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Applications of Informational Correlation in Construction Reliability

Construction safety calculation is essentially based on ths concept of probability as a measure to achieving an aleatory variable given through a statistics of selection.

The considered fundamental variables in the case of structures safety calculation in construction are structural strength (R) and load or burden (Q). The connection between the two variates can be expressed by two ratios:

$$\zeta = R - Q; \quad \eta = \frac{R}{Q} \tag{1}$$

which represents the mathematical expression of the new variables.

The values of R and Q being aleatory have allocation functions:

$$\begin{aligned} F_R(x) &= P(R < x) \\ F_Q(x) &= P(Q < x). \end{aligned} \tag{2}$$

To these allocation functions correspond the allocation densities $f_R(x)$ and $f_Q(x)$ and obviously we have the ratios:

$$\begin{aligned} F_R(x) &= \int_0^x f_R(x)\ \mathrm{d}x \\ F_Q(x) &= \int_0^x f_Q(x)\ \mathrm{d}x. \end{aligned} \tag{3}$$

If the allocation functions f_R, f_Q are independent the probability of structure subsiding can be expressed by the ratio:

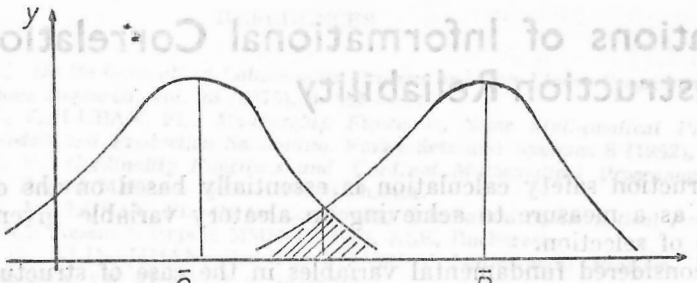
$$P_C = \int_0^\infty F_R(x)\ f_Q(x)\ \mathrm{d}x. \tag{4}$$

An integral form (4) is called integral of convolution and the problem of calculation is rather complicate.

The problem complicates even more, as, in fact, we are going to notice in the following paragraph, when the allocation functions of the corresponding densities belong to different classes of allocation.

For avoiding the difficulties connected with the calculation or estimating by reckoning of some complicate integrals of convolution, as next paragraph will show, we shall analyse a method of structure safety calculation based on the correlation coefficient calculation.

Let us represent graphically, considering the same system of efficiency, the theoretical allocation of the two aleatory variables: structural strength and load:



Analysing the graphic, one can notice the fact that the average value of structural strength is higher than the average value of the load $\bar{R} > \bar{G}$.

The damage probability for a certain structure is equal to the probability that the load size should be higher than the structural strength set

$$P_s = P(Q \geqslant R)$$

and obviously we get the ratios:

$$F_Q(x) = P(Q \leqslant x) = \int_0^x f_Q(\xi)\, d\xi \tag{5}$$

$$1 - F_Q(x) = P(Q > x) = \int_x^\infty f_Q(\xi)\, d\xi$$

where $f_Q(\zeta)$ and $F_Q(x)$ are density and the allocation function of the load or burden.

The probability that structural strength should acquire value x is: $P(R = (x) \leqslant f_R$ (where) $f_R(x)$ represents the density of allocation structural strength.

The probability that in point x, the load value Q should be higher than that of structural strength is:

$$f_R(x) = \int_{\xi=x}^\infty f_Q(\xi)\, d\xi. \tag{6}$$

The damage probability becomes:

$$P_C = P(Q > R) = \int_{x=0}^{\infty} \left[\int_{\xi=0}^{\infty} f_Q(\xi)\, d\xi \right] f_R(x)\, dx. \tag{7}$$

This ratio represents the probability that maximum load should surfass minimal structural strength for all x values.

The ratio in square brackets represents the area of the hachured part in picture (1).

Ratio $f_R(x)\, dx$ is the probability that structural strength R should take value x.

The probability that load should take values higher than structural strength in points x is:

$$\left[\int_{\xi=x}^{\infty} f_R(\xi)\, d\xi \right] f_R(x)\, dx. \tag{8}$$

The probability that load should take values higher than structural strength in all points x is:

$$R_t = P(Q \geqslant R) = \int_{x=0}^{\infty} \left[\int_{t=x}^{\infty} f_Q(\xi)\, d\xi \right] f_R(x)\, dx. \tag{9}$$

In the paragraphs below we are going to introduce a new index of reliability of structure with the help of the informational correlation of the load and structural strength allocations.

Definition 1. We call information al correlation of structural strength and load, the expression:

$$\mathrm{Cov}\ (R, Q) = \int_0^{\infty} f_R(x) f_Q(x)\ dx \tag{10}$$

where $f_R(x)$, $f_Q(x)$ are allocations densities of structural strength and load.

Definition 2. We call empiric informational correlation of selection of structural strength and load, the expression:

$$\widetilde{\mathrm{Cov}}\,(R,\ Q) = \sum_{i=1}^{n} p_R^{(i)} p_Q^{(i)} \tag{11}$$

where $p_R^{(i)}$ and $p_Q^{(i)}$ are relative frequencies of the load and structural strength values.

Definition 3. We call coefficient of informational correlation of load and structural strength allocation, the expression:

$$\mathrm{Cor}\ (R,\ Q) = \frac{\int_0^{\infty} f_R(x) \cdot f_Q(x) dx}{\sqrt{\int_0^{\infty} f_R^2(x)\, dx \cdot \int_0^{\infty} f_Q^2(x)\, dx}}. \tag{12}$$

Definition 4. We call empirical coefficient of correlation of load and structural strength, the expression:

$$\widetilde{\mathrm{Cor}}\ (R,Q)=\frac{\sum_{i}^{n} p_R^{(i)}\cdot p_Q^{(i)}}{\sqrt{\left[\sum_{i}^{n} p_R^{(i)^2}\right]\left[\sum_{i}^{n} p_Q^{(i)^2}\right]}}. \tag{13}$$

Definition 5. We call informational index of safety of a structure, the expression:

$$\widetilde{I}=1-\mathrm{Cor}(R,Q). \tag{14}$$

Definition 6. We call empiric informational index of safety of a construction, the expression:

$$\widetilde{I}=1-\widetilde{\mathrm{Cor}}\ (R,\ Q) \tag{15}$$

where $\widetilde{\mathrm{Cor}}\ (R, Q)$ is the empiric coefficient of correlation of structural strength and load.

If the load and structural strength allocations are identical, then the correlation coefficient has value 1 and in this case the safety index of the structure is zero.

If the correlation coefficient is zero, then the safety index has maximum value and is equal to one.

When a structure is considered a system made up of n elements connected in a series, that is to say the damaging of an element entails the damaging of the whole system we may consider that I_s the informational index of safety for the system is given by the ratio:

$$I_s=\prod_{k=1}^{n} I_k \tag{16}$$

where I_k is the informational index in the case of various allocations of structural strength and load:

a) If the structural strength and load have normal allocations $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$ we have:

$$I(\sigma_1, \sigma_2)=1-\frac{2\sqrt{\sigma_1\sigma_2}}{\sqrt{\sigma_1^2+\sigma_2^2}}. \tag{17}$$

b) The structural strength and load in exponential allocation

$$I(\lambda_1, \lambda_2)=1-\frac{2\sqrt{\lambda_1\ \lambda_2}}{\lambda_1+\lambda_2}=\frac{(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}{\lambda_1+\lambda_2}. \tag{18}$$

If a system is made up of p elements grouped in series, then the safety index is:

$$I_s=\prod_{1}^{n}\frac{(\sqrt{\lambda_1^{(k)}}-\sqrt{\lambda_2^{(k)}})^2}{\lambda_1^{(k)}+\lambda_2^{(k)}} \tag{19}$$

where $\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ are the parameters of exponential allocation for elements k.

c) If structural strength and load have allocations χ^2 having $n_1 > 0$, $n_2 > 0$ degrees of liberty, then we get:

$$I = 1 - \frac{\Gamma\left(\frac{(n_1 + n_2}{2} - 1\right)}{\Gamma^{1/2}(n_1 - 1) \cdot \Gamma^{1/2}(n_2 - 1)}. \tag{20}$$

d) Let us consider two Weibull allocations of structural strength and load:

$$f_R(x, s_1, \lambda) = s_1\lambda_1 e^{-\lambda_1 x^{s_1}} \cdot x^{s_1 - 1}$$
$$f_Q(x, s_2, \lambda) = s_2\lambda_2 e^{-\lambda_2 x^{s_2}} \cdot x^{s_2 - 1} \tag{21}$$

In hypostasis $s_1 = s_2$ the informational index of saftey:

$$I = 1 - \frac{(4\lambda_1\lambda_2)^{\frac{2s-1}{2s}}}{(\lambda_1 + \lambda_2)^{\frac{2s-1}{s}}}$$

When we do not have at our disposal enough data for estimating the character of the theoretical allocations of structural strength and load for a system of structures on the basis of observation data and measurements one can find out the empirical allocation index of safety for each element. The smallest index of safety calculated for all elements is considered global informational index.

Let us exemplify the calculation of the informational index of safety in the case of reliability study for a concrete pole.

For one of the pole parts bar LTR the values of the structural resistence and load are given in the table below:

Q	$n_Q^{(i)}$	R	$n_R^{(i)}$
2	8	32	5
7	17	37	3
12	6	42	10
17	17	47	12
22	2	52	10
27	2	57	7
32	1	62	13
37	1	67	4
42	1	72	1

Informational energies of structural strength and load:

$$E_Q = \sum_{i=1}^{n} \frac{n_Q^2}{N^2} = 0.215, \quad E_R = \sum_{i=1}^{n} \frac{n_R^2}{N^2} = 0.145. \tag{22}$$

Informational correlation: $\text{Cov}\ (R, Q) = \sum_{i}^{n} \frac{n_Q n_R}{N^2} = \frac{1.5 + 1.3 + 1.1}{65^2} = \frac{18}{65^2}$.

The correlation coefficient is:

$$\text{Cor}\ (R, Q) = \frac{\text{Cov}\ (R, Q)}{\sqrt{E_R \cdot E_Q}} = \frac{18}{\sqrt{1613 \cdot 909}} \cong 0{,}024.$$

The informational index of safety is:

$$I \cong 1 - 0.024 \cong 0.985.$$

The value of the safety informational index is almost 1 and therefore from a statistics point of view one can consider the reliability of the structure as being good.

If the various part of the constructions are not connected by a grouping of a series type, we find out the indices of informational safety and we consider the minimal value of the safety indices found as a measure of the safety index of the pole. Following a similar way to that described in the previous model, the safety indices of bars and foundations have been calculated and the following data have been obtained:

Bars	LTR	L-10	Cross-studs ND—TR	Contracted bar	Sandy clay Fondations
Informational safety index	0.986	0.975	0.961	0.909	0.906

The smallest value calculated for safety indices 0.906 is considered informational index of safety for the structure.

For calculating the informational safety index of useful loads in spaces having functional destinations, we reckon that they are best described by the allocation of the extreme values for maximums or Gumbal.

This allocation is characterized by average m_1 and standard deviation σ_1.

The allocation function is given by the ratio

$$F(x) = e^{-e^{\alpha_1 - (x - u_1)}} \tag{23}$$

and the allocation density by the ratio

$$f(x) = \alpha_1 e^{-e^{\alpha_1 (x - u_1)}} \cdot e^{-\alpha_1 (x - u_1)}. \tag{24}$$

From the ratios:

$$\int_0^\infty x f(x)\, dx = m_1, \quad \int_0^\infty (x - m_1)^2 f(x)\, dx = \sigma_1$$

we get: $\alpha_1 = \frac{1.282}{\sigma_1}$; $u_1 = m_1 - 0{,}45\ \sigma_1$.

The informational energy of Gumbel allocation is:

$$E = \int_0^\infty f^2(x)\, dx = \int_0^\infty \alpha_1^2 e^{-2e^{-\alpha_1 (x - u_1)}} \cdot e^{-2\alpha_1 (x - u_1)}\, dx. \tag{25}$$

Through a change of the variable:

$$e^{-\alpha_1(x-u_1)} = t; \quad -\alpha e^{-\alpha_1(x-u_1)}\,dx = dt; \quad \alpha t\,dx = dt$$

becomes

$$I = \int \alpha_1^2 e^{-2e^{-\alpha_1(x-u_1)}} \cdot e^{-2\alpha_1(x-u_1)}\,dx = -\int \alpha_1^2 e^{-2t} t^2 \frac{dt}{\alpha_1 t} =$$

$$= -\alpha_1 \int e^{-2t} t\,dt = \frac{\alpha_1}{4} e^{-2t}(2t+1) = \frac{\alpha_1}{4} e^{-2e^{-\alpha_1(x-u_1)}}(2e^{-\alpha_1(x-u_1)} + 1). \quad (26)$$

Therefore we get:

$$E = \frac{\alpha_1}{4} - \frac{\alpha_1}{4} e^{-2e^{\alpha_2 u_1}}(2e^{\alpha_1 u_1} + 1). \quad (27)$$

The informational covariation of two Gumbel allocations is given by the ratio:

$$\text{Cov}\ (f_1, f_2) = \int_0^\infty f_1(x) \cdot f_2(x)\ dx =$$

$$= \int_0^\infty \alpha_1 \alpha_2 e^{-e^{-\alpha_1(x-u_1)} - e^{-\alpha_1(x-u_2)}} \cdot e^{-\alpha_1(x-u_1)-\alpha_2(x-u_2)}\,dx. \quad (28)$$

In the hypothesis $\alpha_1 = \alpha_2 = \alpha$, we get:

$$I = \int_0^\infty \alpha_1 \alpha_2 e^{-e^{-\alpha x}} \cdot e^{\alpha u_1} - e^{-\alpha x} \cdot e^{\alpha x_2} \cdot e^{-2\alpha x}\, e^{\alpha u_1 + \alpha u_2}\,dx \quad (29)$$

or

$$\text{Cov}\ (f_1, f_2) = \int_0^\infty x e^{-e^{-\alpha x}(e^{\alpha u_1} + e^{\alpha u_2})} \cdot e^{-2\alpha x} \cdot e^{\alpha(u_1+u_2)}\,dx.$$

Let us designate $p = e^{\alpha u_1} + e^{\alpha u_2}$, then we get:

$$\text{Cov}\ (f_1, f_2) = \alpha^2 \int_0^\infty e^{-e^{-\alpha x \cdot p}} \cdot e^{-2\alpha x} \cdot e^{\alpha(u_1+u_2)}\,dx.$$

Let us consider the integral:

$$I = \alpha^2 \int - e^{-e^{\alpha x \cdot p}} \cdot e^{-2\alpha x}\,dx. \quad (30)$$

By changing the variable

$$e^{-\alpha x} = t; \quad -\alpha e^{-\alpha x}\,dx = dt; \quad dx = \frac{dt}{-\alpha t}$$

becomes:

$$I = -\alpha\int e^{-tp}t\mathrm{d}t = -\alpha\left[-t\frac{e^{-tp}}{p} + \frac{1}{p}\int e^{-tp}\,\mathrm{d}t\right] =$$

$$= -\alpha\left[-t\frac{e^{-tp}}{p} - \frac{1}{p^2}e^{-tp}\right] + e = \alpha\frac{e^{-tp}}{p^2}(1+tp). \tag{31}$$

Therefore

$$\mathrm{Cov}\ (f_1, f_2) = \frac{\alpha e^{-e^{-\alpha x}}(e^{\alpha u_1} + e^{\alpha u_2})}{(e^{\alpha u_1} + e^{\alpha u_2})^2}[1 + e^{-\alpha x}(e^{\alpha u_1} + e^{\alpha u_2})]\Big|_0^\infty =$$

$$= \left[\frac{\alpha}{(e^{\alpha u_1} + e^{\alpha u_2})^2} - \frac{\alpha\cdot e^{-(e^{\alpha u_1}+e^{\alpha u_2})}(1 + e^{\alpha u_1} + e^{\alpha u_2})}{(e^{\alpha u_1} + e^{\alpha u_2})^2}\right] e^{\alpha(u_1+u_2)}. \tag{32}$$

In the special case in which $u_1 = u_2$, that is to say the allocations are identical, we get the formula of informational energy and we get:

$$\mathrm{Cov}\ (f_1, f_2) = \frac{\alpha}{4e^{2\alpha u}} - \frac{\alpha e^{-2e^{\alpha u}}(1 + 2e^{\alpha u})}{4e^{2\alpha u}}e^{\alpha u} = \frac{\alpha}{4} - \frac{\alpha}{4}e^{-2e^{\alpha u}}(1 + 2e^{\alpha u}). \tag{33}$$

The informational correlation coefficient becomes:

$$\mathrm{Cov}\ (f_1, f_2) = \frac{\mathrm{Cov}\ (f_1, f_2)}{\sqrt{Ef_1 \cdot Ef_2}} =$$

$$= \frac{\dfrac{\alpha e^{\alpha(u_1+u_2)}}{(e^{\alpha u_1} + e^{\alpha u_2})^2}[1 - e^{-(e^{\alpha u_1}+e^{\alpha u_2})}(1 + e^{\alpha u_1} + e^{\alpha u_2})]}{\dfrac{\alpha}{4}\sqrt{[1 - e^{-2\alpha^{\alpha u_1}}(2e^{\alpha u_1} + 1)][1 - e^{-2e^{\alpha u_2}}(2e^{\alpha u_2} + 1)]}} \tag{34}$$

One can notice that when the allocations are identical, then $\mathrm{Cov}(f_1, f_2) = 1$.

The formula (34) allows us to calculate according to the parameters u_1, u_2 and α of a Gumbel allocation of the structural strength and load, the safety informational index of a structure.

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