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The Onicescu Informational Energy in the Continuous Case

By similarity with the relative entropy an informational energy has been introduced for random variables with distribution densities. Starting form a distribution density for which there is no informational energy this study proves that it is defined only for classes of random variables.

Key words: information, synthesis index, relative entropy, discrete random variables, continuous random variables.

In the study [3] Academician O. Onicescu has introduced the notion of informational energy associated with a discrete random variable

$$X:\begin{pmatrix} x_i \\ p_i \end{pmatrix} i \in I$$

through

$$\mathcal{E}(X) = \sum_{i \in I} p_i^2$$

This measure reflects the grade of organization and disorganization of a system characterized by X. The advantage of using this method instead of entropy

$$H(X) = -\sum_{i \in I} p_i \log p_i$$

is obvious due not only to the expression including only the squares of probabilities p_i but also to the smaller errors propagated in this case.

Many other properties and results obtained by using the entropy have been outlined with the aid of informational energy. We shall reconsider this subject; now, we shall approach the continuous case when a distribution

density $f_x(x)$ exists. As it is known, in this case, we use a relative measure of information introduced through the relative entropy

$$H_x(X) = -\int_{-\infty}^{\infty} f_x(x) \log f_x(x) \, dx$$

and the study on the problems of continuous information theory in the sense of Shannon's ideas.

In case of the existence of the distribution density f_x O. Onicescu suggests as a measure of information the informational energy given by the expression

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} f_x^2(x) \, dx$$

This measure however raises the problem if all functions $f : \mathbb{R} \to \mathbb{R}$ with the properties $f(x) \geq 0$, $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x) \, dx = 1$ (f distribution density of a random variable) are of integrable square.

Considering $f : \mathbb{R} \to \mathbb{R}$, through

$$f(x) = \begin{cases} \dfrac{2}{x} \dfrac{1}{\sqrt{1-x^2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

one immediately notices that it is a distribution density, that is $f(x) \geq 0$ regardless of $x \in \mathbb{R}$, $\dfrac{2}{\pi} \int_2^1 \dfrac{dx}{\sqrt{1-x^2}} = 1$

Nevertheless $\int_{-\infty}^{\infty} f^2(x) \, dx$ is divergent.

The idea of the existence of a distribution density with this characteristic (for which there is no $\mathcal{E}(X)$ has appeared from a practical problem namely:

If we consider a mineralized rock in which the useful mineral substance is dissipated in the form of spherical or approximately spherical elements, the problem is posed of estimating the quantity of useful substance in particular and of the useful mineral deposits in general.

A valuating method might consist in achieving a plane section in which the substance appears in the form of sections of mineralized spheres, sections representing circles with an apparent diameter which will be a random variable with values in an interval $(0, D)$ with the distribution density

$$f_X(x) = \begin{cases} \dfrac{2}{\pi} \dfrac{1}{\sqrt{D^2 - x^2}} & x \in (0, D) \\ 0 & x \notin (0, D) \end{cases}$$

But, as we have seen, in this case there is no informational energy

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} f^2(x)\ \mathrm{d}x.$$

In this way, an approach to the informational energy through the relation suggested by O. Onicescu excludes, out of consideration, all the random variables the distribution densities of which are not of integrable square-thus, for which there is no proposed $\mathcal{E}(X)$.

In order to include the distribution density

$$f_X(x) = \begin{cases} \dfrac{2}{\pi} \dfrac{1}{\sqrt{D^2 - x^2}} & x \in (0,\ D) \\ 0 & x \notin (0,\ D) \end{cases}$$

into the class of densities for which there is an "informational energy" we shall consider it to be given by the expression

$$\mathcal{E}_\alpha(X) = \int_{-\infty}^{\infty} f^\alpha(x)\ \mathrm{d}x$$

and we shall choose $\alpha > 0$ so that there should be

$$\int_0^D \left(\frac{2}{\pi}\right)^\alpha \frac{\mathrm{d}x}{(D^2 - x^2)^{\alpha/2}}$$

However

$$\int_0^D \left(\frac{2}{\pi}\right)^\alpha \frac{\mathrm{d}x}{(D^2 - x^2)^{\alpha/2}} = \frac{1}{\pi^\alpha}\left(\frac{2}{D}\right)^{\alpha-1} \int_0^1 z^{-1/2}\,(1-z)^{-\frac{\alpha}{2}}\,\mathrm{d}z =$$

$$= \frac{1}{\pi^\alpha}\left(\frac{2}{D}\right)^{\alpha-1} B\left(\frac{1}{2},\ 1 - \frac{\alpha}{2}\right)$$

which is convergent for $1 - \dfrac{\alpha}{2} > 0$ that is $\alpha < 2$.

Thus, considering

$$\mathcal{E}_\alpha(X) = \int_{-\infty}^{\infty} f^\alpha(x) \text{ with } \alpha > 2,\ \alpha \neq 1$$

we have extended the class of the continuous type random variables for which there is an "informational energy" and we note that this class also includes those for which there is an O. Onicescu informational energy, as results through a direct computation (see O. Onicescu, V. Ştefănescu, Elements of Informational Statistics with Applications, Ed. Tehnică, 1979).

Let us now consider a Couchy distributed random variable X, that is having the density

$$f_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \quad x \in (-\infty, \infty)$$

Then

$$\mathcal{E}_\alpha(X) = \frac{1}{\pi^\alpha} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^\alpha} = \frac{1}{\pi^\alpha} \int_0^\infty z^{-1/2} (1+z)^{-\alpha} \, dz = \frac{1}{\pi^\alpha} B\left(\frac{1}{2}, \quad \alpha - \frac{1}{2}\right)$$

which is convergent for $\alpha > \frac{1}{2}$.

It results that taking $\alpha \in \left(\frac{1}{2}, \ 2\right)$, $\alpha \neq 1$ we have extended the class of random variables for which an "informational energy" exists

$$\mathcal{E}_\alpha(X) = \int_{-\infty}^{\infty} f^\alpha(x) \, dx$$

The question now arises if there is $\alpha \in \mathbb{R}_+^*$, $\alpha \neq 1$ so that $\mathcal{E}_\alpha(X) = \int_{-\infty}^{\infty} f^\alpha(x) \, dx$ should be convergent for all functions $f : \mathbb{R} \to \mathbb{R}$, $f \geqslant 0$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(x) \, dx = 1$ that is for all possible distribution densities.

The answer to the question is given by the theorem below.

Theorem. There are no $\alpha \in \mathbb{R}$, $\alpha \neq 1$ so that for all distribution densities $f : \mathbb{R} \to, \mathbb{R}$ $f \geqslant 0$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(x) \, dx = 1$, $\int_{-\infty}^{\infty} f(x) \, dx$ be convergent.

Proof. Considering the distribution densities

$$f(x) = \begin{cases} \frac{2}{\pi} \frac{1}{\sqrt{1 - x^2}} & x \in (0, \ 1) \\ 0 & x \in (0, \ 1) \end{cases}$$

and

$$f(x) = \frac{1}{(1+x^2)} \quad x \in (-\infty, \infty)$$

presented above, it results that it suffices to prove the assertion for

$$\alpha \in \left(\frac{1}{2}, \ 2\right) / \{1\}$$

Consider $\alpha \in (1,2)$ and the family of random variable $(X_\alpha)_\alpha \in (1, 2)$ with het distribution densities

$$f_\alpha(x) = \begin{cases} \dfrac{2}{B\left(\dfrac{2}{1}, \dfrac{\alpha - 1}{2}\right)} (1 - x^2)^{-1/2} & x \in (0,\ 1) \\ 0 & x \notin (0,\ 1) \end{cases}$$

Obviously

$$f_\alpha(x) = \geqslant 0,\ x \in \mathbb{R} \text{ and } \int_{-\infty}^{\infty} f_\alpha(x)\, dx = 1.$$

Nevertheless

$$\int_{-\infty}^{\infty} f_\alpha^\alpha(x)\, dx = \frac{2^\alpha}{\left(B\left(\dfrac{1}{2}, \dfrac{\alpha - 1}{2}\right)\right)^\alpha} \int_0^1 (1 - x^2)^{-1}\, dx$$

is divergent. This proves that there is no $\alpha \in (1,\ 2)$ so that $\int_{-\infty}^{\infty} f^\alpha(x)\, dx$ convergent for all distribution densities. Now consider $\alpha \in \left(\dfrac{1}{2}, 1\right)$ and $(X_\alpha)_{\alpha \in \left(\frac{1}{2}, 1\right)}$ a family of random variables with the distribution densities

$$f_\alpha(x) = \begin{cases} \dfrac{1}{B\left(1, \dfrac{1 - \alpha}{\alpha}\right)} (1 + x)^{-\frac{1}{2}} & x \in (0, \infty) \\ 0 & x \notin (0, \infty) \end{cases}$$

Naturally, $f_\alpha(x) \geqslant 0,\ x \in \mathbf{R}$ and

$$\frac{1}{B\left(1, \dfrac{1 - \alpha}{\alpha}\right)} \int_0^\infty (1 + x)^{-\frac{1}{2}}\, dx = 1 \quad \alpha \in \left(\frac{1}{2},\ 1\right)$$

Nevertheless $\dfrac{1}{\left(B\left(1, \dfrac{1 - \alpha}{\alpha}\right)\right)^\alpha} \int_0^\infty (1 + x)^{-1}\, dx$ is divergent and, therefore there is no $\alpha \in \left(\dfrac{1}{2},\ 1\right)$ so that $\int_{-\infty}^{\infty} f^\alpha(x)\, dx$ be convergent for all non-negative functions on $\mathbf{R}$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$. With this, the theorem is completely proved.

In conclusion, whereas for the discrete random variables

$$X : \begin{pmatrix} x_i, & i \in I \\ p_i & \end{pmatrix}$$ the informational energy

$$\mathcal{E}(X) = \sum_{i \in I} p_i^2$$ has always got a sense,

its analogue for the case when X has a distribution density f_x, $\mathcal{E}(X) = \int_{-\infty}^{\infty} f_x^2(x)\, dx$ has a sense only for the class of integrable square distribution densities, a class including all classical distributions known in specialist literature.

We show that a number of results have been obtained in the field of decision theory following that way.

If we want to extend the class of random variables for which an "informational energy" is defined we can introduce the informational energy of the degree α, $\mathcal{E}_\alpha(X) = \int_{-\infty}^{\infty} f^\alpha(x)\, dx\ \alpha > 0, \alpha \neq 1$ comprising the set of variables permitting densities f for which $\int_{-\infty}^{\infty} f^\alpha(x)\, dx$ is convergent.

This definition is in keeping with the informational energy of degree α in the discrete case

$$X : \begin{pmatrix} x_i, & i \in I \\ p_i & \end{pmatrix},\ \mathcal{E}_\alpha(X) = \sum_{i \in 1} p_i^\alpha\ \alpha > 0,\ \alpha \neq 1$$

introduced in [12] for which the results obtained in case $i = 2$, with slight, insignificant changes are maintained.

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