N. RACOVEANU, Prof., Dr. eng.
M. ALTĂR
S. CRUCEANU

A Model of Economic Growth Interpreted in the Theory of Distributions

The idea of considering the total volume of investments as a sum of a continuous function and a distribution has been the starting point of this study. With respect to the evolution of capital goods, the depreciation rate is assumed to be a function of time. The notion of composable families of distributions is used with a view to solve the differential equation describing the evolution of capital goods.

I. The study of models of economic growth has acquired particular importance lately. Utilizing a modern mathematical apparatus, these models are elaborated in view of studying the evolution of national economy and also in view of reaching optimum decisions in a series of problems that are fundamental for its development, such as: determining the optimum ratio between the gross investment and the consumption, the optimum distribution of the net investment between capital goods industries and consumer goods industries, and so on.

For all the models, starting from the first, elaborated towards 1928 by G. A. Feldman and F. Ramsey, and passing on to the latest, it is characteristic that the following elements are used as a starting point:

a. an initial capital stock K_0;
b. an assumption about population growth;
c. an aggregate production function;
d. an optimality criterion.

It should be mentioned that, whereas at the microeconomic stage the choice of the optimum criterion does not raise any special problems since it is, from case to case, the reaching of the maximum physical volume or value of production, of the minimum production costs, of the maximum profit and so on, it has not yet been possible to establish such criteria at the macroeconomic stage. Although some suggestions have been elaborated,

the problem still remains an open one from the standpoint of economic theory.

With respect to the development of the labour force, in most models it is assumed to be of the form:

$$L(t) = \exp(nt) \tag{1}$$

where $L(0) = 1$ has been considered.

II. With respect to the depreciation of capital goods, all the authors consider that this can be characterized by a constant proportional rate, μ. In other words, it is admitted that the evolution of capital goods is described by a differential equation of the form

$$\dot{K}(t) = I(t) - \mu K(t), \qquad \mu = \text{constant}. \tag{2}$$

The authors of this paper feel serious doubts concerning the above. Indeed, to assume, in the case of a model of economic growth having an infinite horizon or even a sufficiently large finite horizon, that the evolution of capital goods is described by equation (2) amounts to more than a simplifying hypothesis.

Taking this into account the following equation is suggested to describe the evolution of capital goods:

$$K(t) = K(0)\,\Psi(t) + \int_0^t I(\theta)\,\Psi(t-\theta)\,d\theta \tag{3}$$

where $\Psi(.)$ is a weight function.

The following hypotheses can be made with reference to the function $\Psi(t)$:

a. it is strictly decreasing;

b. it becomes zero outside an interval

$$(t-\tau,\ t]$$

c. $\Psi(0) = 1$

In this case equation (3) can be written:

$$K(t) = \int_{t-\tau}^{t} I(\theta)\,\Psi(t-\theta)\,d\theta \text{ or} \tag{4}$$

$$K(t) = (I * \Psi)(t) \tag{5}$$

Assuming the weight function $\Psi(.)$ to be derivable, and the function $I(.)$ to be continuous, we obtain from equation (3):

$$\dot{K}(t) = K(0)\,\dot{\Psi}(t) + I(t) + \int_0^t I(\theta)\,\dot{\Psi}(t-\theta)\,d\theta \tag{6}$$

and from equation (4) we obtain:

$$\dot{K}(t) = I(t) + \int_{t-\tau}^{t} I(\theta)\,\dot{\Psi}(t-\theta)\,d\theta \tag{7}$$

In case we assume

$$\dot{\Psi}(t-\theta) = -\mu(t)\,\Psi(t-\theta) \tag{8}$$

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$$\Psi(t) = \exp\left(-\int_0^t \mu(u+\theta)\,du\right) \tag{9}$$

equation (6) becomes:

$$\dot{K}(t) = I(t) - \mu(t)\ K(0)\ \Psi(t) - \mu(t) \int_0^t I(\theta)\ \Psi(t-\theta)\ d\theta$$

or

$$\dot{K}(t) = I(t) - \mu(t)\ K(t) \tag{10}$$

Equation (10) is advantageous in comparison with equation (2) because the depreciation rate of capital goods is no longer considered constant, but a function of t.

III. With respect to the gross investments $I(t)$, the following premises will be used as a starting point:

The total gross investments $I(t)$ are divided into two parts: common or continuous investments (I_c), and special investments, of particular importance, (I_s).

By special investments, noted by I_s, the authors mean the industrial objectives for whose achievement national economy must make a more prolonged effort.

Under these conditions we can write:

$$I(t) = I_c(t) + \sum_k I_s^k\ \delta(t - t_k) \tag{11}$$

where the Dirac distribution was noted by δ.

By substituting in formula (10) we obtain:

$$\dot{K}(t) + \mu(t)\ K(t) = I_c(t) + \sum_k I_s^k\ \delta(t - t_k) \tag{12}$$

In view of solving this linear differential equation with variable coefficients, we shall use the notion of composable families of distributions [1].

It is well known that a family of distributions $\psi = \{\psi_s\}_{s \in R}$ is called composable :

a. if $\forall x \in \mathcal{D}$ the function

$$y(s) = <x(t),\ \Psi_s(t)> \in \mathcal{D}$$

b. if $x_n \in \mathcal{D}$ and $x_n \to 0$ in $\mathcal{D}$,

then the string $y_n(s) = <x_n(t),\ \Psi_s(t)>$ tends towards 0 in $\mathcal{D}$.

It is likewise known that if the family Ψ is composable, then with any distribution f, a distribution $h = \Psi \circ f$ (the result of the composition of the composable family Ψ with a distribution f), thus defined:

$$<x,\ \Psi \circ f> = \ll x(t),\ g_s(t)>, f(s)> \tag{13}$$

can be placed in correspondence, whatever the elementary function $x \in \mathcal{D}$

It is readily seen that the family $\Delta = \{\delta_s\}_{s \in R}$ is composable, and we have:

$$\Delta \circ f = f, \text{ for any } f.$$

The fact will also be used that if D is a linear differential operator with infinitely derivable variable coefficients, then for any composable family $\Psi = \{\Psi_s\}_{s \in R}$ the family $DG = \{D\Psi_s\}_{s \in R}$ is likewise composable and,

whatever the distribution f, we have:

$$D(H \circ f) = (DG) \circ f$$

In our problem we shall only consider "normal" families of distributions $\Psi = \{\Psi_s\}_{s \in R}$, where the distribution Ψ_s has its support on the halfline $[s, +\infty)$. For such distributions the function $y_{(s)} = <g_s(t), x(t)>$ has its support bounded on the left, and the conditions a) and b) of composability become:

a'. the function $y(s) = <g_s(t), x(t)>$ is indefinitely derivable for any $x \in \mathcal{D}$;

b'. from $x_n \in \mathcal{D}$, $x_n \to 0$ in $\mathcal{D}$, there results that the functions $y_n(s) = <g_s(t), x_n(t)>$ as well as their derivatives of any order converge uniformly towards zero along any segment.

It can readily be shown that if G is normal and f has its support bounded on the left, then $G \circ f$ has a meaning.

We shall use the above results to solve equation (12) where $I_c(t)$ is continuous along R and $I_c(t) = 0$ for $t < 0$, $\{I_s^k\}_{k \in N}$ a string of positive numbers and $\{t_k\}_{k \in N}$ an increasing string of positive numbers, convergent at $+\infty$.

The solution to equation

$$K(t) + \mu(t)\, K(t) = \delta_s \tag{12'}$$

is of the form:

$$K(t) = C \exp\left(-\int_0^t \mu(\theta)\, d\theta\right) \tag{14}$$

where C is a distribution.

Taking into account the fact that the distribution C must satisfy the equation

$$C' \exp\left(-\int_s^t \mu(\theta)\, d\theta\right) = \delta_s$$

and that the derivative of the distribution δ_s is Heavyside's distribution $h(t-s)$, defined as follows:

$$h(t-s) = \begin{cases} 0 & t < s \\ 1 & t \geqslant s, \end{cases} \tag{15}$$

by integrating in parts, we have:

$$C = \exp.\left(=\int_s^t \mu(\theta)\, d\theta\right) h(t-s) - \text{prime}\left[\exp\left(\int_0^t \mu(\theta)\, d\theta\right)' h(t-s)\right] \tag{16}$$

where the primitive of the respective distribution was noted by "prime".

It is clear that, for $t < s$, we have $c(t) = 0$, and for $t > s$, we have:

$$C(t) = \exp\left(\int_0^t \mu(\theta)\, d\theta\right) - \exp\left(\int_0^s \mu(\theta)\, d\theta\right) \int_s^t \exp\left(\int_0^v \mu(\theta)\, d\theta\right)' dv$$

hence:

$$C(t) = \exp\left(\int_0^s \mu(\theta)\, d\theta\right) h(t-s) \tag{17}$$

Therefore, a solution to equation (12') is:

$$\Psi_s(t) = \exp\left(-\int_s^t \mu(\theta)\, d\theta\right) h(t-s) \tag{18}$$

The family $\Psi = \{\Psi_s(t)\}_{s\in R}$ is normal, and since the right-hand member of equation (12) has its support bounded on the left, one of its solutions will be of the form:

$$K(t) = \Psi \circ [I_c + \Sigma\, I_s^k \delta(t - t_k)] \tag{19}$$

In the case where we consider

$$\begin{cases} \mu(\theta) = \mu = \text{constant} \\ I_s^k = \begin{cases} K_0 & k = 0 \\ 0, & k \neq 0 \end{cases} \\ I_c(t) = I(t) \end{cases} \tag{20}$$

From (18) there results:

$$\Psi_0(t) = \exp(-\mu t)\, h(t) \tag{21}$$

and formula (20) becomes:

$$K(t) = \Psi_0 * [I + K_0 \delta(0)] \tag{22}$$

or

$$K(t) = K_0 \exp(-\mu t)\, h(t) + \int_0^t I(\theta) \exp[-\mu(t-\theta)]\, d\theta \tag{23}$$

Formula (23), obtained as a particular case of formula (9), is the one utilized in most models of economic growth.

In the general case, the volume of current investments I_c, the volume of special investments I_s, as well as the instants when they start operating, being given, it is possible, on the basis of formula (19), to study the evolution of the main factor of a model of economic growth, viz. that of the capital goods.

To solve equation (12)

$$K(t) = -\mu(t)\, K(t) + I_c(t) \sum_k I_k^s \delta(t - t_k)$$

a series of results obtained by D. Wexler [5] can also be used.

To this end we shall consider equations:

$$\dot{K}_s(t) = \mu(t)\, K_s(t) + \Sigma\, I_s^k \delta(t - t_k) \tag{24}$$

$$\dot{K}_c(t) = \mu(t)\, K_c(t) + I_c(t) \tag{25}$$

If we note by K_s, and by K_c, respectively, the solutions to equations (24) and (25) it is obvious that the solution to equation (12) is:

$$K(t) = K_s(t) + K_c(t) \tag{26}$$

From the economic point of view, $K_s(t)$ represents the volume of capital goods proceeding from special investments, and $K_c(t)$ the volume of the capital goods proceeding from common investments.

The solution $K_c(t)$ to equation (25) is readily found, since the equation is linear and non-homogeneous.

With regard to equation (24), one of its solutions is a distribution with continuous derivatives in every interval, of the form $[t_k, t_{k+1})$. More precisely, we have:

$$K_s(t) = C_k \exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right)$$

with (27)

$$t_k \leqq t < t_{k+1}$$

where C_k satisfies the following difference equation:

$$C_k = \exp\left(-\int_{t_{k-1}}^{t} \mu(\theta)\, d\theta\right) C_{k-1} + I_s^k \tag{28}$$

Substituting C_k from (28) in the expression of $K_s(t)$ given by formula (27) we have:

$$K_s(t) = C_{k-1} \exp\left(-\int_{t_{k-1}}^{t} \mu(\theta)\, d\theta\right) + I_s^k \exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right) \tag{27'}$$

where

$$t_k \leqslant t < t_{k+1}.$$

Making $t = t_k$ in formula (27), we obtain:

$$K_s(t_k) = C_k \tag{29}$$

In other words, constant C_k represents, from the economic point of view, the volume of the special capital goods K_s at the instant $t = t_k$.

Formula (27') can also be written:

$$K_s(t) = K_s(t_{k-1}) \exp\left(-\int_{t_{k-1}}^{t} \mu(\theta)\, d\theta\right) + I_s^k \exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right) \tag{27''}$$

where

$$t_k \leqslant t < t_{k-1}$$

From formula (27'') it can be seen that the volume of capital goods K_s at the instant $t \in [t_k, t_{k+1}]$ is equal to the sum of the volume of capital goods K_s at the instant t_{k-1}, corrected by the weight function

$$\Psi(t) = \exp\left(-\int_{t_{k-1}}^{t} \mu(\theta)\, d\theta\right)$$

and the volume of investments I_s^k having started operating at the instant k, likewise corrected by the weight function

$$\exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right).$$

The solution to equation (24) is:

$$K_s(t) = K_s(0) \exp\left(-\int_0^t \mu(\theta)\, d\theta\right) + \sum_{t_1 \leqslant t_k \leqslant t} \exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right) I_s^k$$

for $t \geqslant t_1$

There results, therefore, that in the hypothesis where at the instants $t_1, t_2, \ldots$, the special investments $I_s^1, I_s^2, \ldots$, start operating, and the continuous investments are $I_c(t)$, the volume of capital goods at the instant $t > t_1$ is:

$$K(t) = K_s(t) + K_c(t) = K_1(0) \exp\left(-\int_0^t \mu(\theta)\, d\theta\right) + \sum_{t_1 \leqslant t_k \leqslant t} \exp\left(-\int_{t_k}^{t} \mu(\theta)\, d\theta\right) I_s^k + \\ + K_c(t) \tag{31}$$

where $K_c(t)$ is the solution to the non-homogeneous linear equation (25).

An interesting case is the one where:

$$I_c(t) = \alpha(t)\, K(t) \tag{32}$$

In this case equation (12′) becomes:

$$\dot{K}(t) = -[\mu(t) - \alpha(t)]\, K(t) + \sum_k I_s^k \delta(t - t_k) \tag{12''}$$

The solution to equation (12″) is

$$K(t) = K_0 \exp\left[-\int_{t_0}^{t} [\mu(\theta) - \alpha(\theta)]\, d\theta\right] + \sum_{t_1 \leqslant t_k \leqslant t} \exp\left[-\int_{t_k}^{t} [\mu(\theta) - \alpha(\theta)]\, d\theta\right] I_k^s \tag{33}$$

In the case where the policy of completing the depreciation capital goods from current investments is adopted, the balance being allotted for special investments, we have:

$$\alpha(t) = \mu(t) \tag{34}$$

and from formula (33) there results:

$$K(t) = K_0 + \sum_{t_1 \leqslant t_k \leqslant t} I_s^k \tag{35}$$

The economic interpretation of the above formula is obvious.

If we note

$$\alpha(t) - \mu(t) = \gamma(t) \tag{36}$$

there results from formula (33)

$$K(t) = K_0 \exp\left(\int_0^t \gamma(\theta)\, d\theta\right) + \sum_{t_1 \leqslant t_k \leqslant t} \exp\left(\int_{t_k}^t \gamma(\theta)\, d\theta\right) I_k^s \qquad (37)$$

Based on the above considerations, the authors have in view to elaborate a model of economic growth whose solution would make it possible to determine the economic policy with regard to the optimum use of the product created: for consumption, current investments and special investments, as well as the optimum use of manpower.

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UN MODELE D'ACCROISSEMENT ECONOMIQUE INTERPRETE DANS LA THEORIE DES DISTRIBUTIONS

Les auteurs de l'ouvrage prennent pour point de départ l'idée du partage des investissements totaux $I(t)$ en deux catégories: les investissements continuels ou ordinaires (I_c) et les investissements spéciaux, d'une importance particulière (I_s). Les investissements spéciaux comprennent les objectifs pour la réalisation desquels l'économie nationale doit fournir des efforts particuliers, à plus long terme.

Dans ces conditions, nous avons comme point de départ la relation:

$$I(t) = I_c(t) + \sum_k I_s^k \delta(t - t_k)$$

où on a noté par δ la distribution Dirac.

Pour ce qui est de la formule décrivant l'évolution des fonds fixes, celle-ci est supposée avoir la forme:

$$K(t) + \mu(t)K(t) = I_c(t) + \sum_k I_s^k \delta(t - t_k)$$

où l'on estime que le quota d'usure $\mu(\cdot)$ est une fonction de t.

Pour la solution de cette équation on utilise la notion de familles composables de distributions ainsi qu'une méthode élaborée par D. Wexler.

МОДЕЛЬ ЭКОНОМИЧЕСКОГО РОСТА, ВЫРАЖЕННАЯ ТЕОРИЕЙ РАСПРЕДЕЛЕНИЙ

В этой работе авторы исходят из идеи разделения общих вложений $I(t)$ на две категории: непрерывные или обычные вложения (I_c) и специальные вложения особого значения (I_s). Под специальными вложениями авторы подразумевают те объекты, для создания которых народному хозяйству приходится делать особые усилия в течение продолжительного времени.

В этих условиях исходят из следующего соотношения:

$$I(t) = I_c(t) + \sum_k I_s^k \delta(t - t_k)$$

где δ обозначено распределение Дирака.

Что касается формулы, дающей картину эволюции основных фондов, она предполагается в следующей форме:

$$K(t) + \mu(t)\, K(t) \sum_k = I_c(t) + \sum_k I_s^k \delta(t - t_k)$$

где норма износа $\mu(\cdot)$ считается функцией t.

Для решения этого уравнения используется понятие сравниваемых семей распределения а также метод, разработанный Д. Векслером.