminus \{i_0, j_0\}$. Let us assume that we have to choose an optimal vector X for the function $F(X)$ which satisfies one of the constraints, $g_{i_0}(X) \leqslant 0$ or $g_{j_0}(X) \leqslant 0$ and all the remaining constraints. It is to be noted that $g_{i_0}(X) \leqslant 0$ represents a group of m_{j_0} linear constraints. So we have the following problem:

$$\text{Minimize (Maximize) } F(X) = \frac{C^tX + \alpha}{D^tX + \beta} \tag{3}$$

Subject to

$$\begin{gathered} g_i(X) \leqslant 0, \quad i \in I^* \\ g_{i_0}(X) \leqslant 0 \text{ or } g_{j_0}(X) \leqslant 0 \\ X \geqslant 0. \end{gathered} \tag{4}$$

The basic idea to deal with disjunctive constraints as put forward by H.P. Williams [12, p. 166] is to connect the given constraint sets through certain *indicator variable* δ_i (boolean variable) so as to construct a new problem for which the solution methods are available and thus its solution would give that of the original problem.

Let $\bar{g}_{i_0}$ and $\bar{g}_{j_0}$ be upper bounds for g_{i_0} and g_{j_0} respectively, and δ_{i_0} and δ_{j_0} be the corresponding indicator variables. Then the problem (3)—(4) is equivalent to the following problem:

$$\text{Minimize (Maximize) } F(X) = \frac{C^tX + \alpha}{D^tX + \beta} \tag{5}$$

Subject to

$$g_i(X) \leqslant 0, \quad i \in I^* \tag{6.1}$$

$$g_{i_0}(X) \leqslant (1 - \delta_{i_0})\, \bar{g}_{i_0} \tag{6.2}$$

$$g_{j_0}(X) \leqslant (1 - \delta_{j_0})\, \bar{g}_{j_0} \tag{6.3}$$

$$\delta_{i_0} + \delta_{j_0} \geqslant 1, \quad \delta_{i_0}, \delta_{j_0} \in \{0, 1\} \tag{6.4}$$

$$X \geqslant 0 \tag{6.5}$$

Now for $\delta_{i_0} = 1$ and $\delta_{j_0} = 0$ the constraints (6.2) and (6.3) become

$$g_{i_0}(X) \leqslant 0 \tag{6.2'}$$

$$g_{j_0}(X) \leqslant \bar{g}_{j_0} \text{ (superfluous)} \tag{6.3'}$$

Hence only (6.2) are to be considered. In this case "or" type is exclusive. If we choose the constraint (6.4) under the form $\delta_{i_0} + \delta_{j_0} = 1$ then "or" type is non-exclusive.

A more general form of this problem is given by,

Problem II. Let us assume that from p constraint sets (2), we wish that at least $k (k < p)$ of them to be picked up. So the problem is,

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{7}$$

Subject to

$$g_i(X) \leqslant (1 - \delta_i)\, \bar{g}_i, \quad i \in I \tag{8.1}$$

$$\sum_{i=1}^{p} \delta_i \geqslant k, \quad \delta_i \in \{0, 1\}, \quad i \in I \tag{8.2}$$

$$X \geqslant 0 \tag{8.3}$$

If we desire that only k constraint sets to be picked up then (8.2) may be modified to the form

$$\sum_{i=1}^{p} \delta_i = k. \tag{8.2'}$$

Let $C_i (i = 1, 2, ..., C_p^k)$ be the all subsets of $I = \{1, 2, ..., p\}$ having k elements. We define

$$G_i = \{X : g_s(X) \leqslant b_s, \ s \in C_i, \ X \geqslant 0\}, \ i = 1, 2, ..., C_p^k$$

then the problem (7) — (8) is equivalent to the following problem:

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta}$$

Subject to

$$X \in G_1 \cup G_2 \cup ..... \cup G_{C_p^k}$$

Problem III. For $X = (x_1, x_2, ..., x_n)$, we wish that $x_i = 0$ or if $x_i \neq 0$ then $\underline{x}_i \leqslant x_i \leqslant \bar{x}_i$, where $\underline{x}_i$ and $\bar{x}_i$ are some given bounds for x_i. In this case we have the problem.

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{9}$$

Subject to

$$g_i(X) \leqslant 0, \quad i = 1, 2, ..., p$$
$$x_i = 0 \text{ or } \underline{x}_i \geqslant \overline{x}_i \leqslant \bar{x}_i \tag{10}$$
$$x_j \geqslant 0, \quad j = 1, 2, ..., n.$$

This problem is equivalent to the following problem:

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{11}$$

Subject to

$$g_i(X) \leqslant 0, \quad i = 1, 2, ..., p \tag{12.1}$$
$$\underline{x}_i - x_i \leqslant \delta_1 \underline{x}_i \tag{12.2}$$
$$x_i \leqslant \delta_2 \bar{x}_i \tag{12.3}$$
$$\delta_1 + \delta_2 = 1, \ \delta_1, \delta_2 \in \{0, 1\} \tag{12.4}$$
$$x_i \geqslant 0, \quad i = 1, 2, ..., n. \tag{12.5}$$

Indeed, if $\delta_1 = 1$ and $\delta_2 = 0$, then from (12.2) and (13.3) we obtain $x_i = 0$ and if $\delta_1 = 0$ and $\delta_2 = 1$, we get $\underline{x}_i \leqslant x_i \leqslant \bar{x}_i$.

Problem IV. Let $J_1, J_2, ..., J_k$ be subsets of the index set $J = \{1, 2, ..., n\}$. Let us suppose that it is required that at least one variable from each of the subsets $J_1, J_2, ..., J_k$ be equal to zero. We have the problem,

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{13}$$

Subject to

$$AX \leqslant b, \ X \geqslant 0 \tag{14}$$

and the additional constraints of the form

$$\begin{aligned} &\prod_{j \in J_1} x_j = 0 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\prod_{j \in J_k} x_j = 0 \end{aligned} \tag{15}$$

Here A is an $m \times n$ matrix and b is $m \times 1$ vector. The constraints (15) can be interpreted as disjunctions, for example, the constraint $\prod_{j \in J_1} \underline{x}_J = 0$ may be rewritten as "there is at least one $j \in J_1$ such that $x_J = 0$". The complementary fractional programming problem, where, from each of the several pairs of variables, at least one must vanish, can be written as the problem (13)—(15). If the objective function $F(X)$ is linear, we obtain the problem considered by G. Owen [9].

Problem V. Now let us consider the following problem termed by F. Radó [10] in general as programming problem with logical conditions:

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{16}$$

Subject to

$$g_i(X) > 0 \rightarrow g_{p+i}(X) \leqslant 0, \quad i = 1, 2, ..., p \tag{17}$$
$$X \geqslant 0$$

Here "$\rightarrow$" is defined such that

$\rightarrow$	0	1
0	1	1
1	0	1

Due to the equivalence $(a \rightarrow b) \Leftrightarrow (\bar{a}$ or $b)$ this problem can be written as,

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{18}$$

Subject to

$$g_i(X) \leqslant 0 \text{ or } g_{p+i}(X) \leqslant 0, \quad i = 1, 2, ..., p \tag{19}$$
$$X \geqslant 0$$

A natural way for solving this problem is to solve 2^p programming problems having the constraint $X \geqslant 0$ and one of the constraints,

$$g_i(X) \leqslant 0 \text{ or } g_{p+i}(X) \leqslant 0 \text{ for each } i = 1, 2, ..., p;$$

and choose the one for which $F(X)$ has the optimal value. Hence we can use the branch and bound method given by F. Radó [10] in linear case but if p is very large then it is a cumbersome process.

Generally, in disjunctive fractional programming we can consider a lot of linear (or nonlinear) constraints unit by connectives $(\cup, ., -, \rightarrow)$, because writing such constraints under the form of conjunction of disjunction, results in a constraint of the form,

$$g_i^{(1)}(X) \leqslant 0 \cup g_i^{(2)}(X) \leqslant 0 \cup \ ... \ \cup g_i^{(s)}(X) \leqslant 0, \quad i = 1, 2, ..., p.$$

4. SOLUTION METHOD WITH ILLUSTRATION

Now we consider a problem which is a more general case of the Problem I and describe the complete method for solving it. The problem is of choosing an optimal vector X for which the function $F(X)$ satisfies one of the constraint (2) i.e.

Problem P

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{20}$$

Subject to

$$A_1X \leqslant b_1 \cup A_2X \leqslant b_2 \cup \ldots \cup A_pX \leqslant b_p \tag{21}$$
$$X \geqslant 0$$

Let us note that this problem results from the Problem II for $k = 1$.

It must be mentioned that the feasible region defined by (21) is disconnected and therefore, nonconvex in nature as brought out by Fig. 1 illustrating the numerical example given later in this section. Hence the available methods of linear fractional programming [1], [7] are not directly applicable.

As it was presented in the above section, we introduce the indicator variable $\delta_i (i = 1, 2, \ldots, p)$ which takes the value 1 if the i^{th} constraint set viz. $A_iX \leqslant b_i$, $X \geqslant 0$ is considered in the solution procedure, otherwise $\delta_i = 0$. Further, the operation of this variable can be illustrated by the following:

For all non-negative x_j, the inequality $\sum_{j=1}^{n} a_j^i x_j \leqslant b_i (a_j^i$ is a column vector of the matrix $A_i)$ holds then δ_i assumes the value 1 i.e.

$$\delta_i = 1 \rightarrow \sum_{j=1}^{n} a_j^i x_j \leqslant b_i \tag{22}$$

Now (22) can also be written as

$$\sum_{j=1}^{n} a_j^i x_j + M_i \delta_i \leqslant M_i + b_i \tag{23}$$

where M_i is an upper bound for the expression

$$\sum_{j=1}^{n} a_j^i x_j - b_i.$$

Similarly for all non-negative x_j, the expression

$$\delta_i = 1 \rightarrow \sum_{j=1}^{n} a_j^i x_j \geqslant b_i \tag{24}$$

can be replaced by

$$\sum_{j=1}^{n} a_j^i x_j + m_i \delta_i \geqslant m_i + b_i \tag{25}$$

where m_i is a lower bound for the expression

$$\sum_{j=1}^{n} a_j^i x_j - b_i.$$

Thus for the Problem (P) using (23), we put

$$\delta_i = 1 \rightarrow A_iX \leqslant b_i, \quad X \geqslant 0, \quad i = 1, 2, \ldots, p$$

in the form

$$A_i X + V_i \delta_i \leqslant b_i', \quad i = 1, 2, ..., p$$
$$X \geqslant 0, \quad \delta_i \in \{0, 1\}, \quad i = 1, 2, ..., p$$

where b_i' denotes the modified value of b_i in light of the coefficient V_i of δ_i.

Finally, in order to ensure that at least one of the constraint sets is picked up, a constraint on δ_i viz. $\sum_{i=1}^{p} \delta_i \geqslant 1$ is to be added. So, the problem to be solved is:

Problem TP

$$\text{Minimize (Maximize) } F(X) = \frac{C^t X + \alpha}{D^t X + \beta} \tag{26}$$

Subject to

$$A_i X + V_i \delta_i \leqslant b_i', \quad i = 1, 2, ..., p \tag{27.1}$$

$$\sum_{j=1}^{p} \delta_i \geqslant 1 \tag{27.2}$$

$$X \geqslant 0, \quad \delta_i \in \{0, 1\}, \quad i = 1, ,2 ..., p. \tag{27.3}$$

Theorem. Let (X°, δ°) be the optimal solution of the Problem (TP) then *X° is the optimal solution of the Problem (P).*

Proof. The constraints (27.2) and (27.3) ensure that one of the constraints given in (21) is satisfied and since (27.1) is the extended form of (21), it follows that X° is feasible for (P). Since the objective function of (P) and (TP) is same, it follows that X° is the optimal solution of the Problem (P).

The Problem (TP) belongs to a class of mixed integer linear fractional programming problems, so the methods given in [2], [4], [5], [6] could be used to obtain the solution which in turn would give that of the Problem (P) as proved by above theorem. The following example illustrates the complete method:

Numerical example

$$\text{Minimize } F(X) = \frac{2x_1 + x_2 + 1}{3x_1 + 5x_2 + 3} \tag{28}$$

Subject to

$$X \in S_1 \cup S_2 \cup S_3 \tag{29}$$

where

$$S_1 = \{(x_1, x_2) : x_2 \leqslant 2, \quad x_1 + x_2 \leqslant 3, \quad x_1, x_2 \geqslant 0\}$$
$$S_2 = \{(x_1, x_2) : -x_1 + 2x_2 \leqslant 1, \quad 2x_1 - x_2 \leqslant 4, \quad x_1, x_2 \geqslant 0\}$$
$$S_3 = \{(x_1, x_2) : x_1 \leqslant 4, \quad 3x_2 \leqslant 2, \quad x_1, x_2 \geqslant 0\}$$

It is clear from the Fig. 1 that although S_i, $i = 1, 2, 3$ is a convex set, the union of these as represented by ABCDEFGO, expressing (29) forms a nonconvex region.

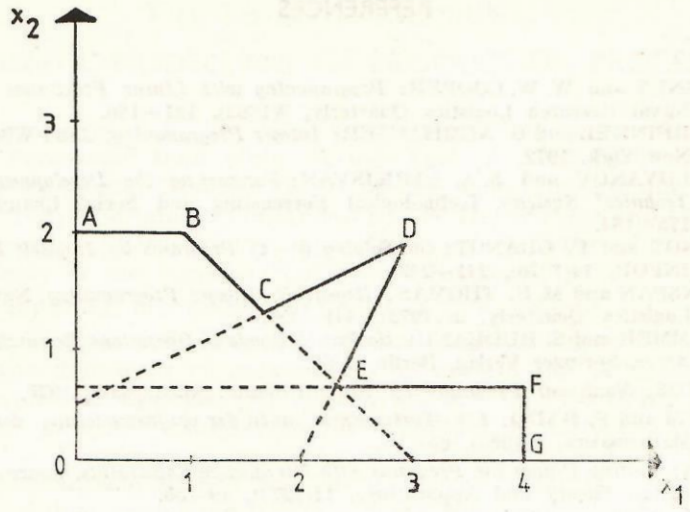


Fig. 1

Now introducing the indicator variable δ_i and assuming $\delta_i = 1$ implies S_i is included and then using (23) and including (27.2), the constraints of the new problem are given by

$$\begin{aligned} x_2 + \delta_1 &\leqslant 3 \\ x_1 + x_2 + 4\delta_1 &\leqslant 7 \\ -x_1 + 2x_2 + 5\delta_2 &\leqslant 6 \\ 2x_1 - x_2 + 4\delta_2 &\leqslant 8 \\ x_1 &\leqslant 4 \\ 3x_2 + 2\delta_3 &\leqslant 3 \\ \delta_1 + \delta_2 + \delta_3 &\leqslant 3 \\ x_1, x_2 \geqslant 0, \quad \delta_i \in \{0, 1\}, \quad i &= 1, 2, 3. \end{aligned} \tag{30}$$

The optimal solution of this new problem is $x_1 = 0$, $x_2 = 2$ and $\delta_1 = 1$ giving the minimum value of $F(X) = 3/13$.

5. CONCLUSION

The method proposed here can easily be extended to the problems having other nonlinear objective function with disjunctive linear constraints. This however brings forth the need for developing efficient algorithms for solving mixed integer nonlinear programming problems. The problems with nonlinear disjunctive constraints would also require attention and it is hoped that further research would be done in this direction.

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LE PROBLEME DE PROGRAMMATION LINÉAIRE FRACTIONNAIRE DISJONCTIVE

Les auteurs s'occupent d'une classe de problèmes de programmation linéaire fractionnaire où les restrictions sont du type „ou". L'ouvrage est divisé en cinq sections. La première section a un caractère introductif. La seconde section présente quelques situations offrant cette sorte de problèmes. La section trois contient une série de variantes du problème de programmation linéaire fractionnaire disjonctive et chaque variante est transformée en un problème de programmation linéaire fractionnaire entière mixte. La quatrième section présente, pour une telle variante, une méthode de résolution qui est illustrée par un exemple. La dernière section contient quelques conclusions.

К ПРОБЛЕМЕ ДРОБНОГО РАЗДЕЛИТЕЛЬНОГО ЛИНЕЙННОГО ПРОГРАММИРОВАНИЯ

В настоящей работе речь идёт о классе задач дробного линейного программирования в которых выступают ограничения типа «или». Работа разделена на 5 частей. Первая часть имеет характер введения. Во второй части приведены некоторые случаи, когда проявляются такие задачи. Третьяя часть содержит в себе ряд вариантов задач дробного разделительного линейнного программирования и каждый вариант преобразован в задачу смешанного целочисленного дрогного линейнного программирования. В четвёртой части приведён метод разрешения задачи, который проилюстрирован примером. В последней части работы приведены выводы.