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A Computer Simulation Model for Estimating the Optimum Renewal Moment of an Equipment

1. INTRODUCTION

The problem is to determine the optimum moment of time $\hat{t}$ when an equipment must be renewed, taking into account its depreciation and maintainance expenses.

The simulation model for estimating $\hat{t}$ is built-up based upon a known classical deterministic model [2].

Let $A_0 > 0$ be the value of the equipment at the time $\hat{t} = 0$ and let $A_0 \cdot \varphi(t)$ be the value of cummulative depreciation of the equipment at the time $t > 0$, where $\varphi(t)$ is a decreasing function such as $\varphi(0) = 1$ and $\lim_{t \to \infty} \varphi(t) = 0$. The function $\overline{\varphi}(t) = 1 - \varphi(t)$ can be interpreted such as $A_0 \cdot \overline{\varphi}(t)$ is the value of the equipment at the time t.

Let $\psi(t)$ be the function representing the value of the maintainance expenses of the equipment over the time interval $(0, t)$, Assume, in a natural way, that $\psi(t) =$ is an increasing function such as $\psi(0) = 0$, $\lim_{t \to \infty} \psi(t) = 0$. Assuming further, that A_0, $\varphi(t)$ and $\psi(t)$ are given, and that functions $\varphi(t)$ and $\psi(t)$ are of the class C^2, in the following we will show how the *optimum renewal moment* t can be calculated. Let

$$C(t) = A_0 \cdot \overline{\varphi}(t) + \psi(t) \tag{1}$$

be the total cost associated to the equipment on the time interval $(0, t)$, Obviously, the function $C(t)$ is an increasing function of t. Therefore it is not suitable to be used as an *objective* function determining the optimum renewal moment $\hat{t}$, instead the function

$$\gamma(t) = \frac{C(t)}{t} \tag{2}$$

which represents the cost per unit of time can be used as an objective function for the mentioned purpose, and consequently the optimum renewal moment $\hat{t}$ must satisfy the conditions

$$\gamma(\hat{t}) = \min_{t \geq 0} \gamma(t). \tag{3}$$

Therefore, in the above assumptions we conclude that $\hat{t}$ must satisfy conditions

$$\hat{t}C'(\hat{t}) - C(\hat{t}) = 0 \tag{4}$$

$$\hat{t}^3\gamma''(\hat{t}) = \hat{t}C''(\hat{t}) - 2\hat{t}C'(\hat{t}) + 2C(\hat{t}) > 0. \tag{5}$$

Therefore the given constant A_0, the functions $\varphi(t)$ and $\psi(t)$ and the relations (4) and (5) define in this case a classical model for determining the optimum renewal moment.

2. EXISTANCE OF AN OPTIMUM RENEWAL MOMENT IN THE CASE OF CLASSICAL DETERMINISTIC MODEL

If $\varphi(t)$ and $\psi(t)$ are deterministic (i.e. non-stochastic) functions, the problem is to define particular forms of these functions so that an optimum renewal moment exists. In the following we will present some particular cases, from which (I) and (II) below refer to two new situations.

(I) Assume that

$$\varphi_1(t) = \frac{\beta}{\beta + t}, \quad \psi_1(t) = kt^n \tag{6}$$

where $\beta > 0, k > 0, n \geq 2$ are given constants. Analysing the minimum conditions (4) and (5) we conclude that $\hat{t}$ satisfies the equation

$$H_1(\hat{t}) = H_2(\hat{t}), \; H_1(t) = \frac{A_0\beta}{(\beta + t)^2}, \; H_2(t) = k(n-1)\,t^{n-2} \tag{6'}$$

and after some calculations one finds

$$t^3\gamma''(t) = \frac{2A_0t^3}{(\beta + t)^3} + kt^n(n^2 - 3n + 2) > 0, \quad (\forall)t \in (0, \infty)$$

which ensures that $\hat{t}$ is the optimum renewal moment.

(II) Assume that

$$\varphi_2(t) = \frac{\beta}{\beta + t}, \quad \psi_2(t) = kte^t(1 - e^{-\mu t}) \tag{7}$$

where β, k, μ are positive constants such as

$$\beta^2 \cdot \mu \cdot k < A_0, \quad 0 < \mu < 1. \tag{7'}$$

We have

$$C(t) = \frac{A_0t}{\beta + t} + kte^t(1 - e^{-\mu t})$$

$$\gamma(t) = \frac{A_0}{\beta + t} + ke^t(1 - e^{-\mu t})$$

which gives

$$H(t) = \gamma'(t) = \frac{-A_0}{(\beta + t)^2} + ke^t + k(\mu - 1)e^{(1-\mu)t}. \tag{8}$$

Since in the assumption (7')

$$H(0) = -\frac{A_0}{\beta^2} + \mu k,\ H(\infty) = \infty,$$

there is $t_0 \in (0,\ \infty)$ such as $H(t_0) = 0$. But

$$\gamma''(t) = \frac{2A_0}{(\beta + t)^3} + ke^t\ (1 - (1 - \mu)^2\ e^{-\mu t}) > 0,\ (\forall)t > 0.$$

Therefore $\hat{t} = t_0$ is the optimum renewal moment.

If the second assumption (7') becomes $\mu > 1$ and $t_0 > 1$, then $\gamma''(t_0) > 0$ and therefore $\hat{t} = t_0$ is also the optimum renewal moment.

(III) In [2] it is shown that if

$$\varphi_3(t) = e^{-\lambda t},\ \psi_3(t) = k(e^{\mu t} - 1) \tag{9}$$

$$\lambda,\ \mu,\ k > 0$$

then there is an optimum renewal moment $\hat{t} > 0$ which satisfies the equation

$$\frac{\Phi(\lambda \hat{t})}{\Phi(-\mu \hat{t})} = \frac{k}{A_0} \tag{10}$$

where

$$\Phi(x) = 1 - e^{-x}\ (1 + x). \tag{10'}$$

Remarks.

1°. In [2] it is shown that if

$$\varphi_4(t) = 1 - \frac{t}{\theta},\ 0 < t < \theta;\ \psi_4(t) = kt,\ t > 0 \tag{11}$$

then $\gamma(t) = \text{const}$ and the optimum renewal moment $\hat{t}$ is undetermined, being every positive real number,.

2°. Also in [2] it is shown that if

$$\varphi_5(t) = e^{-\lambda t},\ \psi_5(t) = kt,\ (\forall)\lambda,\ k > 0$$

then $\hat{t} = \infty$ which means that the renewal moment may be as large as possible.

3. STATISTICAL DATA PROCESSING FOR ESTIMATING THE OPTIMUM RENEWAL MOMENT

Assume that statistical observations on the functions φ and ψ are given, namely $(\varphi_i,\ \psi_i)$ observed at moments i, $1 \leqslant i \leqslant N$. (In practice the unit of time is one year).

The problem is to estimate first the parameters of the functions $\varphi(t)$ and $\psi(t)$, in the particular cases (I), (II), (III) from above, and then to calculate the optimum renewal moment $\hat{t}$.

In the following we will discuss the problem of estimating these parameters in the three mentioned cases.

(I). The parameters k and n of the function $\psi_1(t)$ are estimated by using the least squares method, after a previous transformation of the function $\psi_1(t)$ as follows:

$$z = \ln \psi, \ \alpha = \ln k, \ z = \alpha + nt. \tag{13}$$

The estimate of the parameter β of the function $\varphi_1(t)$ is

$$\hat{\beta} = \frac{N \varphi_N}{1 - \varphi_N}. \tag{14}$$

(II). In the case of the function $\varphi_2(t)$, the estimate of β is given by (14).

In order to estimate parameters k and α of the function $\psi_2(t)$, we note that the following relations hold,

$$\psi_i = \frac{ki^2}{\alpha + i}, \ 1 \leqslant i \leqslant n \tag{15}$$

and after some calculations, these give

$$\hat{\alpha} = \frac{N(\psi_N - \psi_1 . \ N)}{\psi_1 N^2 - \psi_N} \tag{16}$$

$$\hat{K} = 6 \frac{\hat{\alpha} \sum\limits_{i=1}^{N} \psi_i + \sum\limits_{i=1}^{N} i \cdot \psi_i}{N(N+1)(N+2)}. \tag{17}$$

(III). For estimating the parameter λ of the function φ_3, previously we make the transform

$$\ln \varphi_3(t) = -\lambda \cdot t$$

and finally obtain

$$\hat{\lambda} = -\frac{2}{N(N+1)} \ln \left(\prod_{i=1}^{n} \varphi_i \right). \tag{18}$$

In order to estimate the parameters k and μ of the function ψ_3, we note first that

$$\frac{\psi_{i+2} - \psi_{i+1}}{\psi_{i+1} - \psi_i} = e^{\mu}, \ 1 \leqslant i \leqslant N - 2 \tag{19}$$

and then obtain the estimate of μ as

$$\hat{\mu} = \frac{1}{N-2} \ln \left(\frac{\psi_N - \psi_{N-1}}{\psi_2 - \psi_1} \right). \tag{20}$$

The estimate of k is obtained from

$$\psi_i = \mathrm{k}(e^{\hat{\mu} . i} - 1), \ 1 \leqslant i \leqslant N$$

giving finally

$$\hat{k} = \frac{\sum\limits_{i=1}^{N} \psi_i}{\sum\limits_{i=1}^{N} (e^{\hat{\mu} . i} - 1)}. \tag{20'}$$

The estimates of the parameters which are involved in the functions $\varphi_4(t)$, $\psi_4(t)$, $\varphi_5(t)$ and $\psi_5(t)$ are easily derived from the least squares method.

An important practical problem to be solved before calculating the optimum renewal moment $\hat{t}$ is the choice of the model for the functions $\varphi_i(t)$ and $\psi_i(t)$, $1 \leqslant i \leqslant 5$.

This can be done by comparing the empirical variances for the particular functions φ or ψ; these variances are

$$s_\varphi^2 = \frac{1}{N - C_1} \sum_{i=1}^{N} (\hat{\varphi}(i) - \varphi_i)^2$$

$$s_\psi^2 = \frac{1}{N - C_2} \sum_{i=1}^{N} (\hat{\psi}(i) - \psi_i)^2$$

where $\hat{\varphi}(i)$ and $\hat{\psi}(i)$ are the estimates of corresponding functions at the moment i and C_1, C_2 are respectively the number of estimated parameters for the functions φ and ψ.

The choice of the best functions φ and ψ can then be done by using the known F-test based on the ratio of empirical variances.

In order to determine the optimum renewal moment $\hat{t}$ the equations (6') or (10) must be solved. Note that each of these equations is in the form

$$H_1(t) = H_2(t)$$

and the curves $y = H_1(t)$ and $y = H_2(t)$ have a single common point on $(0, \infty)$, Therefore the *algorithm* for determining this point is:

Step 0. Input parameters (α, k, β or μ), input $\varepsilon > 0$ (ε is a given admissible error). Initialize $t = 0$. Calculate $h_1 = H_1(t)$, $h_2 = H_2(t)$, $d = h_1 - h_2$. Calculate:

$$s = sign(d) = \begin{cases} -1 & \text{if } d < 0 \\ 1 & \text{if } d \geqslant 0. \end{cases}$$

Step 1. Take $t \leftarrow t + 1$ and calculate again $h_1 = H_1(t)$, $h_2 = H_2(t)$, $d = h_1 - h_2$, $s_1 = sign\ (d)$.

Step 2. If $|d| > \varepsilon$ and *s.* $s_1 = +1$ then go to step 1, otherwise if $|d| > \varepsilon$ and *s.* $s_1 = -1$ or if $|d| < \varepsilon$, continue.

Step 3. $\hat{t} = t$, Stop.

Thus, the problem of determining the optimum renewal moment in the classical deterministic case is terminated.

4. THE STOCHASTIC APPROACH. A SIMULATION MODEL

Assume that the functions $\overline{\varphi}(t)$ and $\psi(t)$ are stochastic functions, i.e. In the form $\overline{\varphi}(t; \omega)$, $\psi(t; \omega)$ and they are defined on a given probability field (Ω, K, P), $\omega \in \Omega$. Then the functions $C(t)$ and $\gamma(t)$ become also stochastic functions $C(t; \omega)$ and $\gamma(t; \omega)$.

Denote

$$\begin{aligned} C(t) &= E[\varepsilon(t; \omega)] \\ \gamma(t) &= E[\gamma(t; \omega)] \end{aligned} \tag{21}$$

the expectations of the corresponding stochastic functions.

Then the problem (3) reduces to minimizing the average cost per unit of time.

If we consider the problem

$$\min_{t\geq 0}\gamma(t;\ \omega)=\gamma(\hat{t}) \tag{22}$$

then $\hat{t}$ must be considered as $\hat{t}=\hat{t}(\omega)$, i.e. a random variable.

The optimum renewal moment will be estimated in this case as a confidence interval $(t_1,\ t_2)$ such as

$$P(\hat{t}_1\leqslant\hat{t}(\omega)\leqslant\hat{t}_2)=1-\alpha \tag{22'}$$

where α is a given small probability. The interval $(\hat{t}_1,\ \hat{t}_2)$ will be called the *confidence interval of the optimum renewal moment* (CIORM) of the level $1-\alpha$ and to determine it in a direct way based on (22) is a difficult problem. Therefore, we will use a Monte Carlo simulation model to determine such an interval.

Assume that

$$\begin{aligned} X_t(\omega)&=A_0\,\overline{\varphi}(t;\ \omega)\\ Y_t(\omega)&=\psi(t;\ \omega)\end{aligned} \tag{23}$$

are gaussian stationary stochastic processes with the following known characteristics

$$\begin{aligned} &m_1(t)=E[X_t(\omega)],\ \sigma_1^2=Var\,[X_t(\omega)]\\ &m_2(t)=E[Y_t(\omega)],\ \sigma_2^2=Var\,[Y_t(\omega)]\\ &\rho_1(h)=Corr\,[X_t(\omega),\ X_{t+h}(\omega)]=\theta_1^h,\ 0<\theta_1<1,\ h>0\\ &\rho_2(h)=Corr\,[Y_t(\omega),\ Y_{t+h}(\omega)]=\theta_2^h,\ 0<\theta_2<1\end{aligned} \tag{23'}$$

where ρ_1, ρ_2 are the autocorrelation functions, Obviously, $X_0(\omega)=A_0$, $\lim_{t\to 0}X_t(\omega)=0$ with probability one and $m_1(0)=A_0$, $\lim_{t\to\infty}m_1(t)=0$, $Y_0(0)=0$, $m_2(t)<\infty$, $\lim_{t\to\infty}m_2(t)=\infty$, We assume also that $\{X_t\}$, $\{Y_t\}$ are independent stochastic processes. The aim of the following considerations is to determine the CIORM of the level $1-\alpha$ on a given horizon of time N, $(N\in N_+)$. Note that $\hat{t}(\omega)$ is a discrete random variable in the form

$$\hat{t}(\omega):\begin{pmatrix}1 & 2 & \dots & N\\ p_1 & p_2 & \dots & p_N\end{pmatrix},\qquad \begin{aligned}&p_i=P(\hat{t}(\omega)=i)>0\\ &\sum_{i=1}^{N}p_i=1.\end{aligned} \tag{24}$$

Therefore, for determining CIORM $(\hat{t}_1,\ \hat{t}_2)$ we need the probability distribution (24).

The following simulation algorithm gives an estimate of the probability distribution (24) and the estimate of $(\hat{t}_1,\ \hat{t}_2)$ for a given risk α.

Step 0. Select a large sample size n for estimating the probability distribution. Use a frequency table (integer numbers) $F(1), \ldots, F(N)$ and take $F(i) = 0$, $1 \leqslant i \leqslant N$, Initialize algorithms for computer generation of stochastic processes $\{X_t\}$, $\{Y_t\}$, $t > 0$, (Such algorithms are found in [4] and [5]). Initialize α, A_0 and $i = 1$. Initialize MIN (a large positive real constant, MIN $\gg A_0 N$), Use the relative frequency table (real numbers) $R(1), \ldots, R(N)$.

Step 1. (Produce a sampling value of $\hat{t}$, $1 \leqslant \hat{t} \leqslant N$),

Step 1.0. Take $\nu = 0$, $\psi = 0$,

Step 1.1. Take $\nu \leftarrow \nu + 1$, generate (using algorithms from [1], [5] stationary gaussian and independent stochastic processes Z_ν and U_ν with zero means and variances one which have respectively the autocorrelation functions $\rho_1(h)$ and $\rho_2(h)$. Then calculate

$$X_\nu = m_1(\nu) + \sigma_1 . Z_\nu, \; x_\nu = m_2(\nu) + \sigma_2 U_\nu.$$

Step 1.2. Calculate

$$\psi \leftarrow \psi + Y_{\nu\nu} \quad = A_0 X_\nu + \psi$$

$$\gamma_\nu = \frac{C_\nu}{\nu}, \; \gamma_{min} = min \, (\text{MIN}, \; \gamma_\nu)$$

If $\gamma_{min} = \gamma_\nu$ then take $\nu^* = \nu$ and MIN $= \gamma_{min}$.

Step 1.3. If $\nu < N$ then go to step 1.1, else take $\hat{t} = \nu^*$ and continue

Step 2. (Undate frequencies $F(i)$). Take $F[\hat{t}] \leftarrow F[\hat{t}] + 1$. (This is possible because $1 \leqslant \hat{t} \leqslant N$).

Step 3. If $i < n$ then take $i \leftarrow i + 1$ and go to step 1, else continue.

Step 4. (The calculation of relative frequencies).

$$R[i] = F[i]/n, \; 1 \leqslant i \leqslant N.$$

The $R[i]$ are calculated in floating point.

Step 5. (The calculation of $(\hat{t}_1, \hat{t}_2)$).

Step 5.0. Take $\alpha_2 = \alpha/2$.

Step 5.1. (This step calculates $\hat{t}_1$).

5.1.1. Take $\hat{t}_1 = 1$ and $T_1 = R[1]$.

5.1.2. If $T_1 > \alpha_2$ then go to step 5.2., else continue.

5.1.3. Take $\hat{t}_1 \leftarrow \hat{t}_1 + 1$, $T_1 \leftarrow T_1 + R[\hat{t}_1]$ and go to 5.1.2.

Step 5.2. (This step calculates $\hat{t}_2$).

5.2.1. Take $\hat{t}_2 = N$ and $T_2 = R[N]$.

5.2.2. If $T_2 > \alpha_2$ then go to step 6, else continue.

5.2.3. Take $\hat{t}_2 \leftarrow \hat{t}_2 - 1$, $T_2 \leftarrow T_2 + R[t_2]$ and goto 5.2.2.

Step 6. Deliver $\hat{t}_1$, $\hat{t}_2$, Stop.

Note. In the above algorithm can also be included the calculation of the most probable renewal moment $\hat{t}_p$ which is defined as follows

$$R[\hat{t}_p] = \max_{1 \leqslant i \leqslant N} R[i],$$

For doing this the steps 5 and 6 must be modified accordingly.

5. COMPUTER SIMULATION RESULTS

The preceeding simulation model was implemented and tested on the computer in the following particular case (case (II)).

$$m_1(t) = \frac{\beta}{\beta + t}, \ \beta = 0.3; \ A_0 = 10^6$$

$$m_2(t) = kt^2 \cdot \sqrt{t}, \ k = 10^4$$

$$\sigma_1 = 10^5, \ \sigma_2 = 10^3, \ N = 100, \ n = 1000$$

$$\rho_1(h) = \theta_1^h, \ \theta_1 = 0.5; \ \rho_2(h) = \theta_2^h, \ \theta_2 = 0.7.$$

For a given risk $\alpha = 0.05$ the tolerance interval of the optimum renewal moment derived from the simulation model was $\hat{t}_1 = 4$, $\hat{t}_2 = 6$ and the probable renewal moment was found $\hat{t}_p = 5$. The computer time on the Romanian computer FELIX C-256 was about 20 minutes.

Therefore the simulation model is good enough for solving the problem. Furthermore it enables various artificial experiments under various flexible assumptions.

Finally, we note that under suitable interpretations of the functions $\varphi(t)$ and $\psi(t)$ the model can be also applied to determine the optimum capital repair moment of a given complex equipment.

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