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A Non-Linear Model of Optimization for Gasoline Blends with Tetraethyl Lead Additive

The application of mathematical methods and of modern computation techniques in optimization and control of some processes in the Romanian petroleum industry provides the accomplishment of considerable additional benefits.

The application of linear programming in view of obtaining gasoline blends out of their components, [5], [7], has lead to the development of formulas qualitatively satisfying the requirements — as to octane number, specific gravity, etc. — for a minimum cost, or of a maximum benefit.

The problem may be solved with satisfactory results if the restriction system is consistent, i.e. if the components can meet the qualitative requirements set for the product. What is, however, happening if components cannot meet some of the requirements, namely the octane number? It is a well known fact that, if a similar case occurs, some additives, such as tetraethyl lead, can improve the octane number of the fuel, leading to their widespread utilization in the petroleum industry. Inasmuch as these additives are, however, very expensive, oil refineries try to obtain the desired blends by utilizing the least possible quantities of additive. These terms of the problem lead to the development of a mathematical function, relating the octane number increase to the amount of tetraethyl lead added to the blend.

EFFECT OF TETRAETHYL LEAD ETHYLATION

If we let O_i be the basic octane number (in natural condition) of component i, prior to ethylation, and $\bar{O}$ be the octane number of the blend, it is

a well known fact that $\bar{O}$ is the mathematical expectation of the components [5] octane numbers O_i i.e.:

$$\bar{O} = \frac{\sum_{i=1}^{n} O_i x_i}{\sum_{i=1}^{n} x_i} \tag{1.1}$$

where x_i is the amount of component i incorporated in the volume V of the blend.

If an amount $q^0/_{00}$ of tetraethyl lead is added to the fuel, its octane number increases by an amount which we shall call G. In the specific case of $q = 0.5$, the amount G of increase of fuel octane number i is S_i and is called the susceptibility, or responsiveneess, of this fuel. The size of this susceptibility is different for various grades of gasoline, according to their chemical composition and sulphur content. The susceptibility of a mixture is the mathematical expectation of the constituents' susceptibilities, i.e.

$$S = \frac{\sum_{i=1}^{n} S_i x_i}{\sum_{i=1}^{n} x_i} \tag{1.2}$$

H. Maurin [1] solves the ethylation problem by considering the function

$$\frac{G}{S} = 2.85\, q_1 + 1.8\, q_2 + 1.25\, q_3$$

where q_i stands for three fictitious allotropic varieties of thetraethyl lead.

In the present paper we shall make use of the studies performed in our country on the problem of the mathematical determination of the fuels' ethylation responsiveness, contingent upon their chemical composition [8].

On adding the amount $q^0/_{00}$ of tetraethyl lead in a fuel, the latter's octane number increases by the amount G, given by the relation

$$G = \sqrt{2pq} \tag{1.3}$$

where p is a constant characterizing each grade of gasoline.

If we consider $q = 0.5$, the fuel susceptibility will be

$$S = G_{0,5^0/_{00}} = \sqrt{P} \tag{1.4}$$

By combining equations (1.3) and (1.4) we obtain

$$\frac{G}{S} = \frac{\sqrt{2pq}}{\sqrt{p}} = \sqrt{2q} \tag{1.5}$$

with $q \leqslant 0.6$.

Considering equations (1.1), (1.2) and (1.5) we obtain

$$\bar{O} = \frac{\sum_{i=1}^{n} O_i x_i}{\sum_{i=1}^{n} x_i} \cdot \frac{\sum_{i=1}^{n} S_i x_i}{\sum_{i=1}^{n} x_i} \times \sqrt{2q} \tag{1.6}$$

It can be seen that the problem is non-linear, due to the fact that beside the unknowns x_i, the amount $\sqrt{2q}$ of tetraethyl lead incorporated in the blend also interferes. We shall see later that this problem can be solved by considering it as a parametric programme.

MATHEMATICAL MODEL OF BLEND

We have at our disposal the known quantities b_i of the constituents, with the basic octane numbers O_i (without additives), the susceptibilities S_i and the specific gravities d_i. Our purpose is to obtain the amount b, with the octane number higher or equal to $\bar{O}$, the specific gravity less or equal to $\bar{d}$, and with a minimum content q of tetraethyl lead.

If we designate by x_i the amount of constituent i incorporated in the blend and by c_i the cost of constituent i, and if we designate $\sqrt{2q}$ by Q, we obtain the following model :

The function

$$f = \Sigma c_i x_i$$

is to be minimized, with the following restrictions:

$$\text{I} \begin{cases} \Sigma x_i = b, \text{ a condition referring to the amount required;} \\ \Sigma \dfrac{x_i}{d_i} \geqslant \dfrac{\Sigma x_i}{\bar{d}}, \text{ a condition referring to the specific gravity;} \\ x_i \leqslant b_i, \text{ a condition referring to availabilities;} \\ \Sigma(O_i + QS_i)\, x_i \geqslant \bar{O}\, \Sigma x_i, \text{ a condition referring to the octane number } x_i \geqslant 0. \end{cases}$$

The model is non-linear due to the condition referring to the octane number. However, by considering Q as a parameter, the problem may be solved as a linear parametric programme, having all vectors of the restriction matrix, parametrized. Starting with an initial result for parameter $Q = Q_0$ and knowing that $0 < Q < \sqrt{1.2}$, a partition of this interval with the problem solutions for each partition section may be determined, in the ranges where the problem actually has a solution.

Since simultaneously with the function minimization, the minimization of the amount of tetraethyl lead incorporated in the blend is also aimed at, it follows that the answer to the problem will be that one of the determined results, for which Q is a minimum.

The column-vectors of restriction matrix have a unique component each, depending on parameter Q, i.e. the component corresponding to the octane number. If, however, the number of unknowns is higher than two, the solving of the problem leads to elaborate computations, also including the necessary computations to be carried out in view of solving equations having a higher degree than four. Consequently, in order to facilitate the problem solution, we shall try to modify the condition referring to the octane number. If we let S be the blend susceptibility prior to introducing the additive, and using equation (1.5), the condition referring to the octane number becomes

$$\frac{\Sigma O_i x_i}{\Sigma x_i} + S\sqrt{2q} \geqslant 0. \tag{2.1}$$

In this case, a new restriction, referring to S, is added to Model I, namely

$$S = \frac{\Sigma S_i x_i}{\Sigma x_i} \tag{2.2}$$

By further using the notation $\sqrt{2q} = Q$, Model I becomes:

$$\text{II} \begin{cases} \Sigma x_i = b; \\ \Sigma \dfrac{x_i}{d_i} \geqslant \dfrac{\Sigma x_i}{\bar{d}}; \\ x_i \leqslant b_i, \\ \Sigma O_i x_i + SQb > \bar{O}b; \\ \Sigma S_i x_i = Sb; \\ Q \leqslant \sqrt{1{,}2} \text{ (because we must have } q \leqslant 0.60); \\ x_i \geqslant 0. \end{cases}$$

SOLVING THE PROBLEM

By taking into account that in Model II, the required amount b is known, we may consider Q as a new variable, and S as a parameter, which leads to the computation of a parametric programme, where the matrix contains only one parametrized column-vector; in addition, however, the free term vector depends also on this parameter.

The efficiency function will thus be either the total cost of constituents and of incorporated tetraethyl lead, or the amount of tetraethyl lead added to the blend. Consequently,

$$f = \Sigma C_i x_i + K \frac{Q^2}{2{,}000} \Sigma x_i \quad \text{(where } K \text{ is the cost of tetraethyl lead).}$$

or

$$f = Q.$$

The minimization of $f = Q$ is to be preferred.

Similarly to the case of Model I, we shall determine a partition of the variation range of S, with the corresponding solutions depending on S, out of which that programme solution could be selected, for which q is a minimum. Since the restriction matrix of Model II has a unique column-vector parametrized with respect to n, it is much easier to solve the Model.

The method and the formulas developed by H. Maurin [1] will be used in solving this problem.

Before, however, going over to solving an example taken from actual practice, we shall show the method of determining the variation range of parameter S, which is to be subsequently partitioned.

The variation range of S is $[S_1, S_2]$, where

$$S_1 = min\, S = min \frac{\Sigma S_i x_i}{\Sigma x_i}.$$

with the restrictions

$$\Sigma x_i = b, \quad x_i \leqslant b_i,$$

and

$$S_2 = max\, S = max \frac{\Sigma S_i x_i}{\Sigma x_i},$$

with the restrictions

$$\Sigma x_i = b, \quad x_i \leqslant b_i.$$

PRACTICAL EXAMPLE

The following constituents are available, in the amounts and with the characteristics shown in the following Table:

Constituent	Available amount (tons) b_i	Octane number Θ_i	Specific gravity 20/4 d_i	Susceptibility S_i	Cost lei/ton
1	15,000	80	0.750	10.2	672
2	10,000	84	0.758	7.9	692
3	8,000	90	0.764	7.0	722

It is requested to prepare 10,000 tons gasoline, having the following specifications:

— the octane number, at least 94,

— the maximum specific gravity, 0.760, knowing that one ton of tetra-ethyl lead costs 35,000 lei.

Using the notations agreed upon, as well as by applying Model II, we obtain:

The function $f = q$ is to be minimized, with the following restrictive conditions:

$$\begin{aligned}
&x_1 + x_2 + x_3 = 10{,}000;\\
&x_1 \leqslant 15{,}000;\\
&x_2 \leqslant 10{,}000;\\
&x_3 \leqslant \ 8{,}000;\\
&\frac{x_1}{0.750} + \frac{x_2}{0.758} + \frac{x_3}{0.764} \geqslant \frac{10{,}000}{0.760};\\
&80x_1 + 84\,x_2 + 90x_3 + 10{,}000\,S\sqrt{2q} \geqslant 94 \times 10{,}000;\\
&q \leqslant 0.6.\\
&10.2x_1 + 9.7x_2 + 7x_3 = 10{,}000S.
\end{aligned} \tag{4.1}$$

By designating the term $10{,}000\sqrt{2q}$ by p, and taking into account that relation $x_1 + x_2 + x_3 = 10{,}000$ includes $x_1 < 15{,}000$ and $x_2 < 10{,}000$, the Model becomes:

The function $f = p$ is to be minimized, by the following restrictions:

$$\begin{aligned}
&x_1 + x_2 + x_3 = 10{,}000;\\
&x_3 \leqslant 8{,}000;\\
&133x_1 + 132x_2 + 131x_3 \geqslant 1{,}315{,}800;\\
&80x_1 + 84x_2 + 90x_3 + Sp \geqslant 940{,}000;\\
&p \leqslant 10{,}900;\\
&102x_1 + 79x_2 + 70x_3 = 100{,}000S.
\end{aligned} \tag{4.2}$$

(where S is a parameter).

We shall first compute the variation range of parameter S, that is $[S_1, S_2]$.

According to the previous section, we have:

$$S_1 = min\ \frac{10.2x_1 + 7.9x_2 + 7x_3}{10{,}000}$$

with the restrictions:

$$\begin{aligned}
&x_1 + x_2 + x_3 = 10{,}000\\
&x_3 \leqslant 8{,}000;
\end{aligned}$$

$$S_2 = max\ \frac{10.2x_1 + 7.9x_2 + 7x_3}{10{,}000}$$

with the restrictions:

$$\begin{aligned}
&x_1 + x_3 + x_3 = 10{,}000\\
&x_3 \leqslant 8{,}000.
\end{aligned}$$

Solving the two Models, we obtain:

$$S_1 = 7.18 \quad \text{and} \quad S_2 = 10.2.$$

It follows that parameter S may vary in the range [7.18; 10.2]. In solving the problem, we shall start from an initial result, obtained for a specific value of S taken in this range, e.g. for $S = 8$.

In view of facilitating computation, we shall previously make the transformation $S = S' + 8$ and we shall obtain the initial result for $S' = 0$.

With this transformation, the model becomes:

$$min\ f = p$$

with the following restrictions:

$$\begin{aligned}
&x_1 + x_2 + x_3 + e_1 = 10{,}000;\\
&x_3 + y_1 = 8{,}000;\\
&133x_1 + 132x_2 + 131x_3 - y_2 + e_2 = 1{,}315{,}800;\\
&80x_1 + 84x_2 + 90x_3 + (S' + 8)\,p - y_3 + e_3 = 940{,}000;\\
&p + y_4 = 10{,}900;\\
&102x_1 + 79x_2 + 70x_3 + e_4 = 800{,}000 + 100{,}000S'.
\end{aligned} \tag{4.3}$$

If $S' = 0$, by applying the simplex algorithm (Appendix I), we obtain the optimal result

$$\bar{x}_{I_0} = \begin{pmatrix} y_2 \\ y_1 \\ x_3 \\ p \\ y_4 \\ x_1 \end{pmatrix} \begin{pmatrix} 450 \\ 1{,}125 \\ 6{,}875 \\ 35{,}625/4 \\ 7{,}975/4 \\ 3{,}125 \end{pmatrix} \tag{4.4}$$

with

$$[A^{I_0}]^{-1} = \begin{pmatrix} 4{,}052/32 & 0 & -1 & & 0 & 2/32 \\ -102/32 & 1 & 0 & & 0 & 1/32 \\ 102/32 & 0 & 0 & 0 & 0 & -1/32 \\ -895/64 & 0 & 0 & 1/8 & 0 & 5/128 \\ 895/64 & 0 & 0 & -1/8 & 1 & -5/128 \\ -\ 70/32 & 0 & 0 & 0 & 0 & 1/32 \end{pmatrix} \tag{4.5}$$

We have here the case where $A^k = A^4$ is the fourth vector ($q = 4$) of the optimal basis for $S' = 0$.

Using the notations shown in [1] and [6], we have:

$$A^k = A^4 = \overline{A}^4 + S'D_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix} + S' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

where A^4 is the column-vector of the restrictions matrix, including parameter S. Furthermore, the free term vector will have the form:

$$a = \bar{a} + S'a' = \begin{pmatrix} 10{,}000 \\ 8{,}000 \\ 1{,}315{,}800 \\ 940{,}000 \\ 10{,}900 \\ 800{,}000 \end{pmatrix} + S' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100{,}000 \end{pmatrix}$$

According to the given algorithm [1], we shall have to compute the values of parameter S', for which:

— the fundamental matrix becomes singular,
— the result ceases to be feasible,
— the result ceases to be optimal.

We shall now have to compute the variable quantities as a function of S'.

a) The reverse matrix of the basis varies according to the relation

$$[A^I]^{-1} = E^{(q)}[A^{I_0}]^{-1}, \tag{4.6}$$

where the elementary matrix $E^{(q)}$ is built up on the basis of the unit matrix, in which the q vector is substituted by the $V^{(q)}$ vector, having the following constituents

$$V_q^q = \frac{1}{T_{0_q}^k} = \frac{1}{1 + S'W_{0_q}^k}, \tag{4.7}$$

$$V_i^q = -\frac{T_{0_i}^k}{T_{0_q}^k} = -\frac{S'W_{0_i}^k}{1 + S'W_{0_q}^k}, \qquad i \in \{I_0 - q\}$$

In our example, $k = 4$, $q = 4$. Consequently, we shall have:

$$W_0^4 = [A^{I_0}]^{-1} D_4 = \begin{pmatrix} 4{,}052/32 & 0 & -1 & 0 & 0 & 2/32 \\ -102/32 & 1 & 0 & 0 & 0 & 1/32 \\ 102/32 & 0 & 0 & 0 & 0 & -1/32 \\ -895/64 & 0 & 0 & 1/8 & 0 & 5/128 \\ 895/64 & 0 & 0 & -1/8 & 1 & -5/128 \\ -70/32 & 0 & 0 & 0 & 0 & 1/32 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/8 \\ -1/8 \\ 0 \end{pmatrix}$$

$$T_0^4 = [A^{I_0}]^{-1} \bar{A}^4 + S' W_0^4 \begin{pmatrix} 4{,}052/32 & 0 & -1 & 0 & 0 & 2/32 \\ -102/32 & 1 & 0 & 0 & 0 & 1/32 \\ 102/32 & 0 & 0 & 0 & 0 & -1/32 \\ -895/64 & 0 & 0 & 1/8 & 0 & 5/128 \\ 895/64 & 0 & 0 & -1/8 & 1 & -5/128 \\ -70/32 & 0 & 0 & 0 & 0 & 1/32 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix} +$$

$$+ S' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/8 \\ -1/8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + S' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/8 \\ -1/8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (8 + S')/8 \\ -S'/8 \\ 0 \end{pmatrix}$$

It follows that

$$E^{(q)} = E^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8/(S' + 8) & 0 & 0 \\ 0 & 0 & 0 & S'/(S' + 8) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.8}$$

From the expressions (4.6) and (4.8), it follows that the matrix $[A^I]^{-1}$ is singular if

$$S' + 8 = 0 \quad \text{or} \quad S = 0. \tag{4.9}$$

b) In order to obtain the feasibility range of the result, we compute

$$x_{I_0} = E^{(4)}(\bar{x}_{I_0} + S' t'_0), \tag{4.10}$$

where

$$t'_0 = [A^{I_0}]^{-1} a' = \begin{pmatrix} 6{,}250 \\ 3{,}125 \\ -3{,}125 \\ 15{,}625/4 \\ -15{,}625/4 \\ 3{,}125 \end{pmatrix};$$

It follows

$$x_{I_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8/(S'+8) & 0 & 0 \\ 0 & 0 & 0 & S'/(S'+8) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 450 + 6{,}250S' \\ 1{,}125 + 3{,}126S' \\ 6{,}875 - 3{,}125\,S' \\ (35{,}625 + 15{,}625S')/4 \\ (7{,}975 - 15{,}625S')/4 \\ 3{,}125 + 3{,}125S' \end{pmatrix} =$$

$$= \begin{pmatrix} 450 + 6{,}250S' \\ 1{,}125 + 3{,}125S' \\ 6{,}875 - 3{,}125S' \\ \dfrac{2(35{,}526 + 15{,}625S')}{S'+8} \\ \dfrac{15{,}950 - 20{,}350S'}{S'+8} \\ 3{,}125 + 3{,}125S' \end{pmatrix} \tag{4.11}$$

The feasibility range will be:

$$\begin{aligned} 450 + 6{,}250S' &\geqslant 0 \rightarrow S' \geqslant -0.072; \\ 1{,}125 + 3{,}125S' &\geqslant 0 \rightarrow S' \geqslant -0.36; \\ 6{,}875 - 3{,}125S' &\geqslant 0 \rightarrow S' \leqslant 2.2; \\ 35{,}625 + 15{,}625S' &\geqslant 0 \rightarrow S' \geqslant -2.28; \\ S' + 8 &\geqslant 0 \rightarrow S' \geqslant -8; \\ 15{,}950 - 20{,}350S' &\geqslant 0 \rightarrow S' \leqslant 0.78; \\ 3{,}125 + 3{,}125S' &\geqslant 0 \rightarrow S' \geqslant -1. \end{aligned} \tag{4.12}$$

By solving the system (4.12) we obtain:

$$-0.072 \leqslant S' \leqslant \frac{319}{407} \quad \text{or} \quad 7.928 \leqslant S \leqslant 8\frac{[319}{307}. \tag{4.13}$$

c) The optimal result may be obtained by imposing the condition expressed by:

$$\delta^j = -f^{l_0} E^{(4)} T_0^j + f^j, \text{ for } j = 2;\ 9.$$

We have

$$f^{l_0} = (0,\ 0,\ 0,\ 1,\ 0,\ 0) \text{ and } f^2 = 0,$$

$$T_0^2 = [A^{l_0}]^{-1} A^2 = \begin{pmatrix} -14/32 \\ -23/32 \\ 23/32 \\ -51/128 \\ 51/128 \\ 9/32 \end{pmatrix}$$

Hence

$$\delta^2 = -(0,0,0,1,0,0) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8/(S'+8) & 0 & 0 \\ 0 & 0 & 0 & S'/(S'+8) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -14/32 \\ -23/32 \\ 23/32 \\ -51/128 \\ 51/128 \\ 9/32 \end{pmatrix} =$$

$$= \frac{51}{16(S'+8)}.$$

Similarly, we have:

$$T_0^9 = [A^{l_0}]^{-1} A^9 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/8 \\ 1/8 \\ 0 \end{pmatrix} \text{ şi } f^9 = 0;$$

$$\delta^9 = -(0,0,0,1,0,0)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8/(S'+8) & 0 & 0 \\ 0 & 0 & 0 & S'/(S'+8) & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/8 \\ 1/8 \\ 0 \end{pmatrix} = \frac{1}{S'+8}.$$

The range of the optimal result will be:

$$S' = 8 \geqslant 0 \quad \text{or} \quad S \geqslant 0. \tag{4.14}$$

From the expressions (4.9), (4.13) and (4.14), it follows that the variation range [7.18; 10.2] of S is divided in three subintervals, namely:

— for $7.928 \leqslant S \leqslant 8.78$ we have (according to 4.10) the following feasible and optimal result:

$$\begin{aligned} x_1 &= 3{,}125\,(1 + S') = 3{,}125\,(S - 7), \\ x &= 0 \\ x_3 &= 6{,}875 - 3{,}125S' = 31{,}875 - 3{,}125S, \\ p &= \frac{2(35{,}625 + 15{,}625S')}{S'+8} = \frac{2(-89{,}375 + 15{,}625S)}{S}; \end{aligned} \tag{4.15}$$

— for $7.18 < S < 7.928$, as well as for $8.78 < S < 10.2$, the result ceases to be feasible.

We shall now concentrate on the two subintervals, where the result ceases to be feasible. We begin with the interval $8.78 < S < 10.2$, or $0.78 < S' < 2.2$.

It can be seen from relation (4.12) that in this range, y_4 becomes negative, so it must be excluded from the basis, by applying the dual simplex method.

Beginning with the last simplex table (see Appendix 1), giving the feasible and optimal result (4.4) for $S' = 0$, and making the transformation $A^1 = E^{(q)}A^0$, for both the reverse matrix of the basis and for the vectors outside the basis, we obtain the simplex table shown in Appendix 2, and depending on parameter S'.

By inspecting the simplex table obtained and taking into account that $S' + 8 > 0$, it follows that y_4 cannot be excluded from the basis, for any value assumed by parameter S'. Consequently, within the interval considered here ($0.78 < S' < 2.2$), the problem has no solution.

We shall now consider the interval $7.18 < S < 7.928$ or $-0.82 < S' < -0.072$. It can be seen from relations (4.12) that within this interval, y_1 and y_2 become negative, hence they must be excluded from the basis. The analysis of the same simplex table in Appendix 2 shows that this exclusion is possible.

By applying the dual simplex method, we obtain the optimal and feasible solution, shown in the simplex table of Appendix 3, namely:

$$x_{I_1} = \begin{pmatrix} x_2 \\ y_1 \\ x_3 \\ p \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \dfrac{-16(450 + 6{,}250S')}{7} \\ \dfrac{4(675 - 12{,}500S')}{7} \\ \dfrac{4(13{,}325 + 12{,}500S')}{7} \\ \dfrac{475{,}800 - 100{,}000S'}{7(S' + 8)} \\ \dfrac{134{,}600 + 176{,}300S'}{7(S' + 8)} \\ \dfrac{4(5{,}975 + 12{,}500S')}{7} \end{pmatrix} \tag{4.16}$$

It will now be necessary to determine the feasibility and optimal range of the result.

The feasibility range will be:

$$\begin{aligned} 450 + \ 6{,}250S' \leqslant 0 &\rightarrow S' \leqslant -0.072; \\ 675 - \ 12{,}500S' \geqslant 0 &\rightarrow S' \leqslant \ 0.054; \\ 13{,}325 + \ 12{,}500S' \geqslant 0 &\rightarrow S' \geqslant -1.066; \\ 475{,}800 - 100{,}000S' \geqslant 0 &\rightarrow S' \leqslant \ 4.758; \\ 134{,}600 + 176{,}300S' \geqslant 0 &\rightarrow S' \geqslant -0.765; \\ 5{,}975 + \ 12{,}500S' \geqslant 0 &\rightarrow S' \geqslant -0.478. \end{aligned} \tag{4.17}$$

By solving the system of inequations of (4.17), we obtain

$$-0.478 < S' < -0.072.$$

Accordingly, the solution (4.16) ceases to be feasible in the range $-0.82 < S' < -0.478$, since within this range, x_1 and y_4 become negative, as a result of (4.17). In order to obtain a new feasible solution in this range, x_1 and y_4 should be excluded from the basis, but this is not possible, as can be seen on an inspection of the simplex table (see Appendix 3) and by taking into account that $S' + 8 > 0$. Consequently, within the range $-0.82 < S' < -0.478$, the problem has no solution.

The optimal range of solution (4.16) is easily obtained from the simplex table (see Appendix 3), where it can be seen that

$$\delta^7 = \frac{51}{7(S' + 8)} \quad \text{and} \quad \delta^9 = \frac{1}{S' + 8}.$$

As $S' + 8 > 0$, it follows that the solution (4.16) is optimal within the entire range $-0.82 < S' < -0.072$.

To conclude, the variation range of parameter $S'(-0.82;\ 2.2)$ is partitioned as follows:

— within the range $(-0.072;\ 0.78)$, the problem has the optimal and feasible solution (4.11);

— within the range $(-0.478;\ -0.072)$, the problem has the optimal and feasible solution (4.16);

— within the ranges $(-0.82;\ -0.478)$ and $(0.78;\ 2.2)$ the problem has no solution.

As in solving the problem, we aim at a minimum tetraethyl lead consumption, it follows that we should choose the value of S' within the range $(-0.478;\ 0.78)$, minimizing the value of p.

The expressions (4.11) and (4.16) give:

$$p = \begin{cases} \dfrac{2(35{,}625 + 15{,}625S')}{S' + 8} & \text{for } -0.072 \leqslant S' \leqslant 0.78; \\ \dfrac{8(59{,}475 - 12{,}500S')}{7(S' + 8)} & \text{for } -0.478 \leqslant S' \leqslant -0.072. \end{cases} \tag{4.18}$$

Returning to parameter S, we have:

$$p = \begin{cases} \dfrac{1{,}250(25S - 143)}{S} = 1{,}250\left(25 - \dfrac{143}{S}\right) & \text{for } 7.928 \leqslant S \leqslant 8.78; \\ \dfrac{200(6{,}379 - 5{,}000S)}{7S} = \dfrac{200}{7}\left(-500 + \dfrac{6{,}379}{S}\right) & \text{for } 7.522 \leqslant S \leqslant 7.928. \end{cases} \tag{4.19}$$

It results from (4.19) that within the range $(7.928;\ 8.78)$, the minimum value of p is obtained for $S =$ minimum, i.e. for $S = 7.928$, whereas within the range $(7.522;\ 7.928)$ the minimum value of p is obtained for $S =$ maximum, i.e. for $S = 7.928$. Since in the case of $S = 7.928$, the two forms of p in equations (4.19) result in the same value, we obtain:

$$P_{min} = 8{,}703.375.$$

Knowing that $p = 10{,}000\sqrt{2q}$, it follows that the minimum consumption of tetraethyl lead will be:

$$q_{min} = 0.3787 \text{ tons per } 1{,}000 \text{ tons of blend.}$$

By substituting $S = 7.928$ in (4.11) or (4.16), it follows that the 10,000 tons of fuel requested will be obtained out of the following constituents:

$$x_1 = 2{,}900 \text{ tons}$$

$$x_2 = 0 \tag{4.20}$$

$$x_3 = 7{,}100 \text{ tons}$$

to which an amount of 3.787 tons of tetraethyl lead is added.

The solution (4.20) satisfies the requirements referring to the specific gravity and the octane number and, in addition, is accomplished with a minimum consumption of tetraethyl lead. As to the cost, it is 720.75 lei/ton, as compared to 750 lei/ton, the cost resulting from empirical formulas.

In the chart of Appendix 4, the economic function variation is shown as a function of parameter S,

$$f_1 = \frac{f}{10^4} = \sqrt{2q} = \begin{cases} 0.125\left(25 - \dfrac{143}{S}\right), & \text{for } 7.928 \leqslant S \leqslant 8.78; \\ \dfrac{0.02}{7}\left(-500 + \dfrac{6{,}379}{S}\right), & \text{for } 7.522 \leqslant S \leqslant 7.928. \end{cases}$$

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I

			x_1	x_2	x_3	1	M
						p	e_1
0	450	y_2		—14/32			4,052/32
0	1,125	y_1		—23/32			—102/32
0	6,875	x_3		23/32	1		102/32
1	35,625/4	p		—51/128		1	—895/64
0	7,975/4	y_4		51/128			895/64
0	3,125	x_1	1	9/32			— 70/32
		c_j—z_j	0	51/128	0	0	895/64

II

			x_1	x_2	x_3	1	M
						p	e_1
0	$450 + 6{,}250\ S'$	y_2		—14/32			4,052/32
0	$1{,}125 + 3{,}125\ S'$	y_1		—23/32			—102/32
0	$6{,}875 - 3{,}125\ S'$	x_3		23/32	1		102/32
1	$\frac{2(35{,}625 + 15{,}625\ S')}{S' + 8}$	p		$-51/16(S'+8)$		1	$-895/8(S'+8)$
0	$\frac{15.950 - 20{,}350S'}{S' + 8}$	y_4		$51/16(S'+8)$			$895/8(S'+8)$
0	$3125 + 3125S'$	x_1	1	9/32			—70/32

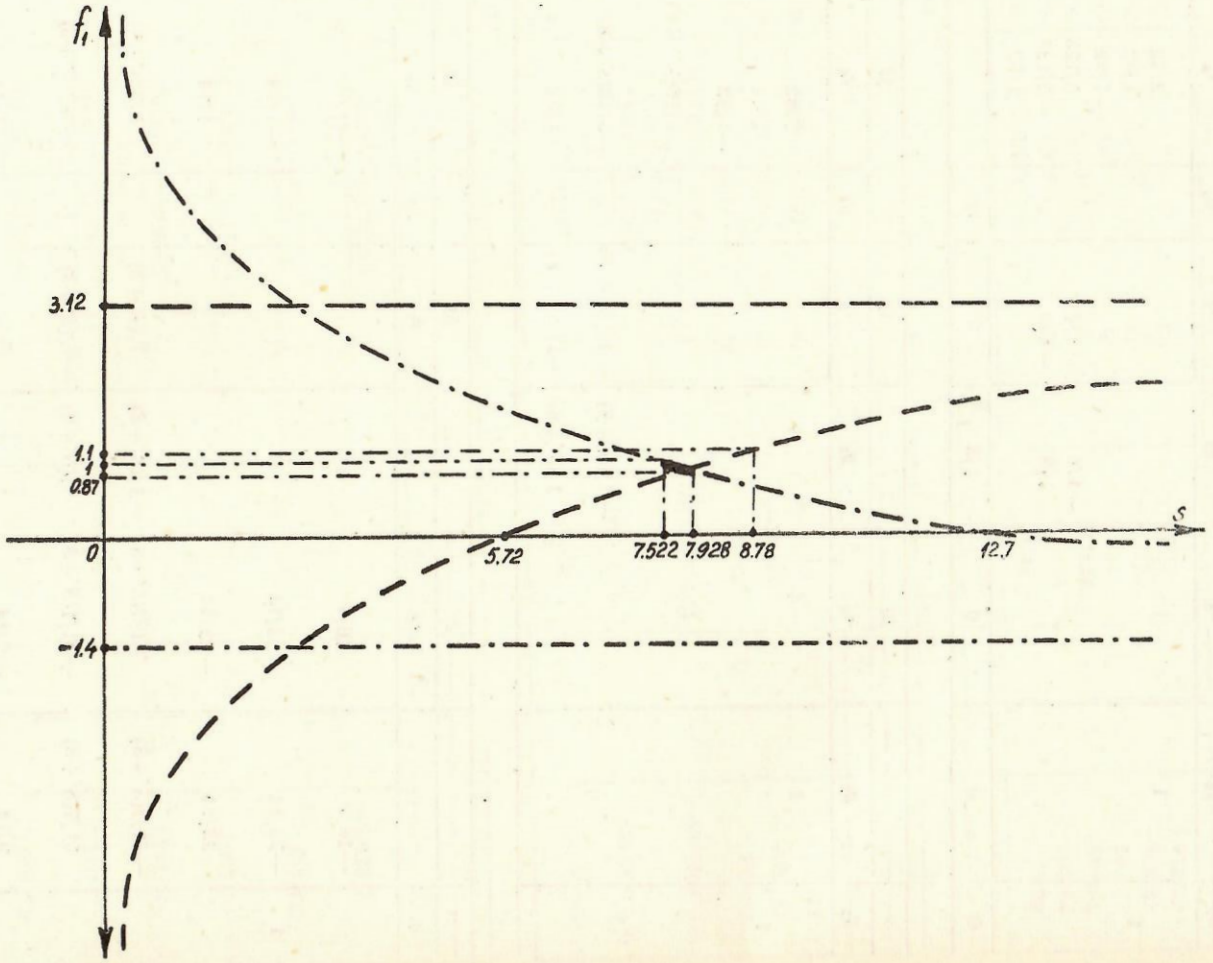
III

			x_1	x_2	x_3	1	M
						p	e_1
0	$\frac{-16(450 + 6{,}250S')}{7}$	x_2		1			—2,026/7
0	$\frac{4(675 - 12{,}500S')}{7}$	y_1					—2,957/14
0	$\frac{4(13{,}325 + 12{,}500S')}{7}$	x_3			1		2,957/14
1	$\frac{475{,}800 - 100{,}000S'}{7(S' + 8)}$	p				1	$-7{,}241/7(S'+8)$
0	$\frac{134.600 + 176{,}300S'}{7(S' + 8)}$	y_4					$7{,}241(S' + 8)$
0	$\frac{4(5{,}975 + 12{,}500S')}{7}$	x_1	1				1,109/14
		c_j—z_j	0	0	0	0	

		M		M		M_1^1
y_1	y_2	e_2	y_3	e_3	y_4	e_4
	1	—1				2/32
1						1/32
						—1/32
			—1/8	1/8		5/128
			1/8	—1/8	1	—5/128
						1/32
0	0	0	1/8		0	

		M		M		M
y_1	y_2	e_2	y_3	e_3	y_4	e_4
	1	—1				2/32
1						1/32
						—1/32
					1	$5/16(S'+8)$
			$-1/(S'+8)$	$1/(S'+8)$		$-5/16(S'+8)$
			$1/(S'+8)$	$-1/(S'+8)$		1/32

		M		M		M
y_1	y_2	e_2	y_3	e_3	y_4	e_4
	—16/7	16/7				—1/7
1	—23/14	23/14				—1/14
	23/14	—23/14				1/14
	$-51/7(S'+8)$	$51/7(S'+8)$	$-1/(S'+8)$	$1/(S'+8)$		$1/7(S'+8)$
	$51/7(S'+8)$	$-51/7(S'+8)$	$1/(S'+8)$	$-1/(S'+8)$	1	$-1/7(S'+8)$
	9/14	—9/14				1/14
0	$51/7(S'+8)$		$1/(S'+8)$		0	



APPENDIX IV