G. BOLDUR

On Group Decision

Indecisiveness occurs often in group decisions. This article analyzes some aspects of this phenomenon and suggests a way out of the dilemma.

INTRODUCTION

The problem of group decisions, which has important implications in economics, deals with the way in which the individual decisions of the members of a group are "summed up", in order to obtain a decision valid for the whole [5], [6], [11].

Further we shall briefly present the general formulation of the problem.

Let us consider a set of alternatives $\mathbf{A} = \{A_1, A_2, ..., A_m\}$ and a set of individuals $\mathbf{H} = \{H_1, H_2, ..., H_n\}$. Every individual effects a decision process concerning the alternatives of set **A**. The decision process has a particular structure and consists in the ordering of all the alternatives according to their usefulness and not only in the choice of the alternative of maximum usefulness[1]). Since usefulness is a subjective function [4], [9], the ordering of alternatives differs from individual to individual. The problem consists in obtaining a decision valid for the set **H**, and based on the decisions of the members $H_1, H_2, ..., H_n$. The group decision may consist in the choice of a single alternative; however, many authors consider that a complete ordering of the alternatives presents interest too.

Let H_i be an arbitrary individual of the group who effects a complete ordering of the alternatives belonging to set **A**. Between two arbitrary

---

[1]) This way o c nsidering the decision process is specific to works which treat of group decision problems. It is justified, in general, by the fact that a complete ordering of the alternatives offers the information required for the deduction of the group decision.

alternatives A_x and A_y ($A_x, A_y \in \mathbf{A}$), there may exist, after H_i has effected the ordering, one and only one of the following relations [5], [6], [11]:

— H_i prefers A_x to A_y ($A_x, \mathcal{P}_i, A_y$);

— H_i prefers A_y to A_x ($A_y, \mathcal{P}_i, A_x$);

— H_i does not prefer either alternative, or in other words, is indifferent to A_x and A_y ($A_x \mathcal{J}_i A_y$).

Thus, two basic relations in the theory of group decisions have been introduced: the preference and the indifference relation. The first relation is transitive $[(A_x \mathcal{P}_i A_y) \& (A_y \mathcal{P}_i A_z) \Rightarrow (A_x P_i A_z)$, and the second is transitive: $[(A_x \mathcal{J}_i A_y) \& (A_y \mathcal{J}_i A_z)] \Rightarrow (A_x \mathcal{J}_i A_z)$ and symmetric: $(A_x \mathcal{J}_i A_y) \Leftrightarrow$ $\Leftrightarrow (A_y \mathcal{J}_i A_x)$ Sometimes the exclusive disjunction of the two relations is used and will be noted by: $(A_x \mathcal{S} A_y) = (A_x \mathcal{P} A_y) \veebar (A_x \mathcal{J} A_y)$.

For the m alternatives belonging to set $\mathbf{A}$, we can construct non-contradictory orderings and contradictory orderings.

An ordering of m alternatives is non-contradictory if, and only if, it does not include a group of r alternatives $(A_{i_1}, A_{i_2} \ldots A_{i_r})$, $r = 3, 4, \ldots, m$, for which relation $(A_{i_1} \mathcal{S} A_{i_2}) \& (A_{i_2} \mathcal{S} A_{i_3}) \& \ldots \& (A_{i_{r-1}} \mathcal{S} A_{i_r}) \& (A_{i_r} \mathcal{S} A_{i_1})$, can be written, where at least one of the binary relations of the type $A_i \mathcal{S} A_j$ has the meaning of $A_i \mathcal{P} A_j$. Let us examine the definition for the case $m = 3$, i.e. when $\mathbf{A} = \{A_1, A_2, A_3\}$. The ordering of the alternatives may be graphically represented by writing them one above the other in the case of the preference relation, and one after the other, joined by a hyphen, in the case of an indifference relation.

Thus, $\begin{matrix} A_x \\ A_y \\ A_z \end{matrix}$ means: $(A_x \mathcal{P} A_y) \& (A_y \mathcal{P} A_z) \& (A_x \mathcal{P} A_z)$, and $\begin{matrix} A_x \\ A_y — A_z \end{matrix}$ means $(A_x \mathcal{P} A_y) \& (A_y \mathcal{J} A_z) \& (A_x \mathcal{P} A_z)$.

In the case of three alternatives, the possible non-contradictory orderings are [6]:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
A_x	A_x	A_y	A_y	A_z	A_z	A_x	A_y	A_z	$A_x — A_y$	$A_y — A_z$
A_y	A_z	A_x	A_z	A_x	A_y	$A_y — A_z$	$A_x — A_z$	$A_x — A_y$	A_z	A_y
A_z	A_y	A_z	A_y	A_y	A_x					

R_{12}	R_{13}
$A_y — A_z$	$A_x — A_y — A_z$
A_x	

The number p of non-contradictory orderings increases rapidly with the number of alternatives.

Trying to order three alternatives in any other way leads to a contradictory ordering. Thus: $(A_x \mathcal{P} A_y) \& (A_y \mathcal{P} A_z) \& (A_z \mathcal{P} A_x)$, as well as $(A_x \mathcal{P} A_y) \& (A_x \mathcal{P} A_z) \& (A_z \mathcal{J} A_x)$, are contradictory orderings.

In general, let $\mathbf{R} = \{R_1, R_2, ..., R_p\}$ be the set of non-contradictory orderings, which may exist for m alternatives. Each of the n individuals belonging to set $\mathbf{H}$ will adopt one of the orderings belonging to set $\mathbf{R}$. If we denote by R_{ij} the ordering R_i adopted by the individual H_j, we can introduce the notion of preference characteristic of the individuals belonging to the group: $\Pi = \{R_{i_1 1}, R_{i_2 2}, ..., R_{i_n n}\}$, where $i_1, i_2, ..., i_n \in \{1, 2, ..., p\}$.

For m alternatives and n individuals there will exist a set of all the possible preference characteristics given by the cartesian product: $\mathbf{R}^n = \mathbf{R} \times \times \mathbf{R} \times ... \times \mathbf{R}$, the number of elements of the set being p^n. With the help of the notions described, the function of group decision may be defined as a rule whereby to each element of set $\mathbf{R}^n$ of the preference characteristics, one can associate an ordering R^*, ($R^* \in \mathbf{R}$), which will represent the group decision.

Denoting this rule by $\mathcal{F}$ we can write: $\Pi \xrightarrow{\mathcal{F}} R^*$, which is read: the rule $\mathcal{F}$ deduces R^* from Π [6].

The fundamental problem of the theory of group decision is the formulation of a satisfactory rule whereby the group decision (the ordering R^*) can be deduced from the set of decisions of the group members (set Π). The basic study of these problems has been made by Arrow. He considers that a decision rule must fulfil, in the case of $m \geqslant 3$ and $n \geqslant 2$, the following conditions:

C_1. It must be defined for all the preference characteristics belonging to set $\mathbf{R}^n$;

C_2. If the decision function requires that for a preference characteristic Π_1, an ordering R^* be deduced for which $A_x \mathcal{P} A_y$, and if a preference characteristic Π_2, which complies with the following conditions, is considered:

a. the relations between alternatives other than A_x and A_y are the same as in Π_1;

b. the relation between A_x and A_y remains for all orderings of the preference characteristic the same as in Π_1, or is modified in favour of A_x, then it is necessary that the decision function should deduce from Π_2 an ordering R_2^* for which relation $A_x \mathcal{P} A_y$ remains valid;

C_3. If the decision function requires that, for a preference characteristic Π_1, an ordering R_1^* be deduced for which there exist certain relations between the alternatives A_x belonging to a subset $\mathbf{A}^\bullet \subset \mathbf{A}$, then it is necessary that for a preference characteristic Π_2 differing from the first by modifications concerning the relations between the alternatives $A_y \in \mathbf{C}\mathbf{A}^\bullet$, an ordering R_2^* be obtained for which the relations between alternatives $A_x \in \mathbf{A}^\bullet$. remain the same as in R_1^*.

C_4. For every pair of alternatives (A_x, A_y) there exists at least one preference characteristic Π for which the group prefers A_x to A_y;

C_5. There does not exist a privileged member for whom, if he prefers A_x to A_y ($x, y \in \{1, 2, ..., m\}, x \neq y$), the decision function will deduce an ordering R^* wherein $A_x \mathcal{P} A_y$, whatever the preferences of the other members of the group may be.

The main result of Arrow's investigations consists in his proving the inconsistency of conditions C_1 — C_5 or, in other words, that a decision function F complying with conditions C_1—C_5 does not exist, and where from a non-con-

tradictory ordering R^* could be deduced for any preference characteristic when $m \geqslant 3$ and $n \geqslant 2$. To illustrate this we shall consider the simple majority decision rule which consists in the adoption by the group of the preference or indifference relation between the alternatives, only if the majority of the members of the group express that relation. (The simple majority decision complies with conditions C_1—C_5).

Let us consider for instance, the preference characteristic

R_1	R_2	R_5
x	y	z
y	z	x
z	x	y

and let us try to apply the simple majority rule.

We note that for two out of the three members of the group (hence for the majority), there exist relation $x \mathcal{P} y$ and also relations $y \mathcal{P} z$ and $z \mathcal{P} x$. The ordering that we wish to construct with these relations is obviously contradictory since, by transitivity $[(x \mathcal{P} y) \,\&\, (y \mathcal{P} z)] \Rightarrow (x \mathcal{P} z)$, which contradicts the third relation.

The foregoing conclusion (known also as Arrow's paradox) has been much discussed and analyzed and a number of suggestions have been made to find a way out of the dilemma. Among these suggestions we mention:

— forsaking Arrow's condition C_1 and excluding a number of possible orderings, so that for the remaining orderings a decision function could be defined leading to a non-contradictory ordering R^* [5]; [6];

— forsaking Arrow's condition C_3 and taking into consideration the "intensity" of the preference.

All the foregoing theoretical considerations are verified when a group decision is sought, represented by a complete ordering of alternatives; it has been also assumed that the group decision occurs under certainty conditions.

OUTLINE OF A MATRICEAL TREATMENT OF THE PROBLEM

The complete ordering of m alternatives can be represented by a square matrix of order m. As it will be shown below, this mode of representation has many advantages for the definition and application of group decision rules.

We shall consider therefore that an ordering of m alternatives is represented by a matrix of order m:

$$\left\| \begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mm} \end{array} \right\|$$

where $a_{ij} = 1$ if $A_i \mathcal{P} A_j$, $a_{ij} = 0$ if $A_i \mathcal{J} A_j$ and $a_{ij} = -1$ if $A_j \mathcal{P} A_i$, (obviously we will have $a_{ij} = 0$ for $i = j$).

As an example, we shall write the ordering matrices for three alternatives:

$$M_1 = \begin{Vmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{Vmatrix} \quad M_2 = \begin{Vmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{Vmatrix} \quad M_3 = \begin{Vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{Vmatrix} \quad M_4 = \begin{Vmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{Vmatrix} \quad M_5 = \begin{Vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{Vmatrix}$$

$$M_6 = \begin{Vmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{Vmatrix} \quad M_7 = \begin{Vmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{Vmatrix} \quad M_8 = \begin{Vmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix} \quad M_9 = \begin{Vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{Vmatrix} \quad M_{10} = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{Vmatrix}$$

$$M_{11} = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{Vmatrix} \quad M_{12} = \begin{Vmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{Vmatrix} \quad M_{13} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

The above matrices $M_1 - M_{13}$ correspond to the non-contradictory orderings $R_1 - R_{13}$ presented in the first part of the paper.

Contradictory orderings too, can be represented matriceally. Thus, the two contradictory orderings presented as examples in the first part of the paper, can be represented matriceally as follows:

$$(A_x \mathcal{P} A_x) \,\&\, (A_y \mathcal{P} A_z) \,\&\, (A_z \mathcal{P} A_x): \quad \begin{Vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{Vmatrix}$$

and

$$(A_x \mathcal{P} A_y) \,\&\, (A_y \mathcal{P} A_z) \,\&\, (A_x \mathcal{I} A_z): \quad \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}$$

A first application of the matriceal representation of orderings refers to the simple majority decision rule.

When the number of group members is great, the determination of the preference or indifference relations adopted by the majority of the members becomes difficult. The matriceal method suggested partly removes this drawback.

Let there be m alternatives and n individuals.

We consider a preference characteristic expressed matriceally:

$$\Pi = \{M_{i_1 1},\ M_{i_2 2},\ ...,\ M_{i_n n}\},\ \text{with}\ i_1,\ i_2,\ ...,\ i_n \in \{1,\ 2,\ ...,\ p\}.$$

Each element of the ordinary matrix is provided with a third index, k, $(k = 1, 2, ..., n)$, indicating the order number of the individual:

$$\left\| \begin{matrix} a_{11}^k \cdots & a_{ij}^k \cdots & a_{1m}^k \\ \vdots & \vdots & \vdots \\ a_{i1}^k \cdots & a_{ij}^k \cdots & a_{im}^k \\ \vdots & \vdots & \vdots \\ a_{m1}^k \cdots & a_{my}^k \cdots & a_{mm}^k \end{matrix} \right\|$$

the ordering R^* representing the group decision will be represented, when the simple majority rule is applied, by the matrix

$$M^* = \left\| \begin{matrix} a_{11}^* \cdots & a_{1j}^* \cdots & a_1^{*m} \\ \vdots & \vdots & \vdots \\ a_{i1}^* \cdots & a_{ij}^* \cdots & a_i^{*m} \\ \vdots & \vdots & \vdots \\ a_{m1}^* \cdots & a_{mj}^* \cdots & a_{mm}^* \end{matrix} \right\|$$

where $a_{ij}^* = 1$ if $\sum_{k=1}^{n} a_{ij}^k \geqslant 1$, $a_{ij}^* = 0$ if $\sum_{k=1}^{n} a_{ij}^* = 0$ and $a_{ij}^* = -1$ if $\sum_{k=1}^{n} a_{ij}^k \leqslant$ $\leqslant -1$.

It is clear that this way of applying the simple majority does not exclude the possible occurrence of contradictory orderings as group decisions. For this reason special stress is laid on the analysis of contradictory and non-contradictory orderings.

The theorem stated below will help to solve the problem:

An ordering of m alternatives is non-contradictory if, and only if, it does not include a group of three alternatives forming a contradictory ordering.

Proof

Let us consider a group of r alternatives, A_{i_1}, A_{i_2}, ... A_{i_r} forming a contradictory ordering, hence an ordering for which we can write: $(A_{i_1}\ \mathcal{S}\ A_{i_2})$ & & $(A_{i_2}\ \mathcal{S}\ A_{i_3})$ & ... & $(A_{i_{r-1}}\ \mathcal{S}\ A_{i_r})$ & $(A_{i_r}\ \mathcal{S}\ A_{i_1})$ where at least one of the binary relations has the meaning $A_i\ \mathcal{P}\ A_j$. Let us assume that the first

$s - 1$ relations are of the type $A_i \mathcal{J} A_j$ and relation s is of the type $A_i \mathcal{P} A_j$. Then we can write:

$$(A_{i_1} \mathcal{J} A_{i_2}) \,\&\, (A_{i_2} \mathcal{J} A_{i_3}) \,\&\, \ldots \,\&\, (A_{i_{s-1}} \mathcal{J} A_{i_s}) \,\&\, (A_{i_s} \mathcal{P} A_{i_{s+1}}).$$

With the first s alternatives we can form C_s^3 groups of three alternatives representing non-contradictory orderings of the type:

$$A_{i_x} \mathcal{J} A_{i_y} \mathcal{J} A_{i_z}, \quad (x \neq y \neq z;\ x, y, z \in \{1, 2, \ldots, s\}).$$

We assume that the following group of three alternatives (A_{i_s}, $A_{i_{s+1}}$, $A_{i_{s+2}}$) forms also a non-contradictory ordering. Since by transitivity $A_{i_1} \mathcal{J} A_{i_s}$, and it has been assumed that $A_{i_2} \mathcal{P} A_{i_{s+1}}$, it is necessary, in order that the group may form a non-contradictory ordering, that we have $A_{i_1} \mathcal{P} A_{i_{s+1}}$. If we consider now the group of alternatives (A_{i_1}, $A_{i_{s+1}}$, $A_{i_{s+2}}$) and require that they, too, should form a non-contradictory ordering, then it will be necessary to have $A_{i_2} \mathcal{P} A_{i_{s+2}}$.

By continuing the argument step by step and requiring that all the groups of three alternatives should form non-contradictory orderings we obtain relation $A_{i_1} \mathcal{P} A_{i_r}$ which contradicts the hypothesis and determines a non-contradictory ordering of the r alternatives [1]).

This theorem can be transposed in matriceal terms, as follows: *an ordering of* n *alternatives is non-contradictory if, and only if, all square submatrices of order three, which can be formed from the matrix representing the ordering, belong to the group of the 13 non-contradictory* ($M_1 - M_{13}$).

The characteristics of the contradictory and non-contradictory orderings established by the foregoing theorem may be applied when trying to obviate the difficulties due to "Arrow's paradox".

Below, we present a method for removing these difficulties which differs from the usual methods outlined in the first part of the paper inasmuch as it is not necessary to forgo Arrow's conditions C_1 and C_2.

The starting point is the remark that while the expression by the members of the group of a complete ordering of all the alternatives is particularly useful to substantiate the decision, there are many cases when the group has to choose one alternative only and hence no complete ordering is required.

This is equivalent to determining a "leader" of the alternatives. The operation is possible if in the matrix M^*, on denoting by $S_i^* = \sum_{j=1}^{m} a_{ij}^*$ there is a certain S_i^* greater than all the others. When such an S_i^* does not exist, then we have indecisiveness which may occur in the following three cases:

1. All the elements of matrix M^* are zero;
2. In matrix M^* all S_i^* are equal;
3. The matrix M^* includes two or more S_i^* equal to one another and greater than all the others.

---

[1]) A suggestive proof of the theorem may be given by using a graph associated to the ordering matrix of the alternatives.

In order to eliminate indecisiveness it is always necessary to designate as referee one member of the group. The designation shall be effected before indecisiveness occurs and the referee shall make use of the due "prerogatives" only in case of need. His intervention will consist in adding the initially selected ordering to the preference characteristic Π and then applying the simple majority rule. If the indecisiveness persists the referee shall repeat once more the selected ordering or an ordering other than that initially selected, so that the new selection should admit the determination of a new" leader" of alternatives and thus induce a group decision (Arrow's condition C_5 being strictly adhered to).

In conclusion, the referee's intervention represents the additional weight which turns the scales.

The matriceal treatment suggests the possibility of a systematic analysis and decision rules other than the simple majority rule; it also shows that the problem of group decision under risk or uncertainty conditions could be tackled by the same method.

CONCLUSIONS

Group decisions occur very frequently and are important in business organization and management where most decisions are taken by councils of specialists. This justifies fully the theoretical analysis of the phenomenon and the attempt of a mathematical modelling, even if only partial for the time being.

At present, the decision process in councils of specialists proceeds sometimes according to spontaneously generated rules which are often modified from one meeting of the council to the next. Such a procedure is deeply harmful to the group activity and sometimes entails the occurrence of such phenomena as diletantism, obstinacy and lack of responsibility.

A proper working of the decision process implies in the first place a clear statement of the decision rule which is valid for the group. Obviously, the range of possible decision rules is very large and depends on the nature of the group and of the decision. Thus, in certain cases the weight of the judgement of a member of the group may depend on qualification, experience, or other criteria.

The determination of such weights is in itself a decision problem. But the most important requirement is a clear statement of the decision rule and in this respect theoretical analysis and mathematical modelling may bring a valuable contribution such as, for instance, the absence of indecisiveness.

There are also other aspects to be considered for the correct working of group decisions such as organizational circumstances (time allotted to meetings, a clear formulation of the problems under discussion, a definite delimitation of arguments and deliberation, etc.) and psychological factors (an environment of earnestness, but devoid of constraint, co-operation and mutual respect etc.). A many-sided, scientific approach to the problem of group decision will undoubtedly bring substantial improvements to organization and management methods.

REFERENCES

1. ARROW, K.J., *Mathematical Methods in the Social Sciences*, "Proceedings of the First Stanford Symposium", Stanford University Press, 1959.
2. BOLDUR, G., *Posibilităţi de aplicare a teoriei deciziei în construcţii*, "Studii de Statistică", — Lucrările privind aplicarea metodelor matematice în economie, Iaşi, 1964, DCS, 1966.
3. CHERNOFF, H. and MOSES L., *Elementary Decision Theory*, Wiley, 1959.
4. FISHBURN, P.C., *Methods of Estimating Additive Utilities*, "Management Science" — Series A, March 1967.
5. KEN-ICHI IANADA, *A Note on the Simple Majority Decision Rule*, "Econometrica", vol. 32, No. 4, October, 1964.
6. LUCE, R.D. and RAIFFA, H., *Games and Decisions*, Wiley, 1958.
7. MATALON, B., *L'analyse hiérarchique*, Metra, vol. I, No. 3, 1962.
8. MILLER, D.W. and STARR, M.K., *Fonction de direction et recherche operationnelle*, Dunod, 1964.
9. NEUMANN, J. von, and MORGENSTERN, O., *Theory of Games and Economic Behavior*, Princeton University Press, 1947.
10. ROŞU, ALEXANDRU, *Teoria jocurilor strategice*, Ed. Militară, 1967.
11. YASUSUKE MURAKANI, *Formal Structure of Majority Decision*, "Econometrica", vol. 34, No. 3, July 1966.

SUR LES DECISIONS EN GROUPE

La première partie de l'ouvrage présente les notions de base de la théorie des décisions en groupe, ainsi que les idées d'Arrow au sujet de l'indécidibilité.

La seconde partie esquisse un traitement matriciel du problème et analyse à l'aide de cet instrument certains aspects de l'indécidibilité, en établissant un théorème concernant les conditions que doivent remplir les ordinations non-contradictoires. L'auteur propose ensuite un moyen d'éviter l'indécidibilité.

О ГРУППОВЫХ РЕШЕНИЯХ

В первой части статьи представляются основные понятия теории групповых решений и идей Аррова относительно «нерешительности».

Во второй части намечается матричное объяснение вопроса и при помощи этого инструмента анализируются некоторые стороны нерешительности и устанавливается теорема условий, которые должны выполняться непротиворечивыми приказами. Затем предлагается путь избежания нерешительности.