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An Algorithm for Determining the Chromatic Number of a Finite Graph

The present paper provides a generalization of the procedure given by D. Welsh and M. Powell for obtaining an upper bound of the chromatic number of a finite graph, as well as an algorithm allowing accurate measurements of the chromatic number and of the partition realizing it, by building a rooted tree which takes into account the maximal independent sets of the graph. The algorithm, as well as the application of the concept of pushdown stack, are described.

We shall consider in this paper a finite graph $G = (\mathbf{X}, \Gamma)$ undirected and loop-free, with n vertices, $\mathbf{X} = \{x_1, x_2, ..., x_n\}$.

A subset $\mathbf{E}$ of the set of vertices $\mathbf{X}$, is said to be independent if $\mathbf{E} \cap \Gamma(\mathbf{E}) = \emptyset$, i.e. if any two vertices of $\mathbf{E}$ are not connected by an arc.

An independent set is said to be maximal if there is no independent set having a greater number of elements to contain it. The chromatic number of graph G represents the minimum number of colours which may be assigned to the vertices of the graph, in such a way that two vertices connected by an arc have always different colours.

It follows that the sets of vertices of graph G coloured with the same colour are independent sets.

The chromatic number determination of graph G, denoted by $K(G)$, is thus reduced to finding a partition of set $\mathbf{X}$ containing the vertices of the graph and constituted by independent sets, including a minimum number of classes.

Among the practical applications provided by the determination of the chromatic number of a graph, there are, for instance, certain problems of production scheduling.

As an illustration, we shall consider a production process constituted of n activities: $A_1, A_2, ..., A_n$, to which we associate an undirected graph,

having n vertices $x_1, x_2, ..., x_n$, with an arc existing between nodes x_i and x_j, if and only if activities A_i and A_j cannot be carried out simultaneously.

The problem of determining a partition of these graph vertices, with a minimum number of classes, being independent sets of the graph, is precisely the problem of determining the minimum number of stages necessary for the production process (which may for instance correspond to a minimum manufacturing time), all activities included in the same stage being simultaneously carried out.

Another example, illustrating the use of ascertaining the chromatic number, is as follows:

Let $\mathbf{X} = \{x_1, x_2, ..., x_n\}$ be a set of coded pieces of information. As a result of some errors made in the transmission, transcription, or other manipulating procedure, a certain amount of confusion may be aroused among the elements of set $\mathbf{X}$.

We shall denote by $\Gamma(x_i)$ the set of codes which may be mistaken for x_i and by $G = (\mathbf{X}, \Gamma)$ the graph characterized in this way.

Consequently, an independent set of graph G will correspond to a subset of pieces of information which cannot be mistaken for one another.

In order to reduce the possibilities of making errors on decoding, a complementary part, called a discriminant or checking part, is associated to each code of the set $\mathbf{X}$. In view of obtaining a correct decoding, the states assumed by the discriminant associated to each code of $\mathbf{X}$ may be equal for all codes belonging to one and the same independent set of G, but they have to be distinct in the reverse case.

It follows that the minimum number of states the discriminant is able to assume (corresponding, for instance, to a code having a minimum number of binary or decimal positions), is equal to the chromatic number of graph G.

Going back to the general case, on set $\mathbf{X}$ of vertices belonging to graph G, a reflexive and symmetric relation can be defined, denoted by $\mathbf{R}_G$ and conceived as a subset of the cartesian product $\mathbf{X} \times \mathbf{X}$, so that, by definition, $(x_i, x_j) \in \mathbf{R}_G$ or x_i and x_j are part of relation $\mathbf{R}_G$ if the vertices x_i and x_j are not connected by an arc, that is if $x_i \notin \Gamma x_j$ and hence $x_j \notin \Gamma x_i$.

The problem of determining the chromatic number of graph G is reduced, in terms of the relation theory, to the problem of determining an equivalence relationship $\mathbf{R}_G^* \subset \mathbf{R}_G$, on condition that $\mathbf{R}_G^*$ has a minimum number of classes.

If we let $\mathbf{A} \setminus \mathbf{B}$ be the difference of sets $\mathbf{A}$ and $\mathbf{B}$, a minimal independent set containing element x_k, denoted $\mathbf{S}x_k$, can be defined by the following two properties:

— for any $x_i, x_j \in \mathbf{S}x_k$, $(x_i, x_j) \in \mathbf{R}_G$;

— for any $x_e \in \mathbf{X} \setminus \mathbf{S}x_k$, there exists $x_m \in \mathbf{S}x_k$, so that $(x_e, x_m) \notin \mathbf{R}_G$.

Moreover, it is to be noticed that a maximal independent set (or a maximal class with respect to relation R_G) corresponds to a complete maximal subgraph (maximal clique) of the complementary graph $G = (\mathbf{X}, \Gamma)$, defined by $x_i \in \bar{\Gamma} x_j$, if and only if $x_i \notin \Gamma x_j$ for any $x_i, x_j \in \mathbf{X}$.

In order to determine all maximal independent sets belonging to graph $G = (\mathbf{X}, \Gamma)$, the algorithm of Bednarek and Taulbee [1] may be used. It consists of the following:

For any $1 \leqslant k \leqslant n$, we let $\mathbf{X}_k = \{x_1, x_2, ..., x_k\}$ and $\mathcal{L}_k$ be the family of maximal independent sets for the subgraph of graph G, whose set of vertices is the set $\mathbf{X}_k$. Let $\mathbf{R}_G x_k$ be the set of all vertices of graph G, connected to x_k by an arc, i.e. $R_G\, x_k = \{y \mid y \in \mathbf{X}, (x_k, y) \notin \mathbf{R}_G\}$ and $\mathbf{Y}_k = \mathbf{R}_G\, x_k \cap \mathbf{X}_k$.

Step 1. $\mathbf{Y}_1$, $\mathcal{L}_1 = \{x_1\}$.

Step 2. The family $\mathcal{J}_k = \{\mathbf{S} \mid \mathbf{S} = \mathbf{M} \cap \mathbf{Y}_{k+1}, \mathbf{M} \in \mathcal{L}_k\}$ is determined.

Step 3. $\mathcal{J}'_k$ = the set of elements of $\mathcal{J}_k$, maximal with respect to the set inclusion (no longer contained by other sets of $\mathcal{J}_k$), is determined.

Step 4. For each $\mathbf{M} \in \mathcal{L}_k$:

$$\text{if } \mathbf{Y}_{k+1} \supset \mathbf{M}, \text{ then } \mathbf{M} \cup x_{k+1} \in L_{k+1},$$

$$\text{if } \mathbf{Y}_{k+1} \not\supset \mathbf{M}, \text{ then } \mathbf{M} \in \mathcal{L}_{k+1} \text{ and } x_{k+1} \cup (\mathbf{M} \cap \mathbf{Y}_{k+1}) \in \mathcal{L}_{k+1}$$

if and only if $\mathbf{M} \cap \mathbf{Y}_{k+1} \in \mathcal{J}'_k$.

Step 5. Steps 2, 3, 4 are repeatedly performed for $k = 2, 3, ..., n-1$.

After completing the application of the algorithm, the family $\mathcal{L}_n$ is obtained, that is the class of maximal independent sets of graph G.

If we let $d(x_k)$ be the degree of vertex $x_k \in \mathbf{X}$, i.e. the number of incident arcs with that vertex, we shall assume that the graph G vertices have been assigned numbers in such a way that the relationship: $d(x_1) \geqslant d(x_2) \geqslant ... \geqslant d(x_n)$ should exist.

Let a partition $\Delta = \{\mathbf{X}_k\}_{1 \leqslant k \leqslant q}$ of $\mathbf{X}$, $(q \leqslant n)$, contain independent sets of vertices, namely, for any x_i, $x_j \in \mathbf{X}_k$ we have $(x_i, x_j) \in \mathbf{R}_G$.

We denote $\alpha(G) = \max\limits_{k=1, ..., n} (min(k, 1 + d(x_k)))$ and

$$\beta_\Delta(G) = \max_{k=1, ..., q} (min(k, 1 + \max_{x_p \in X_k} (d(x_p)))) = \max_{k=2, ..., q} (min(\mathrm{k}, 1 + \max_{x_p \in X_k} (d(x_p))))$$

$$\text{if } \sum_{p=1}^{n} \mathrm{d}(x_p) \neq 0.$$

R. Brooks [3] determines the inequality $K(G) \leqslant 1 + \max\limits_{p=1, ..., n} (d(x_p))$ while D. Welsh and M. Powell have shown [12] that $K(G) \leqslant \alpha(G) \leqslant 1 + \max\limits_{p=1, ..., n} (d(x_p))$

Theorem 1. Relation $K(G) \leqslant \beta_\Delta(G)$ exists for any partition Δ induced by an equivalence relation $\mathbf{R}^*_G \subset \mathbf{R}_G$ and condition $\max\limits_{x_p \in X_1}(d(x_p)) \geqslant \max\limits_{x_p \in X_2} (d(x_p)) \geqslant ... \geqslant \max\limits_{x_p \in X_q} (d(x_p))$ implies the inequality:

$$K(G) \leqslant \beta_\Delta(G) \leqslant \alpha(G) \leqslant 1 + \max_{p=1, ..., n} (d(x_p)).$$

By considering $\mathbf{X}_1 = \{x_1\}$, $\mathbf{X}_2 = \{x_2\}$, ..., $\mathbf{X}_n = \{x_n\}$, then $\beta_\Delta(G) = \alpha(G)$ and $\beta_\Delta(G) = K(G)$ if partition Δ, for instance, has a minimum number of classes, since in this case, the reverse inequality, $\beta_\Delta(G) \leqslant \max\limits_{k=1, ..., q} (\mathrm{k}) = q = K(G)$ also exists, hence $K(G) = \min\limits_{\Delta} (\beta_\Delta(G))$.

Proof. If set $\mathbf{X}_1$ is a maximal class with respect to relation $\mathbf{R}_G$, then we put $\mathbf{X}_1' = \mathbf{X}_1$ and check if $\mathbf{X}_2$ is amaximalclass with respect to the restriction of relation $\mathbf{R}_G$ for set $\mathbf{X} \setminus \mathbf{X}_1$, designated by $\mathbf{R}_G \cap (\mathbf{X} \setminus \mathbf{X}_1)^2$.

If $\mathbf{X}_1$ is not a maximal class with respect to $\mathbf{R}_G$, then $\mathbf{X}_1$ is completed up to a maximal class $\mathbf{X}_1'$ by deleting certain elements from classes $\mathbf{X}_2$, ..., $\mathbf{X}_q$, which become $\mathbf{X}_{2,1}$, ..., $\mathbf{X}_{q,1} (\mathbf{X}_{2,1} \subset X_2; \ldots; \mathbf{X}_{q,1} \subset \mathbf{X}_q)$.

We denote by $\mathbf{X}_{2,2}$, $\mathbf{X}_{3,2}$, ..., $\mathbf{X}_{q_1,2}$ the nonempty sets of the sequence $\mathbf{X}_{2,1}$, ..., $\mathbf{X}_{q,1}$ so that $\mathbf{X}_{2,2} = \mathbf{X}_{r_1,1}$; $\mathbf{X}_{q_1,2} = \mathbf{X}_{r_{q_1-1},1}$ and $2 \leqslant r_1 < r_2 < \ldots < r_{q_1-1} \leqslant q$. If $\mathbf{X}_{2,2}$ is a maximal class with respect to relation $\mathbf{R}_G \cap (\mathbf{X} \setminus X_1')^2$, we let $\mathbf{X}_{2,2} = \mathbf{X}_2'$ and check if $\mathbf{X}_{3,2}$ is a maximal class with respect to relation $\mathbf{R}_G \cap (\mathbf{X} \setminus (X_1' \cup \mathbf{X}_2'))^2$, etc. If $\mathbf{X}_{2,2}$ is not a maximal class with respect to $\mathbf{R}_G \cap (\mathbf{X} \setminus \mathbf{X}_1')^2$, then $\mathbf{X}_{2,2}$ is completed up to a maximal class $\mathbf{X}_2'$ by deleting certain elements out of classes $\mathbf{X}_{3,2}$, ..., $\mathbf{X}_{q_1,2}$. After completing a finite number of steps, a new partition $\Delta' = \{\mathbf{X}_k'\}_{1 \leqslant k \leqslant q'}$, of $\mathbf{X}$ with $q' \leqslant q$ is obtained, having the following properties:

Class $\mathbf{X}_1'$ is maximal with respect to $\mathbf{R}_G$, class $\mathbf{X}_k'$ is maximal with respect to relation $\mathbf{R}_G \cap \left(\mathbf{X} \setminus \bigcup_{i=1}^{k-1} \mathbf{X}_i'\right)^2$ for any $k = 2, 3, \ldots, q'$ and $\mathbf{X}_k \subset \bigcup_{i=1}^{min(k,q')} \mathbf{X}_i'$ for any $k = 1, 2, \ldots, q$.

If $x_m \in \mathbf{X} \setminus \bigcup_{i=1}^{k} \mathbf{X}_i'$, then $d(x_m) \geqslant k$, with the result $x_p \in \bigcup_{i=1}^{\varphi(p)} \mathbf{X}_i'$ with $\varphi(p) = min\ (q', 1 + d(x_p))$ for any $x_p \in \mathbf{X}$. Assuming $x_p \in \mathbf{X}_k$, we obtain $x_p \in \bigcup_{i=1}^{min(k,q')} \mathbf{X}_i'$. But $x_p \in \bigcup_{i=1}^{\psi(p)} \mathbf{X}_i'$ implies $x_p \in \bigcup_{i=1}^{\psi(k)} \mathbf{X}_i'$ with $\psi(k) = min(q', \max\limits_{x_p \in X_k} (\varphi(p))) = min\ (q', \max\limits_{x_p \in X_k} (1 + d(x_p))) = min(q', 1 + \max\limits_{x_p \in X_k} (d(x_p)))$.

Consequently, $x_p \in \bigcup_{i=1}^{\eta(k)} \mathbf{X}_i'$ for any $x_p \in \mathbf{X}_k$, where $\eta(k) = min(min(k, q'), \psi(k)) = min(q', min(k, 1 + \max\limits_{x_p \in X_k} (d(x_p))))$.

We have proved that $x_p \in \bigcup_{i=1}^{min(q',\ \beta_\Delta(G))} \mathbf{X}_i'$ for any $x_p \in \mathbf{X}$, by admitting the notation $\beta_\Delta(G) = \max\limits_{k=1, \ldots, q} (min(k, 1 + \max\limits_{x_p \in X_k} (d(x_p))))$, because $min(q', \max\limits_{k=1, \ldots, q} (min(q', min(k, 1 + \max\limits_{x_p \in X_k} (d(x_p))))))) = min(q', \beta_\Delta(G))$.

In concluding, $X = \bigcup_{i=1}^{min(q',\ \beta_\Delta(G))} \mathbf{X}_i' = \bigcup_{i=1}^{q'} \mathbf{X}_i'$, with the result that $min\ (q', \beta_\Delta(G)) = q'$, hence $\beta_\Delta(G) \geqslant q' \geqslant K(G)$.

In order to prove the second part of the theorem, we shall consider relation $d(x_1) \geqslant d(x_2) \geqslant \ldots \geqslant d(x_n)$, and as $\max\limits_{x_p \in X_1} (d(x_p)) \geqslant \max\limits_{x_p \in X_2} (d(x_p)) \geqslant \ldots \geqslant \max\limits_{x_q \in X_q} (d(x_p))$, we deduce: $d(x_1) = \max\limits_{x_p \in X_1} (d(x_p))$; $d(x_2) \geqslant \max\limits_{x_p \in X_2} (d(x_p))$; ...; $d(x_q) \geqslant \max\limits_{x_p \in X_q} (d(x_p))$.

In these conditions, $min(k, 1 + \max\limits_{x_p \in X_k} (d(x_p))) \leqslant min\,(k,\ 1 + d(x_k))$ for any $k = 1, 2, ..., q$, hence $\beta_\Delta(G) \leqslant \max\limits_{k=1,\ ...,\ q} (min(k, 1 + d(x_k))) \leqslant \alpha(G)$. Q.e.d.

Note that there are instances when $\beta_\Delta(G) = K(G)$, although Δ is not a partition with a minimum number of classes (equal to $K(G)$).

In view of the accurate determination of the chromatic number $K(G)$, a rooted tree $H(\tilde{\mathbf{X}}, \Lambda)$ may be built, allowing to find at least one partition having a minimum number of classes of the set of vertices $\mathbf{X}$, a partition associate to an equivalence relation $\mathbf{R}_G^* \subset \mathbf{R}_G$.

Consider element $x_i \in \mathbf{X}$, whose maximal classes with respect to $\mathbf{R}_G$ are $\mathbf{S}x_i^{(1)}$, $\mathbf{S}x_i^{(2)}$, ..., $\mathbf{S}x_i^{(r(i))}$.

An arbitrary vertex $y_{i,1} \in \mathbf{X} \setminus \mathbf{S}x_i^{(1)}$ is selected, whose maximal classes with respect to relation $\mathbf{R}_G \cap (\mathbf{X} \setminus \mathbf{S}x_i^{(1)})^2$ are $\mathbf{S}y_{i,1}^{(1)}$, $\mathbf{S}y_{i,1}^{(2)}$, ..., $\mathbf{S}y_{i,1}^{(r_{i,1}(y_{i,1}))}$.

The rooted tree H has the root x_i and the application Λ is recurrently defined as follows:
$\Lambda(x_i) = \{x_{i,k} \mid k = 1, 2, ..., r(i)\}$; $\Lambda(x_{i,1}) = \{x_{i,1,k} \mid k = 1, 2, ..., r_{i,1}(y_{i,1})\}$ etc. Taking into account the path $(x_i, x_{i,n_1}, x_{i,n_1,n_2}, ..., x_{i,n_1}, ..., {}_{n_s})$ from root x_i to vertex $x_{i,n_1}, ..., {}_{n_s}$, if $X = \mathbf{S}_{i,n_1}, ..., {}_{n_s}$ where $\mathbf{S}_{i,n_1}, ..., {}_{n_s} = \mathbf{S}x_i^{(n_1)} \cup \mathbf{S}y_{i,n_1}^{(n_2)} \cup \cup ... \cup \mathbf{S}y_{i,n_1, ..., n_{s-1}}^{(n_s)}$, then $\Lambda(x_{i,n_1}, ..., {}_{n_s}) = \emptyset$ is defined. In this case we have $n_s = 1$.

In the reverse case, we consider an element $y_{i,n_1,...,n_s} \in \mathbf{X} \setminus \mathbf{S}_{i,n_1}, ..., {}_{n_s}$ whose maximal classes with respect to relation $\mathbf{R}_G \cap (\mathbf{X} \setminus \mathbf{S}_{i,n_1}, ..., {}_{n_s})^2$ are $\mathbf{S}y_{i,n_1, ..., n_s}^{(1)}$,..., ..., $\mathbf{S}y_{i,n_1, ..., n_s}^{(r_{i,n_1, ..., n_s}(y_{i,n_1, ..., n_s}))}$ and we define $\Lambda(x_{i,n_1, ...\, n_s}) = \{x_{i,n_1, ..., n\ s,k} \mid k = 1, 2, ..., r_{i,n_1, ..., n_s}(y_{i,n_1, ..., n_s})\}$.

The elaboration of the rooted tree is completed when all final vertices are obtained.

Theorem 2. The chromatic number $K(G)$ of graph G equals the minimal length (minimal number of arcs) of the paths from root x_i to the final vertices of rooted tree H.

Proof. To each path $(x_i, x_{i,n_1}, ..., x_{i,n_1,n_2,...,n_s})$ containing s arcs, starting from x_i to a final vertex $x_{i,n_1, ..., n_s}$ $(n_s = 1)$ we associate a partition of set $\mathbf{X}$, constituted of independent sets $\mathbf{S}x_i^{(n_1)}$, $\mathbf{S}y_{i,n_1}^{(n_2)}$,...,$\mathbf{S}y_{i,n_1,...n_{s-1}}^{(n_s)}$ realizing an equivalence relation included in $\mathbf{R}_G$. We have then to prove that the partition set of $\mathbf{X}$ associated to all paths connecting root x_i to the final vertices of rooted tree H, contain at least one minimal partition, induced on set $\mathbf{X}$ by an equivalence relation $\mathbf{R}_G^* \subset \mathbf{R}_G$.

Let $\{\mathbf{X}_j\}_{1 \leqslant j \leqslant r}$ be a partition of the vertices set having a minimal class number, each $\mathbf{X}_j$ being an independent set; in other words, $\mathbf{X}_j \times \mathbf{X}_j \subset \mathbf{R}_G$ for any $j = 1, 2, ..., r$.

Vertex x_i is included in a class $\mathbf{X}_q$; we are entitled, however, to assume that $x_i \in \mathbf{X}_1$, hence there exists an index $1 \leqslant m_1 \leqslant r(i)$ so that $\mathbf{X}_1 \subset \mathbf{S}x_i^{(m_1)}$.

If $\mathbf{X}_1 \neq \mathbf{S}x_i^{(m_1)}$, then class $\mathbf{X}_1$ is completed up to $\mathbf{S}x_i^{(m_1)}$, by deleting from the remaining classes, the elements of $\mathbf{S}x_i^{(m_1)}$ which do not belong to $\mathbf{X}_1$.

We obtain thus a new partition $\mathbf{S}x_i^{(m_1)}$, $\mathbf{X}_2'$, ..., $\mathbf{X}_r'$ of set $\mathbf{X}$, constituted of independent sets (hence $\mathbf{X}_j' \times \mathbf{X}_j' \subset \mathbf{R}_G$) and $\mathbf{X}_j' \subset \mathbf{X}_j$, $\mathbf{X}_j' \neq \emptyset$ for any $j = 2, ..., r$, as a result of the minimality assumption for partition $\{\mathbf{X}_j\}_{1 \leqslant j \leqslant r}$.

We can now assume that $y_{i, m_1} \in \mathbf{X}_2'$, hence there exists an index $1 \leqslant m_2 \leqslant r_{i, m_1}(y_{i, m_1})$ for which $\mathbf{X}_2' \subset \mathbf{S}y_{i, m_1}^{(m_2)}$, and so forth.

In this way we obtain a partition of X:
$\mathbf{S}x_i^{(m_1)}$, $\mathbf{S}y_{i, m_1}^{(m_2)}$, ..., $\mathbf{S}y_{i, m_1, ..., m_{r-1}}^{(m_r)}$ with $m_r = 1$, having a minimal number of classes (equal to r), associated to path $(x_i, x_{i, m_1}, ..., x_{i, m_1, m_2, ..., m_r})$ of tree H. Q.e.d.

If $\mathbf{S}x_q^{(1)}$, $\mathbf{S}x_q^{(2)}$, ..., $\mathbf{S}x_q^{(r(q))}$ are all maximal classes of $x_q \in \mathbf{X}$ with respect to the symmetric and reflexive relation $\mathbf{R}_G \subset \mathbf{X} \times \mathbf{X}$ (i.e. all maximal independent sets of graph G, containing vertex x_q) and if $\mathbf{X}_1 \subset \mathbf{X}$, then the maximal classes of x_q with respect to relation $\mathbf{R}_1 = \mathbf{R}_G \cap \mathbf{X}_1^2$ (i.e. the maximal independent sets of the subgraph constituted with the set of vertices $\mathbf{X}_1$, containing vertex x_q) are the sets of sequence $\mathbf{X}_1 \cap \mathbf{S}x_q^{(1)}$, $\mathbf{X}_1 \cap \mathbf{S}x_q^{(2)}$, ..., $\mathbf{X}_1 \cap \mathbf{S}x_q^{(r(q))}$, maximal with respect to the inclusion.

This fact allows the building of rooted tree H starting merely with the maximal classes of elements of $\mathbf{X}$, with respect to relation $\mathbf{R}_G$, that is with maximal independent sets of graph G.

We have seen in the above sections that the determination of maximal independent sets may be achieved with the aid of the algorithm suggested by Bednarek and Taulbee.

There are still other possible methods, such as the Boolean computation procedure suggested by K. Maghout [5] and by J. Weissman [11], or the combinatorial methods developed by Paull and Unger and by Ioanin and reported in [4], which have been proposed in order to determine the maximal sets of internal compatible states appearing in the problem of minimizing the incompletely specified Mealy type sequential automata or the Boolean equations theory [8].

In order to put the algorithm given in theorem 2 in a form amenable for computer programming, we shall use the concept of "pushdown stack".

By pushdown stack we mean a batch of information of the same kind, stored one under the other, on consecutive levels, numbered downwards, beginning with the upper level. This batch of information can be changed only by adjoining an element on the top level, which leads to the downward shifting of the remaining levels (hence the already existing level numbers in the stack increase respectively by a unit) or, by removing the element placed on the top level, in which case the shifting of the other levels has an upward trend (namely, the figures designating the various levels, are respectively decreasing by a unit).

The algorithm given in Theorem 2, becomes:

α) Determine all maximal independent sets of graph G, stored at the top of the stack (on the first level). Proceed to the next instruction.

β) Determine the vertex of the graph belonging to a minimal number of sets placed on the first level and mark with an asterisk all independent sets on the first level, containing this vertex.

If there are many vertices exhibiting this feature, select the vertex with the lowest index. Place the sets marked with an asterisk at the top of the list of sets on the first level. Proceed to the next instruction.

γ) Mark the set placed at the top of the list of sets (formerly marked with one asterisk) with two asterisks and proceed to point ε.

δ) Check if on the first level, the set marked with two asterisks is followed by an unmarked set. In the affirmative case, proceed to point θ; otherwise, mark with two asterisks the set formerly marked with only one asterisk, placed on the right hand of the set marked with two asterisks, while the set formerly marked with two asterisks will now be marked with one asterisk only. Proceed to the next instruction.

ε) Remove the elements of the set marked with two asterisks from all other sets (marked and unmarked) placed on the first level, and out of the sets obtained in this way retain the maximal sets with respect to the inclusion. If the maximal set is unique, proceed to point η; if not, place these maximal sets on the first level and proceed to point β.

η) Print the class consisting of the unique maximal set obtained at point ε, as well as all sets marked with two asterisks contained in the stack.

This class is a partition of set **X**, consisting of independent sets. Proceed to point δ.

θ) Remove the first level of the stack. If the stack is empty, stop the computation. In the reverse case, proceed to point δ.

At point β we have marked the sets containing a vertex which appears a minimum number of times, in order to shorten the computation and to obtain as simple a rooted tree as possible (with less branches). Note that on assigning some kind of a mark (one or two asterisks) to a set in the stack, the old mark is instantaneously obliterated.

Theorem 2 ensures that the set of partitions printed during the computations contains at least one partition having a minimal number of independent sets, this being precisely the chromatic number of graph G.

The course of the above computations also shows that the chromatic number of graph G is equal to the minimal number of levels contained in the pushdown stack at the moment of obtaining, at point ε of the algorithm a unique maximal set, though increased by a unit. If graph G has n vertices, in order to store the information contained on the levels of the stack (the stack may contain a maximum number of n levels, with the nth level having a sole independent set, at the most), we may use $n-1$ rectangular matrices A_i, with n columns and a sufficient number of lines (not exceeding the number of maximal independent sets of graph G), so that on each line of a matrix be deposited the vertex indexes which appear in an independent set placed on the respective level, the rest of the matrix being completed with nils.

In view of ascertaining the set marked with two asterisks, a marking vector v can be used, as the constituent of the ith order shows which set on level i (line of matrix A_i) is marked with two asterisks; thus, the procedure described at point δ, which obliterates and adds an asterisk, is reduced to the assignment instruction $(v)_i := (v)_i + 1$.

There is another vector showing every moment how many sets, starting with the top of the list, are respectively marked with one asterisk on each stack level, consequently, how many lines have been marked with an asterisk in each of matrices $A_i (1 \leqslant i \leqslant n-1)$.

If we are interested only in $K(G)$ and not in the partition realizing it as well, there is no more need to print the partitions obtained during the calculations. The computer has to compare instead the number of non-empty levels in the pushdown stack, present at the time of obtaining at point ε of the procedure a unique maximal set, then to select out of these numbers the smallest one, to increase it by a unit and to print it.

If n is large, the difficulty in applying the algorithm lies in the rapid increase of the number of maximal independent sets of graph G.

In order to obviate this difficulty, it is possible to use an alternative based on the modified algorithm of Bednarek and Taulbee, which determines only the maximal independent sets containing a given vertex of graph G, for each node of the rooted tree.

Application. Let the graph with 7 nodes be defined as follows:

$$\Gamma x_1 = \{x_2, x_3, x_5, x_7\}$$
$$\Gamma x_2 = \{x_1, x_3, x_4\}$$
$$\Gamma x_3 = \{x_1, x_2, x_4, x_5\}$$
$$\Gamma x_4 = \{x_2, x_3, x_5, x_6, x_7\}$$
$$\Gamma x_5 = \{x_1, x_3, x_4, x_7\}$$
$$\Gamma x_6 = \{x_4, x_7\}$$
$$\Gamma x_7 = \{x_1, x_4, x_5, x_6\}$$

It follows that the vertices degrees are respectively, $d(x_1) = 4$; $d(x_2) = 3$; $d(x_3) = 4$; $d(x_4) = 5$; $d(x_5) = 4$; $d(x_6) = 2$ and $d(x_7) = 4$.

We shall first obtain a rough chromatic number estimation for this graph by selecting the following independent sets:

$$\mathbf{X}_1 = \{x_1, x_4\}; \quad \mathbf{X}_2 = \{x_5, x_2\}; \quad \mathbf{X}_3 = \{x_7, x_3\}; \quad \mathbf{X}_4 = \{x_6\}$$

and we check if $\max\limits_{x_p \in X_1} (d(x_p)) \geqslant \max\limits_{x_p \in X_2} (d(x_p)) \geqslant \max\limits_{x_p \in X_3} (d(x_p)) \geqslant \max\limits_{x_p \in X_4} (d(x_p))$, since $5 \geqslant 4 \geqslant 4 \geqslant 2$.

According to Theorem 1, $K(G) \leqslant \max\limits_{k=2,3,4} (\min(k, 1 + \max\limits_{x_p \in X_k} (d(x_p)) =$ $= \max (\min (2, 5), \min (3, 5), \min (4, 3)) = \max(2, 3, 3) = 3$, i.e. chromatic number of the above-defined graph G is 3 at most.

In order to accurately compute the chromatic number, as well as to find a partition with a minimum number of classes of the vertex set belonging to this graph, we determine first the maximal independent sets, by using the algorithm of Bednarek and Taulbee, while keeping in mind that the binary relation symmetric and reflexive $\mathbf{R}_G$ is defined by the following pairs of vertices which are not connected by arcs:

(x_1, x_1); (x_1, x_4); (x_1, x_6); (x_2, x_2); (x_2, x_5); (x_2, x_6); (x_2, x_7); (x_3, x_3); (x_3, x_6); (x_3, x_7); (x_4, x_1); (x_4, x_4); (x_5, x_2); (x_5, x_5); (x_5, x_6); (x_6, x_1); (x_6, x_2); $(x_6; x_3)$; (x_6, x_5); (x_6, x_6); (x_7, x_2); (x_7, x_3) (x_7, x_7).

If we use the notations introduced on the occasion of describing the Bednarek and Taulbee algorithm, we obtain successively:

$\mathcal{L}_1 = \{x_1\}$; $\mathcal{J}_1 = \mathcal{J}'_1 = \emptyset$; $\mathcal{L}_2 = \{x_1\}, \{x_2\}$; $\mathcal{J}_2 = \mathcal{J}'_2 = \emptyset$; $\mathcal{L}_3 = \{x_1\}, \{x_2\}, \{x_3\}$; $\mathcal{J}_3 = \mathcal{J}'_3 = \{x_1\}$; $\mathcal{L}_4 = \{x_1, x_4\}, \{x_2\}, \{x_3\}$; $\mathcal{J}_4 = \mathcal{J}'_4 = \{x_2\}$; $\mathcal{L}_5 = \{x_1, x_4\}, \{x_2, x_5\}, \{x_3\}$; $\mathcal{J}_5 = \mathcal{J}'_5 = \{x_1\}, \{x_2, x_5\}, \{x_3\}$; $\mathcal{L}_6 = \{x_1, x_4\}, \{x_1, x_6\}, \{x_2, x_5, x_6\}, \{x_3, x_6\}$; $\mathcal{J}_6 = \mathcal{J}'_6 = \{x_2\}, \{x_3\}$; $\mathcal{L}_7 = \{x_1, x_4\}, \{x_1, x_6\}, \{x_2, x_5, x_6\}, \{x_2, x_7\}, \{x_3, x_6\}, \{x_3, x_7\}$.

It follows that the family of maximal independent sets of graph G is precisely $\mathcal{L}_7$. The rooted tree built on the basis of Theorem 2 assumes the aspect shown in Fig. 1, where near each vertex we have encircled the vertex of graph G, whose maximal independent sets have been computed in graph G, or in a subgraph of graph G.

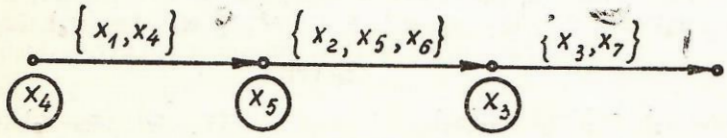

Fig. 1

Since in each step we have selected the vertex belonging to a minimal number of maximal classes, we have obtained a linear tree, a result which is not generally true.

As the unique path from the root to the final node of the rooted tree has three arcs, it follows that $K(G) = 3$ and the resulting partition is: $\{x_1, x_4\}$; $\{x_2, x_5, x_6\}$; $\{x_3, x_7\}$.

It is thus possible to assign to vertices x_1 and x_4 the colour α, to vertices x_2, x_5 and x_6 the colour β and to vertices x_3 and x_7 the colour γ and, in this way, two vertices connected by an arc are not coloured with the same colour.

Another rooted tree resulting if we start the computation with different initial vertices is shown in Figure 2, for which we obtain the same partition with a minimal number of classes associated to the shortest path connecting the root with the final rooted tree nodes.

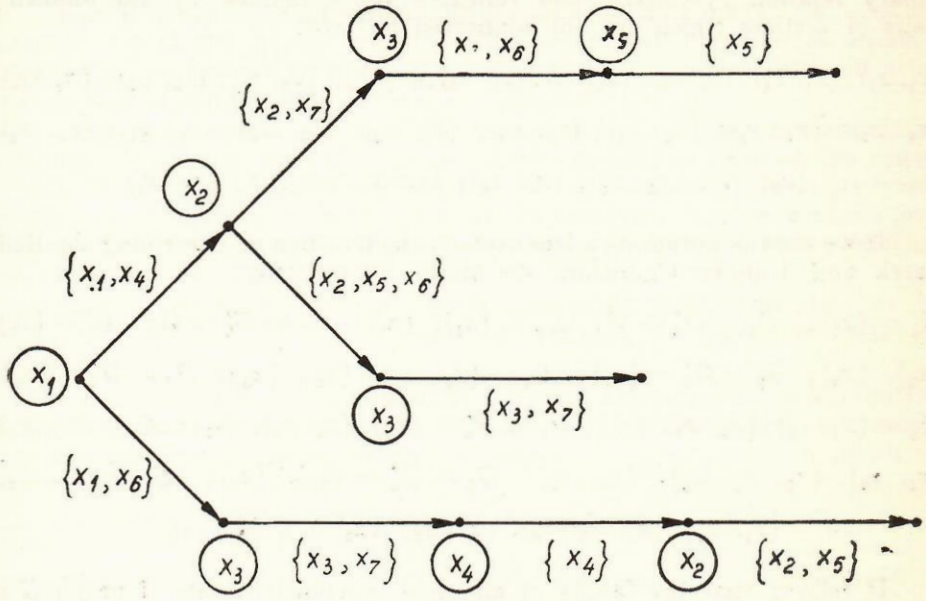

Fig. 2

The application of the algorithm given in Theorem 2 and described by resorting to the concept of pushdown stack, successively leads, during the computation, to the stack configurations shown below:

$(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6),$

Level 1

$(x_2, x_7), (x_3, x_6), (x_3, x_7)$

a)

$*(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6)$

Level 1

$(x_2, x_7), (x_3, x_6), (x_3, x_7)$

b)

$**(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6)$

Level 1

$(x_2, x_7), (x_3, x_6), (x_3, x_7)$

c)

$(x_2, x_5, x_6), (x_2, x_7)$

Level 1

$(x_3, x_6), (x_3, x_7)$

$**(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6),$

Level 2

$(x_2, x_7), (x_3, x_6), (x_3, x_7)$

d)

$*(x_2, x_5, x_6), (x_2, x_7)$

Level 1 $(x_3, x_6), (x_3, x_7)$

$**(x_1, x_4), (x_3, x_6), (x_2, x_5, x_6)$

Level 2 $(x_2, x_7), (x_3, x_6), (x_3, x_7)$

e)

$**(x_2, x_5, x_6), (x_2, x_7)$

Level 1 $(x_3, x_6), (x_3, x_7)$

Level 2 $**(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6), (x_2, x_7), (x_3, x_6), (x_3, x_7)$

h)

To print: $(x_1, x_4), (x_2, x_5, x_6), (x_3, x_7)$

g)

$**(x_1, x_4), (x_1, x_6), (x_2, x_5, x_6)$

Level 1 $(x_2, x_7), (x_3, x_6), (x_3, x_7)$

f)

The stack is empty. Computation is stopped.

i)

Since the maximal class obtained in step *g* is unique, namely (x_3, x_7), this set was printed, together with all sets marked with two asterisks, contained in the stack, and a partition of the vertices set resulted, constituted by independent sets, namely, (x_1, x_4), (x_2, x_5, x_6), (x_3, x_7).

Since a unique partition (with three classes) was printed out, it follows that the chromatic number of the given graph is three, and this partition achieves a colouring of the graph G vertices with a minimal number of colours; if more partitions would have been printed, we would have had to select a partition with a minimum number of classes.

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UN ALGORITHME POUR LA DETERMINATION DU NOMBRE CHROMATIQUE D'UN GRAPHE FINI

Cet ouvrage étudie la détermination exacte du nombre chromatique d'un graphe fini, non-orienté et sans boucles, ainsi que la méthode pour trouver une majoration satisfaisante du nombre chromatique par le calcul des degrés des sommets du graphe, c'est-à-dire du nombre d'arcs incidents en ces sommets.

C'est ainsi qu'en partant des ensembles intérieurement stables maximaux du graphe (qui peuvent être déterminés par exemple à l'aide de l'algorithme de Bednarek et de Taulbee) on donne une méthode pour la construction d'une arborescence qui permet de trouver au moins une partition avec un nombre minimal de classes, qui nous offre donc la possibilité de colorier les sommets du graphe d'un nombre minimal de couleurs.

Un autre but de cet ouvrage est de généraliser la majoration du nombre chromatique d'un graphe fini, obtenue par Welsh et Powell (Computer Journal, May 1967), à partir d'une partition des sommets du graphe formée d'ensembles intérieurement stables.

La majoration ainsi obtenue permet une approximation optimalle du nombre chromatique, dans le sens qu'il existe des partitions de l'ensemble de sommets du graphe, qui, bien qu'ils n'ayent pas nécessairement un nombre minimal de classes intérieurement stables, mènent à une approximation qui coïncide avec la valeur exacte du nombre chromatique du graphe.

Pour faciliter la programmation de l'algorithme sur un ordinateur, l'auteur le présente également à l'aide de la notion de pile susceptible de modéler le processus arborescent décrit antérieurement.

En conclusion l'auteur présente un exemple de calcul pour un graphe à 7 sommets. Il mentionne en même temps les possibilités d'appliquer cette théorie à certains problèmes d'ordonnancement de la production ou à la codification d'un système d'informations utilisant un nombre minimal de symboles de vérification.

АЛГОРИТМ ДЛЯ ОПРЕДЕЛЕНИЯ ХРОМАТИЧЕСКОГО ЧИСЛА ОКОНЧАТЕЛЬНОГО ГРАФА

В этой работе изучается вопрос точного определения хроматического числа окончательного графа, неориентированного и без петель и методы отыскивания достаточно хорошего ограничения хроматического числа путем расчета степеней вершин графа, т.е. числа случайных ребер в этих вершинах.

Так, исходя из максимальных независимых множеств графа (которые могут быть определены, например, при помощи алгоритма Беднарека и Толби или путем

решения системы неравенств Була) представляется (на языке теории бинарных отношений) метод построения разветвления, способствующего нахождению по крайней мере одного разбиения с как минимальным числом классов, которое, следовательно, дает нам возможность окраски вершин графа минимальным числом цветов.

Другой целью работы является обобщение ограничения хроматического числа окончательного графа, полученного Уэльшом и Поуэлом (Computer Journal, May, 1967), исходя из разбиения вершин графа, составленное из независимых множеств.

Таким образом, полученное увеличение позволяет сделать сколько угодно точное приближение хроматического числа в том смысле, что существуют разбиенння множества вершин графа, которые, хотя и не обязательно, имеют минимальное число независимых классов, ведут к приближению, совпадающему с точным значением хроматического числа графа.

Для облегчения программирования алгоритма на счетной машине,он представляется и при помощи понятия "pushdown stack", которое моделирует процесс разветвления, описанный выше.

В заключение дается пример расчета для графа с семью вершинами. Одновременно в работе отмечаются возможности применения этой теории к некоторым задачам руководства производством или к кодированию системы информации, используя минимальное число контрольных символов.