Dr. MARGARETA MIHALYI
Dr. C. RAISCHI
Department of Mathematics
Academy of Economic Studies

An Algorithm for Optimal Ordering of the Arcs of a Network

In this paper, one considers the networks where for every vertex x, the set of arcs incident out of x is totally ordered. To such a network one associates a new one, by changing the value of every arc with the sum of the values of the precedent arcs. In this paper we give an algorithm of ordering the arcs incident out of every vertex x so that the greatest value of the paths of the new network to be minimal.

We give also an example of a question whose solution requires such an ordering.

Let us assume that we have to determine the shortest length of time necessary to fulfill a certain objective (a program of goods production, a program of development or a research program). One considers that one knows the operations required to achieve the objective and for every operation one knows its duration and the set of precedent operations. We can figure our problem by a graph of network type whose arcs represent the operations and the value of any arc is the duration of the corresponding operation [2].

If, excepting the prescribed precedences, the operations can be done independently one from another, then the duration of performing the objective is given by the value of the critical path. But, sometimes the operations with the same initial endpoint must be done succesively (they are achieved by the same team of workers, or on the same machine etc.). For instance, let $O_1, O_2, ..., O_n$ be the operations with the same initial endpoint x and $d(O_1)$, $d(O_2), ..., d(O_n)$ their durations. If the operations preceding $O_1, ..., O_n$ end at the moment t and if these operations are done in the order of their indices, then O_1 will end at the moment $t + d(O_1)$, O_2 at the moment $t + d(O_1) + + d(O_2)$, etc. For this reason, the value of the arc corresponding to O_2 will be $d(O_2) + d(O_1)$, the value of the arc corresponding to O_3 will be $d(O_3) + d(O_2) + + d(O_1)$, and so on. Then, the duration for performing the objective is the

value of the critical path in the network with the values of the arcs modified as above. The question is to find an order of the operations, so that the value of the critical path to be minimal.

In the following we present notations and definitions used in this paper.

We call *network* a finite graph without circuits

$G = \langle x_0, X, x_{n+1}, \Gamma, v\rangle$, where:

— $X = \{x_0, x_1, ..., x_{n+1}\}$ is the set of graph vertices;

— Γ is a function $\Gamma: X \to \mathcal{P}(X)$; one denotes by U the set of the arcs: $U = \{(x_i, x_j) \mid x_i, x_j \in X, x_j \in \Gamma(x_j)\}$;

— x_0 is the unique vertex x_i with the property $\Gamma^{-1}(\{x_i\}) = \varnothing$;

— x_{n+1} is the unique vertex x_i with the property $\Gamma(x_i) = \varnothing$;

— $v: U \to \mathbb{R}_+$ is the value function; it associates to any arc a non-negative number, called the value of the arc.

From now on, all graphs will be networks.

A function $O: U \to \mathbb{N}$ is called *an order in the graph G* if for any vertex x the constraint of O to $\{x\} \times \Gamma(x)$ is a bijection onto the set $\{1, 2, ..., \text{card } \Gamma(x)\}$.

A graph G and order O in it being given, by $G(O)$ we mean the graph $\langle x_0, X, x_{n+1}, \Gamma, v_0\rangle$ where x_0, X, x_{n+1}, Γ are as in G, but $v_0: U \to \mathbb{R}$ is given by the formula:

$$v_0(x_i, x_j) = \sum_{\{k \mid 1 \leqslant O(x_i, x_k) \leqslant O(x_i, x_j)\}} v(x_i, x_k) \qquad (*)$$

namely, to the "initial" value of the arc (x_i, x_j) one adds the values of all the arcs with the initial endpoint x_i and which procede (x_i, x_j) in the order O.

For a graph G and an order O in it, one denotes by $V(G(O))$ the greatest value of the paths of $G(O)$, [1]. One says that an order O' is at least as good as an order O if $V(G(O')) \leqslant V(G(O))$; an order O_0 is called *optimal (order)* if

$$V(G(O_0)) = \min_{O \in \mathcal{O}} V(G(O))$$

where $\mathcal{O}$ is the set of all the orders in G. By *optimal path* from the vertex x_i to the vertex x_j one means a path of maximal value from x_i to x_j in $G(O)$, where O is an optimal order in G.

Any network being progresively finite, there exists an ordinal function on X, $o: X \to \mathbb{N}$, according to [1]. In the case of a network $o(x)$ represent the greatest number of arcs in a path from x to the unique final vertex x_{n+1}.

Proposition. An optimal ordering in G and the value of an optimal path from x_0 to x_{n+1} can be obtained in the following way:

1) the number $\lambda_{n+1} = 0$ is attached to the vertex x_{n+1}

2) we consider the set $\{\lambda_k \mid x_k \in \Gamma(x_i)\}$ for any vertex x_i, $i \neq n+1$ and we order the set of arcs which have x_i as initial endpoint by decreasing order of the values λ_k, more precisely: if $\Gamma(x_i) = \{x_{k_1}, ..., x_{k_r}\}$ and $\lambda_{k_1} \geqslant ... ..., \geqslant \lambda_{k_r}$ then one considers $O(x_i, x_{k_j}) = j$; the number $\lambda_i = \max_{x_k \in \Gamma(x_i)} \{v_0(x_i, x_k) + \lambda_k\}$ is attached to the vertex x_i;

The order thus obtained is optimal and λ_0 is the value of an optimal path from x_0 to x_{n+1}.

Proof. Let $o(x_0) = N$, where o is the ordinal function on the set X. The sets $X_k = \Gamma^{-1}(\{k\})$, $k \in \{1, 2, ..., N\}$ give a partition of the set $X/\{x_{n+1}\}$. Let x_i be a vertex of the graph G and consider that k is the index such that $x_i \in X_k$. We shall prove by induction on k the existence of the value λ_i and the fact that λ_i is the value of an optimal path from x_i to x_{n+1}.

For $k = 1$, it results $\Gamma(x_i) = \{x_{n+1}\}$ and from x_i to x_{n+1} the only path is the arc (x_i, x_{n+1}); in this case $\lambda_i = v(x_i, x_{n+1})$, $v_0(x_i, x_{n+1}) = v(x_i, x_{n+1})$ and the above assertions are obviously true.

Assume the above assertions true for any $k \leqslant p$.

If $k = p + 1$, then any path from x_i to x_{u+1} is of the form (x_i, x_j), $[x_j, x_{n+1}]$ where $x_j \in \Gamma(x_i)$ and $[x_j, x_{n+1}]$ denote a path from x_j to x_{n+1}; since $x_j \in \Gamma(x_i)$ we have $o(x_j) < o(x_i) = p + 1$ and so there is $s < p + 1$ such that $x_i \in X_s$. To obtain an optimal path from x_i to x_{n+1} we must choose an optimal path from every $x_j \in \Gamma(x_i)$ (by inductive assumption the chosen path has a certain value λ_j). Hence, to determine an optimal path from x_i to x_{n+1} one has to find an optimal one from x_i to y in the graph G_i:

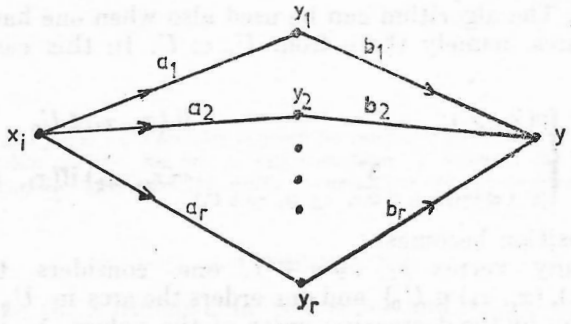

Fig. 1

where $\{y_1, ..., y_r\} = \Gamma(x_i)$, $a_j = v(x_i, y_j)$, b_j is the number corresponding to the vertex y_j, y is an auxiliary vertex and we assume that the indexes are chosen such that $b_1 \geqslant , ..., \geqslant b_r$.

We shall prove that an optimal order in G_i is given by

$$O(x_i, y_j) = j, O(y_j, y) = 1, \qquad j \in \{1, ..., r\}.$$

Let O' be an arbitrary order in G_i and consider $\{y_1, ..., y_r\} = \{y_{i_1}, ..., y_{i_r}\}$ such that $O'(x_i, y_{i_j}) = j$. We chose the integer m minimal with the property $i_m \neq m$ and we denote by t the number for which $i_t = m$; obviously, $t > m$. We shall show that the order O'' which differs from O' only by the interchange of the arcs (x_i, y_{i_m}) and (x_i, y_{i_t}) (namely: $O''(x_i, y_{i_m}) = O'(x_i, y_{i_t})$, $O''(x_i, y_{i_t}) = O'(x_i, y_{i_m})$ is at least as good as the order O'.

The graphs $G_i(O')$ and $G_i(O'')$ differ only by the values of the arcs (x_i, y_{i_j}) for j in $\{m, m + 1, ..., t\}$; among the paths from x_i to y which contain such arcs, that of maximal value is $[x_i, y_{i_t}, y]$ in $G_i(O')$, because it has the value

$$b_m + \sum_{j=1}^{t} a_{i_j} = \max_{j \in \{m,\ m+1,\ ...,\ t\}} b_j + \sum_{j=1}^{t} a_{i_j};$$

hence, the order O'' is at least as good as O'.

By the above procedure, in at most $r-1$ steps, one obtains from the order O' the order O; it follows that O is at least as good as O'. The value of an optimal path from x_i to y on G_i, hence of an optimal path from x_i to x_{n+1} in G will be:

$$\max_{x_j \in \Gamma(x_i)} \{v_0(x_i, x_j) + \lambda_j\} = \lambda_i.$$

Remark 1. In order to use the algorithm from the Proposition, it is better that the vertices of the graph be ordered upon levels [2], namely to be determined the sets $o^{-1}(\{k\}), k \in \{0, 1, ..., N\}$ by one of the methods shown in [2]; one orders arbitrarily the vertices in $O^{-1}(N)$ and then one considers that for two vertices x_i, x_j one has $i < j$ if and only if $o(x_i) > o(x_j)$. The value λ_k are computed by decreasing order of the indexes k.

Remark 2. For the determination of an optimal order of the arcs incident into every vertex, one can use the above algorithm in the graph obtained by reversing all the arrows.

Remark 3. The algorithm can be used also when one has to order only a part of the arcs, namely those from $U_0 \subset U$. In this case, the formula (x^*) becomes

$$\varphi_0(x_i, x_j) = \begin{cases} v(x_i, x_j) & \text{if } (x_i, x_j) \notin U_0 \\ \sum\limits_{\{k \mid 1 \leqslant O(x_i, x_k) \leqslant O(x_i, x_j),\ (x_i, x_j) \in U_0\}} v(x_i, x_k) & \text{if } (x_i, x_j) \in U_0 \end{cases}$$

and 2) in proposition becomes:

2)′ for any vertex x_i, $i \neq n+1$, one considers the set $A_i = \{\lambda_k \mid x_k \in \Gamma(x_i), (x_i, x_k) \in U_0\}$ and one orders the arcs in U_0 which have x_i as initial endpoint by the decreasing order of the values λ_k in A_i, namely: if $B_i = \{x \in \Gamma(x_i) \mid (x_i, x) \in U_0\}$ and $B_i = \{x_{k_1}, ..., x_{k_r}\}$ such that $\lambda_{k_1} \geqslant ... \geqslant \lambda_{k_r}$ then $O(x_i, x_{k_j}) = j$ for $j \in \{1, ..., r\}$; one associates to the vertex x_j the number

$$\lambda_i = \max_{x_k \in \Gamma(x_i)} \{v_0(x_i, x_k) + \lambda_k\}.$$

REFERENCES

1. BERGE, C.: *Teoria grafurilor și aplicațiile ei* (Graphs Theory and Its Applications), Ed. Tehnică, Bucharest, 1969.
2. KAUFMANN, A., DESBAZEILLE, G.: *Critical Path Method*, Ed. Tehnică, Bucharest. 1971.