I. NĂDEJDE

On the Ergodicity of Finite Markov Chains

The aim of this paper is to determine the minimum number of positive elements and their position in a stochastic matrix on condition that the Markov chain so defined be ergodic.

Let be a system S capable of showing a finite set of states, $\Omega = \{S_i\}$; $i = 1, 2, \ldots r$. It will be admitted that the state of the system can only change at finite time intervals, i.e. at discrete points of the time axis. Adopting a suitable unit of measurement for the time intervals that elapse between the changes of state, we can note by $t = 0, 1, 2, \ldots$ the instants of the eventual transitions from one state to another; in the notion of "transition" is also included the particular case where the system remains in the same state.

If the state of the system at the instant t_0 determines completely the states of the system at any later instant $t > t_0$, the evolution of the system is *deterministic*. The evolution is random if the system that is in the state S_t at the instant t will be in the state S_i at the instant $t + s$, $s > 0$ with a certain probability. The process is purely random if the evolution of the system is independent of its past. Noting by $X(t)$ the random variable that characterizes the state of the system S at the instant t, and by e a subset of states in Ω, a purely random process is characterized by

$$P_r\{X(t) \in e \mid X(\theta_1) = S_1, \ldots, X(\theta_n) = S_n\} = P_r\{X(t) \in e\} \tag{1}$$

It is said that the evolution of the system is Markovian if

$$P_r\{X(t) \in e \mid X(\theta_1) = S_1, \ldots, X(\theta_n) = S_n\} = P_r\{X(t) \in e \mid X(\theta_n) = S_n\} \tag{2}$$

$$\forall_n \text{ and } \forall\ \theta_1, \ldots, \theta_n \text{ so that } \theta_1 < \ldots \theta_n < t$$

i.e. if the state of the system at the instant t depends on its previous evolution by the instrumentality of its state at the instant immediately pre-

ceding the instant t. The process consisting in changes of state of the system S characterized by the equality (2) is called a *discrete Markov process.*

A Markov process is *homogeneous in time* if

$$P_r\{X(t+T)\in e \mid X(\theta_n+T)=S_n\}=P_r\{X(t)\in e \mid X(\theta_n)=S_n\} \qquad (3)$$

i.e. if the transition probabilities of the system are not influenced by a translation in time or, otherwise said, if the probability of transition from S_i to S_j is independent of the number of preceding transitions.

A *Markov chain* is a *discrete* Markov process, *homogeneous in time*, for which *the instants of eventual transition are equidistant.*

A Markov chain is *finite* if the set of states of the system contains a finite number of elements.

If, instead of events of the type $X(t)\in e$, where e is a subset of Ω, we consider the events $X(t)=S_i$, where S_i is an element of Ω, the Markov chain is characterized by

$$P_r\{X(t_{n+1})=S_j \mid X(t_n)=S_i=p_{ij} \qquad (4)$$

which is a function of two variables S_i and S_j. These variables can acquire r values if r is the number of elements of the set Ω. Consequently, function (4) can acquire r^2 values that may constitute the elements of a $r\times r$ size matrix. This matrix, called *transition probability matrix* will be noted by

$$P=(p_{ij})\ \ i,j=1,\ 2,...,\ r \qquad (5)$$

PROPERTIES OF THE TRANSITION PROBABILITY MATRIX

The matrix $P=(p_{ij})\ i,\ j=1,\ 2,\ ...,\ r$ has two basic properties:

1°. All the elements of matrix P are non-negative, $p_{ij}\geqslant 0, \forall\ i,j=1,...,r$.

2°. The sum of the elements of any line of matrix P is equal to the unit

$$\sum_{j=1}^{r} p_{ij}=1, \qquad i=1,...r$$

A matrix having properties 1° and 2° is called *stochastic matrix* or *Markov matrix.*

If the vector-column having r elements equal to the unit is noted by η, property 2° will be written

$$P\eta=\eta \qquad (6)$$

indicating that η is a proper vector to the right, of matrix P, corresponding to its eigenvalue 1.

Using this formulation of property 2° it can readily be proved that:

3°. The product of two stochastic matrices is a stochastic matrix.

Particularly, all the powers of a stochastic matrix are stochastic matrices.

For a discrete Markov process the state of system S at an instant t whatever, is a random variable $X(t)$ that may acquire the values $i = 1, 2,...,r$. Let us note by $x_i(t)$ the probability for the system to be in the state i at the instant t. Then the probability $x_i\ (t+1)$ for the system to be in the state i at the instant $(t+1)$ is

$$x_i(t+1) = \sum_{j=1}^{r} p_{ij}\ x_j\ (t); \quad i = 1,\ 2,\ ...,\ r \tag{7}$$

If we note by $\mathbf{x}(t)$ the line vector having the elements $x_i(t)$; $i = 1, ..., r$ relations (7) may be written

$$\mathbf{x}(t+1) = \mathbf{x}(t)P$$

or, generally,

$$\mathbf{x}(t+n) = \mathbf{x}(t)P^n \tag{8}$$

Vector $\mathbf{x}(t)$ is called the *probability vector* at the instant t.

A vector $\mathbf{x}(x_1,\ x_2,\ ...,\ x_r)$ is a probability vector if it satisfies the conditions:

1°. All its components are non-negative

$$x_i \geqslant 0, \qquad i = 1, ... r$$

2.° The sum of the vector components is equal to the unit,

$$\sum_{i=1}^{r} x_i = 1.$$

For $t = 0$, relation (8) becomes

$$\mathbf{x}(n) = \mathbf{x}\ (0) \times P^n \tag{9}$$

where $\mathbf{x}(0)$ is *the probability vector of the initial state* of the system or the initial probability distribution.

From (9) there results that given the initial probability distribution, $\mathbf{x}(0)$, the subsequent behaviour of the system is determined by the matrix of transition probabilities P.

ERGODICITY CONDITIONS FOR FINITE MARKOV CHAINS

If $\lim_{n\to\infty} \mathbf{x}(n)$ exists, let $\mathbf{x}^*$ be the limit vector. From (9) we have

$$\mathbf{x}^* = \lim_{n\to\infty} \mathbf{x}(n) = \mathbf{x}(0) \cdot \lim_{n\to\infty} P^n \tag{10}$$

which shows that, generally, $\mathbf{x}^*$ is dependent on both P and $\mathbf{x}(0)$.

A Markov chain is called *ergodic* if the limit probability distribution $\mathbf{x}^*$, is independent on the initial distribution $\mathbf{x}(0)$. It is proved that a finite Markov chain is ergodic if, and only if the following conditions are fulfilled:

1°. The transition matrix in n stages, P^n, tends towards a limit P^* when $n \to \infty$.

2°. P^* has all its lines identical to the vector of limit probabilities, $\mathbf{x}^*$.

The necessary and sufficient ergodicity conditions of the Markov chains can be expressed contingent upon the transition matrix P spectre.

It is proved that a finite Markov chain is *ergodic* if, and only if:

1°. The unit is the simple eigenvalue of the transition probability matrix P and

2°. All the other eigenvalues of the matrix P have their modulus strictly smaller than the unit.

A finite Markov chain is called ergodic in the mean if for $n \to \infty$, $\mathbf{x}(n)$ tends, in the Cesaro sense, to a limit distribution $\mathbf{x}^*$, independent of the distribution $\mathbf{x}(0)$.

From the definition of ergodicity in the mean and from relation (10) there results that a finite Markov chain is ergodic in the mean if $P^n \to P^*$ for $n \to \infty$, in the Cesaro sense.

It is proved that this condition is fulfilled if and only if the unit is a simple eigenvalue of the transition probability matrix.

There results from the above that to determine the ergodicity of a finite Markov chain the $\left|\lim_{n\to\infty}\right| P^n$ must be computed, or a polynominal equation of degree r must be solved. For r sufficiently large both procedures entail difficulties.

Another method that may be used to determine the ergodicity of finite Markov chains is to study the graph associated to the matrix of transition probabilities P.

Let $G = (S, \Gamma)$ be the graph associated to matrix P. The set of graph vertices S, has the system states as elements. We shall note by i the vertex of graph G corresponding to state S_i, such that the set S of the graph associated to a finite Markov chain with r states be $S = \{1, 2, ..., r\}$. The set Γ of the graph arcs is formed of elements $p_{ij} > 0$ of the matrix P.

For example, the graph associated to the Markov chain having the transition probability matrix P_1 is shown in Fig. 1. It can be noticed

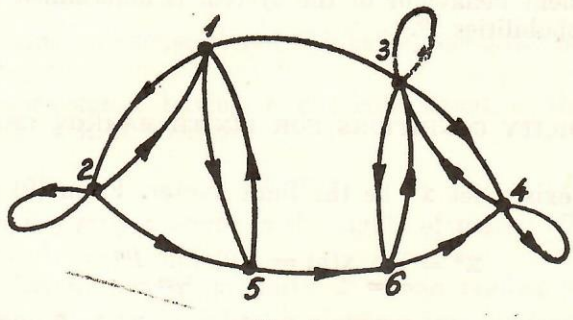


Fig. 1

that in view of plotting the graph associated to matrix P it is not necessary to know the values of the elements p_{ij}; it is sufficient to know which of these elements are different from zero (non-null).

$$P_1 = \begin{bmatrix} \cdot & p_{12} & p_{13} & \cdot & p_{15} & \cdot \\ p_{21} & p_{22} & \cdot & \cdot & p_{25} & \cdot \\ \cdot & \cdot & p_{33} & p_{34} & \cdot & p_{36} \\ \cdot & \cdot & p_{43} & p_{44} & \cdot & \cdot \\ p_{51} & \cdot & \cdot & \cdot & \cdot & p_{56} \\ \cdot & \cdot & p_{63} & p_{64} & \cdot & \cdot \end{bmatrix}$$

Note: The elements $p_{ij} = 0$ in P_1 have been replaced by dots.

The relation $\mathcal{P}$ is defined on the set S of the vertices of graph G associated to a finite Markov chain. It consists of the statement:

$\mathcal{P}$: "there is a path from i to j"

agreeing that any vertex is connected with itself by a path of length [1]) zero. This relation will be noted by $i\mathcal{P}_j$.

Relation $\mathcal{P}$, defined on the set of vertices of a graph G containing circuits is *reflexive* [2]) and *transitive* [3]); it therefore constitutes a *pre-order.*

Let $\mathcal{S}$ be the relation consisting of the statement:

$\mathcal{S}$: "i and j are on the same circuit or they coincide."

It is easy to establish that

$$i\mathcal{P}i \wedge j\mathcal{P}i \Leftrightarrow i\mathcal{S}j$$

Relation $\mathcal{S}$ is *reflexive* [4]), *transitive* [5]) and *symmetric* [6]) so that it constitutes a relation of equivalence. It is well known that any relation of equivalence determines a partition of the elements of the set on which it is defined, the subsets of the partition being called *classes of equivalence.*

Relation $\mathcal{S}$ partitions graph G in classes of equivalence which are very connect components of G.

A new graph $g = (s, \gamma)$ can now be plotted. The set s of the vertices has as elements the very connect components of G; the arc (u, v), $\forall u, v \in s$, belongs to the set γ if, and only if, there is at least one arc joining a vertex of the very connect component u to a vertex of the very connect component v. The graph $g = (s, \gamma)$ is called the *reduced graph* associated to graph G. It can readily be seen that graph g has no circuits. Indeed

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[1]) By length of a path is understood the number of arcs which make up the path.

[2]) $i\mathcal{P}i$ is real for any $i \in S$ by the definition of the path of length zero.

[3]) Relation $\mathcal{P}$ is transitive since $i\mathcal{P}j \wedge j\mathcal{P}z \Rightarrow \forall\, i, j, z \in S$.

[4]) The reflectivity of relation $\mathcal{S}$ is assured by its definition.

[5]) $i\mathcal{S}j \wedge j\mathcal{S}k \Rightarrow i\mathcal{S}k$ in $\mathcal{S}$ is transitive.

[6]) Symmetry is obvious since, if i is on the same circuit as j, j is likewise on the same circuit as i.

if $(u, v) \in \gamma$, then $(v, u) \notin \gamma$ since otherwise u and v would constitute a single very connect component of G.

Relation $\mathcal{P}$ defined on the set s is *reflexive*, *transitive* and *antisymmetrical* so that it constitutes a relation of order.

An element $u \in s$, i.e. a very connect component of graph G, is called *minimal element* if relation $x \mathcal{P} u$ is true for any $x \in s$; if the minimal element is unique it bears the name of *minimum element.*

In short, it can be said that the relation of equivalence $\mathcal{E}$ partitions the set of vertices of graph G into classes of equivalence and that relation $\mathcal{P}$ determines the minimal classes of equivalence.

The classes of equivalence are very connect components of graph G so that from any vertex i of a very connect component, any other vertex j belonging to *the same* very connect component can be reached; from a vertex of graph G, belonging to a minimal class, *only* the vertices of the graph belonging to the same minimal class can be reached.

It can readily be seen that any finite graph has at least one minimal element.

In the example in Fig. 1 the vertices of the graph — hence the states of the Markov chain represented by this graph — are partitioned into two classes of equivalence u and v. Class u is formed by 1, 2 and 5, and v by 3, 4, and 6. The reduced graph is shown in Fig. 2; v is the minimum element.

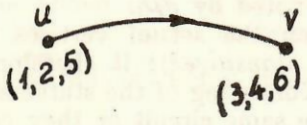


Fig. 2.

In conclusion, the graph representing the transition matrix of a finite Markov chain can be used to determine the states constituting the minimal elements. These states are called *final states* of the Markov chain; all the other states are called *transition states.*

In the above example the final states are 3, 4, 6 and the transition states are 1, 2, 5.

Since any finite graph has at least one minimal element, there results that any finite Markov chain has at least one final class.

If the minimal elements of g are formed each by a single peak of graph G, the states corresponding to these peaks are called *absorbing states.*

It can readily be seen that if graph g has $s < r$ minimal elements, the matrix associated to graph G is of the form

$$P = \begin{bmatrix} A_1 & 0 & \ldots 0 & 0 \\ 0 & A_2 & \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots A_s & 0 \\ B_1 & B_2 & \ldots B_s & A_{s+1} \end{bmatrix} \tag{11}$$

where: A_l, $l = 1 \ldots, s + 1$ are irreductible square matrices. If some of the matrices A_l, $l = 1 \ldots, s$ are formed of a single element, this element has the value 1 since P is a stochastic matrix. If matrix B_l, corresponding to matrix A_l which reduces to a single element, is different from zero (non null), the Markov chain has an absorbing state.

The matrices A_l, $l = 1,\ldots,s$, are called final matrices because they are associated to the final states of the Markov chain. These matrices cannot be partitioned since the minimum elements of graph g are very connect components of the graph G.

The matrices, B_k, $k = 1, \ldots, s$ are called transition matrices.

If at least one of the matrices B_k is zero, the corresponding Markov chain is partitionable. Indeed, let us admit that B_1 is a zero matrix. Let $r_l \times r_l$, $l = 1\ldots, s$, be the dimensions of the matrices A_l, $l = 1, \ldots, s$. Matrix B_1 then has the dimensions $t \times r_1$, $t = r - \sum_{l=1}^{s} r_l$. From the hypothesis $B_1 = 0$ there results that $p_{ij} = 0$ for all $i = r - t,\ldots,r$ and $j = 1,\ldots, r_1$ so that the transitions between the first r_1 states of the Markov chain and the other $r - r_1$ states are impossible. Otherwise said, the first r_1 states form an isolated system.

Hereafter we shall only deal with non-partitionable Markov chains, so that we shall admit that $B_k, k=1,\ldots,s$ are non-null matrices different from zero.

Matrix P_1 in the above example can be brought to form (11) by suitably permuting the lines and columns. Thus we have:

$$P_1' = \left[\begin{array}{ccc:ccc} p_{33} & p_{34} & p_{36} & & & \\ p_{43} & p_{44} & \cdot & & 0 & \\ p_{63} & p_{64} & \cdot & & & \\ \hdashline p_{13} & \cdot & \cdot & \cdot & p_{12} & p_{15} \\ \cdot & \cdot & \cdot & p_{21} & p_{22} & p_{25} \\ \cdot & \cdot & p_{56} & p_{51} & \cdot & \cdot \end{array}\right] = \left[\begin{array}{cc} A_1 & 0 \\ B_1 & B_2 \end{array}\right].$$

There stand out from P_1' that we have a single final matrix A_1 and hence a single set of final states and a transition matrix B_1.

From the graph associated to a finite Markov chain the following can also be deduced:

Proposition 1. For any final state i the probability for the system never to return to i after having once left the state i, is zero.

Indeed, i being a final state, it belongs to a minimal class u. There results from the definition of the minimal classes that all the successors to the state i belong to class u since the system, leaving the state i, passes into a state j of the same final class. The final classes being very connect graphs, i and j are on the same circuit. The statement of proposition 1 is thus proved. The reciprocity of this statement is easy to prove.

Proposition 2. For any state of transition i, the probability for the system never to return to i after having once left the state i is positive.

Indeed, starting from the state of transition i, the system can pass to a state belonging to a final class, so that its return to the state i becomes impossible.

The reciprocity of this statement is likewise true.

To summarize propositions 1 and 2, let us note by E_i the event so that once the system has left the state S_i it should never return to this state.

With this notation it is possible to write:

$$\boxed{i \text{ is a final state}} \Leftrightarrow \boxed{P_r(E_i) = 0}$$

$$\boxed{i \text{ is a transition state}} \Leftrightarrow \boxed{P_r(E_i)}$$

Proposition 3. A finite Markov chain is ergodic in the mean if, and only if, the graph associated to the chain has a minimal class. In this case, the transition matrix, P can be

a) unpartitionable, or

b) of the form $P = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$

In both hypotheses the unit is a simple characteristic root of matrix P, hence the chain is ergodic in the mean.

Let us now assume that the chain has s minimal classes. In this case, after suitable permutation of the lines and columns, the transition matrix can be written under the form (11). Partitioning the unit matrix E, of order r, in the same way in which the transition matrix P of (11) is partitioned and noting by E'^h, $h = 1, \ldots, s+1$, the unit matrices thus obtained, the characteristic equation is written

$$\left| \lambda E - P \right| = \left| \lambda E'^{s+1} - B_{s+1} \right| \prod_{h=1}^{s} \left| \lambda E'^h - A_h \right| = 0 \qquad (12)$$

There readily results from (12) that, in the case considered, $\lambda = 1$ is the characteristic root of order s of the matrix P, hence the chain is not ergodic in the mean.

Proposition 4. A finite Markov chain is ergodic if, and only if the graph associated to the chain has a non-periodic minimal class.

If the graph fulfils the required condition, matrix P can be:

a) a primitive matrix

or

b) a regular matrix

In both cases $\lambda = 1$ is a simple root of the characteristic equation and $|\lambda| = 1 \Rightarrow \lambda = 1$, hence the chain is ergodic.

Let us admit that the minimal class is periodic, of period h. In this case there is a permutation matrix A, so that $P_A = A'PA$ acquire the normal form

$$P_A = A'PA = \begin{bmatrix} 0 & P_1 & 0 & .... & 0 \\ 0 & 0 & P_2 & .... & 0 \\ . & . & . & .... & . \\ 0 & 0 & 0 & .... & P_{h-1} \\ P_h & 0 & 0 & .... & 0 \end{bmatrix} \tag{13}$$

where $P_k (k = 1,..., h-1)$ are matrices of the dimensions $r_k \times r_{k+1}$ and P_k has the dimensions $r_h \times r_1$ so that the diagonal blocks are square and $\sum_{k=1}^{h} r_k = r$. For all integers $\alpha \geqslant 0$, $P_A^{\alpha p+1}$ is of the form

$$P_A^{\alpha h} = \begin{bmatrix} C_1^{\alpha} & 0 & ..... 0 \\ 0 & C_2^{\alpha} & ..... 0 \\ 0 & 0 & C_h^{\alpha} \\ . . . & . . . . . & \end{bmatrix} \tag{14}$$

where C_k^{α}, $k = 1, ..., h$ are primitive matrices of the dimensions $r_k \times r$.

From (13) and (14) there readily results that $P_A^{\alpha p+1}$ is

$$P_A^{\alpha p+1} = \begin{bmatrix} 0 & P_1 C_2^{\alpha} & 0 & .... 0 \\ 0 & 0 & P_2 C_3^{\alpha} & .... 0 \\ .... & .... & ..... & .... \\ 0 & 0 & 0 & P_{h-1} C_h^{\alpha} \\ P_h C_1^{\alpha} & 0 & 0. & .... 0 \end{bmatrix} \tag{15}$$

From (15) one deduces that there is no integer number t so that $P_A^t > 0$, hence P_A is a non-primitive matrix of index h of non-primitivity, which proves that the Markov chain associated to this matrix is not ergodic.

Propositions 3 and 4 can be summarized as follows:

Graph with a periodic minimum element	$\Leftrightarrow$	Ergodic Markov chain in the mean	$\Leftrightarrow$	$\lambda = 1$ simple root of the characteristic equation
Graph with a non-periodic minimum element	$\Leftrightarrow$	Ergodic Markov chain	$\Leftrightarrow$	$\lambda = 1$ simple root of the characteristic equation and all other roots of strict modulus smaller than 1

RELATIONS BETWEEN THE NUMBER AND POSITION OF POSITIVE ELEMENTS OF A STOCHASTIC MATRIX AND THE ERGODICITY OF THE ASSOCIATED MARKOV CHAIN

There results from the above that the discussion of the subject set forth only amounts to the determination of the minimum number of positive elements of the transition matrix $P = (p_{ij})$ and to the determination of the position of these elements in such a manner that P should be unpartitionable.

Proposition 5. The stochastic matrix $P = (p_{ij})$, $i, j = 0, ..., r$ which has only r positive elements is unpartitionable if, and only if there exists a permutation matrix A so that the elements g_{ij} of matrix $P_a = A'PA$ satisfy the conditions

$$g_{ij} \begin{cases} > 0 \text{ if } j - i = 1, \\ = 0 \text{ if } j - i \neq 1; \end{cases} \tag{16}$$

$$p_{r1} > 0. \tag{17}$$

The corresponding Markov chain is ergodic in the mean.

The condition is sufficient. Indeed, if g_{ij}, $i = 1, ... r$; $j = 1, ..., r$ satisfy (16) and (17) then

$$p_{12} \times p_{23} \times ... \times p_{k-1,\,k} \times ... \times p_{r1} \neq 0$$

since there is a path from i to j for any $i = 1, ..., r$ and $j = 1, ... r$.

The condition is necessary. Let us admit that $p_{k-1,\,k} = 0$. Since there must exist a non-null (different from zero) element on every line, let us admit that on line k we shall have $p_{k-1,\,t} > 0$, $t \neq k$. In this case $p_{ik} = 0$, for all $i = 1, ..., r$ since there is no path from a vertex whatever $i \neq k$ to the vertex k, hence the matrix is partitionable.

It is obvious that the minimum number of positive elements of a stochastic matrix of order r is equal to r. If these r positive elements are defined by conditions (16) and (17) the graph of the matrix is very connect and cyclic so that the corresponding Markov chain is ergodic in the mean.

Generalizing proposition 5 it may be stated that any irreductible matrix can be set under the form (13) by a suitable permutation of the lines and columns. If $h > 2$, the corresponding Markov chain is ergodic in the mean. If $h \leqslant 2$, P_1 is unpartitionable.

We shall now study the stochastic matrices of dimensions $r \times r$ with $r + 1$ positive elements.

Proposition 6. A stochastic matrix with $r + 1$ positive elements is unpartitionable if, by suitable permutation of its lines and columns, r of the positive elements satisfy conditions (16) and (17). In this case, the $(r + 1)^{\text{th}}$ positive element may occupy a position whatever. If, after the permutation, $p_{k-1,\ k} = 0$, the matrix is unpartitionable if, and only if,

$$p_{k-1,\ t} \times p_{t,\ k} > 0;\ t \neq k \tag{18}$$

The condition is sufficient. We have by hypothesis

$$p_{1,2} > 0;\ p_{2,3} > 0 \ldots p_{k-2,\ k-1} > 0;\ p_{k,\ k+1} > 0, \ldots p_{r,1} > 0 \tag{19}$$

and

$$p_{k-1,\ t} > 0, p_{t,\ k} > 0. \tag{20}$$

If (i and j) $\notin \{k - 1,\ k\}$, there results from (19) that there exists a path from i to j. If (i and j) $\in \{k - 1,\ k\}$, the existence of a path from i to j is assured by condition (20).

The condition is necessary. Let us assume that we replace (20) by

$$p_{k-1,\ t} > 0;\ p_{s,\ k} > 0;\ t \neq s \tag{21}$$

In this case there is no path from i to j for (i and j) $\in \{k - 1,\ k\}$, since the matrix is partitionable.

In the hypothesis that the matrix elements satisfy (19) and (20) the associated Markov chain has a single periodic final class for any t and s, hence it is ergodic in the mean.

If conditions (19) and (20) are satisfied, a sufficient condition of ergodicity is for $(r + 1)$ to be a prime number. Indeed, in this case the associated graph is very connect, having two elementary circuits. Let us note by l_1 and l_2 the length of these circuits, $l_1 + l_2 = r + 1$. If $(r + 1)$ is a prime number, l_1 and l_2 have no common divider, consequently the graph is non-periodic and the associated Markov chain is ergodic. If $(r + 1)$ is not a prime number, the ergodicity depends on the choice of t, a choice which determines the length of the two circuits. Let us admit that in a stochastic matrix having $r + 1$ positive elements, we have, after suitable permutations $p_{r,\ 1} > 0$, $p_{ij} > 0$ for $j - i = 1$, $j \neq k$ and s, $s > k$ but $p_{k-1,k} = 0$ and $p_{s-1,\ s} = 0$. Since the matrix is stochastic, we must have at least one positive element on each of the lines $k - 1$ and $s - 1$, hence the graph has at least one outward incident arc on each of the peaks $k - 1$ and $s - 1$. Let $p_{k-1,\ t} > 0$ and $p_{s-1,\ u} > 0$. It can readily be seen from Fig. 3 that the graph with these r arcs, even for $t = u$ is not very connect, hence the matrix is partitionable. Moreover, the graph cannot become very connect if only one arc is added to it, which is equivalent to the statement that, however the $(r + 1)^{\text{th}}$ positive element were chosen, the matrix would remain partitionable if $t \neq u \neq s - 1$.

The matrix is unpartitionable if, and only if,

$$p_{k-1,\ s} \times p_{k-1,\ s} \times P_{s-1,\ k} > 0.$$

One can establish by similar considerations that if n of the p_{ij} elements, with $j - 1 = 1$, are zero, the matrix is unpartitionable if it has at least

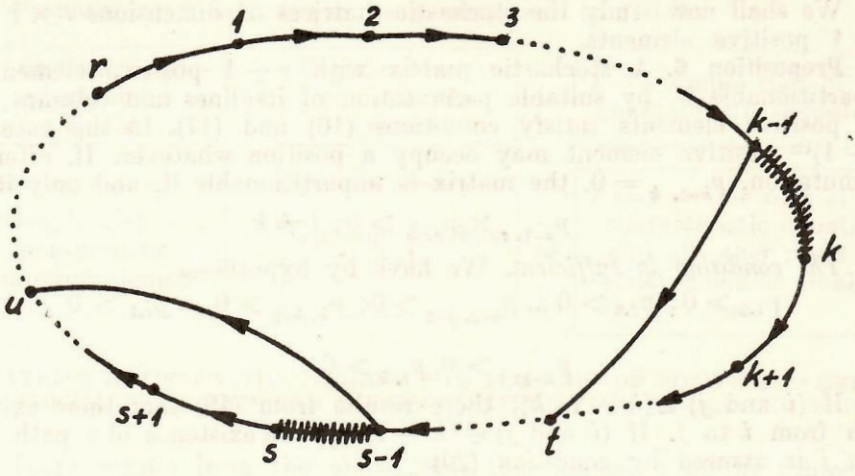


Fig. 3

$r + n + 1$ positive elements thus located that the graph should again become very connect.

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DE L'ERGODICITE DES CHAINES MARKOV FINIES

Soit $P = (p_{ij})$; $i = 1,..., r$; $j = 1,...,r$ la matrice du premier passage d'une chaîne Markov à r positions et $G = (S, \Gamma)$ le graphe associé à cette matrice.

Les conditions d'ergodicité d'une chaîne Markov peuvent être établies en utilisant aussi bien les informations contenues dans la matrice $P(p_{ij})$ que le graphe $G = (S, \Gamma)$. Le graphe G n'étant déterminé que par la position des éléments non-nuls de P, on en tiré la conclusion que l'ergodicité d'une chaîne Markov dépend uniquement du nombre et de la position des éléments positifs de la matrice de premier passage P.

L'auteur établit les conditions nécessaires et suffisantes pour qu'une chaîne Markov avec r positions soit ergodique lorsque le nombre des éléments positifs de la matrice qui le définit est r ou $r + 1$.

ОБ ЭРГОДИЧНОСТИ ЗАКОНЧЕННЫХ ЦЕПЕЙ МАРКОВА

Допустим, что $P = (p_{ij})$; $i = 1, ... r$; $j = 1, ..., r$ матрица первого прохода цепи Маркова с r состояниями и $G = (S, \Gamma)$ — граф, приобщенный к этой матрице.

Условия эргодичности у цепи Маркова могут быть определены путем использования сведений, содержащихся в матрице $P(p_{ij})$ или графа $G = (S, \Gamma)$. Так как граф G определяется только позицией ненулевых элементов из P, приходим к выводу, что эргодичность цепи Маркова зависит только от числа и позиции положительных элементов из матрицы первого прохода P.

Устанавливаются условия, необходимые и достаточные, чтобы цепь Маркова с r. состояниями была бы эргодичной, когда число положительных элементов матрицы которые определяют эту цепь, равно r или $r + 1$.