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An Option Problem Associated to a Graph

The quality of a specific task does not depend so much on the proper value of performers, as on the optimal correlation between their abilities and the requirements of each separate activity constituting the task.

We assume a directed, antisymmetrical graph, $G = (\mathbf{X}, \Gamma)$ describing the achievement of a task. Consequently, a time interval $d_i \geqslant 0$ is ascribed to each peak $x_j \in \mathbf{X}$, standing for an activity.

The specific task requires both its performance during the shortest possible time and the achievement of each constituent activity in particular qualitative conditions. Simultaneously, the elaboration of a certain task allows each activity to be performed by one or more members of the staff assigned to the respective task; the eventual performers, or operators, liable of accomplishing the activity, are different for various activities.

In each activity the performers capable of achieving it are interchangeable, by their carrying out the activity in equal times, though obtaining various qualitative performances in its achievement.

Each activity shall, however, be accomplished by a sole operator of the staff, and each operator will be limited by his available time in the achievement of one or more activities, in which he appears as a potential performer, but under no circumstances will he simultaneously work for two or more activities.

In this context, assuming a project to be collectively carried out by a number n of distinct operators j, $(j = 1, 2, ..., n)$, we consider that each performer j is able to accomplish only part of the activities, designated by x_{ij}, $(x_{ij} \in \mathcal{A}_{ji} \subset \mathbf{X})$. In these activities x_{ij}, operator j may obtain qualitative performances differing from one activity to other. It is thus possible to associate to each activity x_i an exclusive disjunction $\mathcal{D}_i$ such as $(A \veebar C \veebar ... \veebar B)$, describing:

— the operators, or performers, A, C, ..., B capable of accomplishing the activity, though only one among them is being selected to achieve it;

— the priorities in selecting the operators, by taking into account the performing variability displayed by them for the same activity; in view of describing this variability, the priority of entering them in the disjunction associated to the said activity is used.

Using the notation t_j for the total available time of performer j, in the interval D (where D is the critical path duration of the graph, computed as a function of d_i), it is requested to calculate:

"the achievement distribution of activities x_i according to operators j, so as the total duration of carrying out the activities should be a minimum and the task be accomplished in optimum qualitative conditions, i.e., the total qualitative performance should be a maximum".

In order to define the concept of "total or task qualitative performance" we can proceed as follows: we let q_i be the number of operators comprised in the disjunction $\mathcal{D}_i$. In each activity x_i, a priority index η_{ij} is assigned to each operator $j \in \mathcal{D}_i$, indicating the position of operator j in the disjunction $\mathcal{D}_i$; it is obvious that; $1 \leqslant \eta_{ij} \leqslant q_i$

(Resuming the reasoning, it may be shown that $\left[\sum_{i=1}^{n} A_n^i\right]$ possible disjunctions could be developed for n operators.)

By letting x_{ij}^0 stand for the activities (or activity) achieved in optimal conditions by operator j, the following definition can be derived:

$$P = \sum_{i=s}^{m} \sum_{j=1}^{n} \eta_{ij}^0$$

for the total (task) qualitative performance.

Consequently, the maximum qualitative performance will be

$$P_{max} = [MIN] \sum_{i} \sum_{j} \eta_{ij}^0 \tag{1}$$

The available times t_j will have to satisfy the condition:

$$t_j > (d_{ij})_{max}, \quad \text{for } j = 1, 2, ..., n \tag{2}$$

Moreover, the following condition can be formulated

$$(d_{ij})_{max} \leqslant t_j \leqslant \sum_{i} d_{ij},$$

allowing the derivation of the existence condition of this problem, namely

$$\sum_{j} (d_{ij})_{max} \leqslant \sum_{j} t_j \leqslant \sum_{i} d_i \tag{3}$$

This condition remains also true when $\sum_{j} t_j > \sum_{i} d_i$ without changing the problem contents.

THE ALGORITHM USED IN SOLVING THE PROBLEM

In order to solve the problem, it is necessary to draw up the priority matrix:

$$M = \{\eta_{ij}\}, \text{ where for } j \notin \mathcal{D}_i \Rightarrow \eta_{ij} = \infty$$

The Gantt graph is next plotted, based on the early terms (*ID*, *TD*) and on the late terms (*IT*, *TT*) of achieving the activities, computed by means of the critical path method.

The following matrix is thus obtained:

$M =$

Operators \ Activities	x_1	x_2	...	x_i	...	x_m
1	η_{11}	η_{21}	...	η_{i1}	...	η_{m1}
2	η_{12}	η_{22}	...	η_{i2}	...	η_{m2}
⋮	⋮	⋮	...	⋮	...	⋮
j	η_{1j}	η_{2j}	...	η_{ij}	...	η_{mj}
⋮	⋮	⋮	...	⋮	...	⋮
n	η_{1n}	η_{2n}	...	η_{in}	...	η_{mn}

It may be seen that, as a rule, the problem admits an assignment algorithm allowing the achievement of the optimal distribution of operators j among the x activities, according to the criterion given by equation (1).

The adequate mathematical model is as follows:

$$\sum_i y_{ij} = 1;$$

$$\sum_j y_{ij} = 1;$$

$$[MIN]\, Z = \sum_i \sum_j \eta_{ij} \times y_{ij}$$

$$y_{ij}^2 = y_{ij} (i = 1, 2, ..., m; \quad j = 1, 2, ..., n).$$

It follows from the solution obtained that

$$\forall_i : |x_{ij}^0| = 1. \tag{4}$$

This last relationship satisfies the solving conditions previously prescribed for the problem, but also leads to:

$$\forall_j : |x_{ij}^0| = 1, \tag{5}$$

that is, a number of only n activities x_i out of the m existing ones may admit an operator selected from disjunction $\mathcal{D}_i$, the remainder of $(m - n)$ activities being unsatisfied, although the time conditions (1), (2), allow a complete solution to the problem.

In the first case, when $n \geqslant m$, the assigning problem on matrix M is solved, namely, operator j, capable of achieving the maximum qualitative performance in the activity x_i, is assigned to the respective activity. Obviously, for this end:

$$\forall_i : |x_{ij}^0| = 1;$$

$$\forall_j : |x_{ij}^0| = 0 \vee 1$$

In the case of $m > n$, in order to eliminate the disadvantage brought about by (5), we suggest in the following section an heurystical procedure of solving the problem, based on the accomplishment of the optimum proposed, by means of some "successive assignments" in a maximum number of $(m-n)$ iterations. This procedure further allows to avoid the assignment of one and the same operator to two or more activities to be simultaneously performed during some limited period of time, or during the full duration of their achievement.

THE SUCCESSIVE ALLOCATION PROCEDURE

Starting from matrix M, the following rules are being successively applied:

1° The matrix is supplemented, by introducing a number of $(m - n)$ new lines. On these $(m - n)$ new lines, the hypothetical performers E_{n+1}, E_{n+2}, ..., E_m, are gradually recorded.

2° For each operator E_{n+k}, $(k = 1, 2, ..., m - n)$, the priority index $\eta_{i, n+k}$ corresponding to column i, $(i = 1, 2, ..., m)$ will be computed as follows:

$$\eta_{i, n+k} = max_i(\eta_{ij}) + k \tag{6}$$

3° The assignment algorithm is applied by using the quadratic matrix drawn up according to rules **1°** and **2°**.

4° The values t_j are again computed as follows:

$$t_j = t_j - d^0_{ij} \tag{7}$$

Thus, a number of n' performers ($n' \leqslant n$) will result, for whom, on recalculating:

$$t_j > 0$$

5° A new matrix $M = \{\eta_{ij}\}$ is drawn up, in whose columns the activities x are entered, which correspond to the values $x^0_{i,\,n+k}$, resulting from the application of the assignment algorithm according to rule 3°. On the lines of this matrix the performers are recorded for whom, after recalculation according to rule 4°, it follows that $t_j > 0$.

6° The priorities of the matrix are changed by taking the infinite value ($\eta_{ij} = \infty$), for each of the following cases:

a) $j \notin \mathcal{D}_i$;

b) $j \in \mathcal{D}_i$ and $t_j < d_{ji}$;

c) $j \in \mathcal{D}_i$; $t_i \geqslant d_{ji}$ and when activity x_i and activity $x^0_{i'j}$, ($i' \neq i$), determined by applying rule 3°, are critical and are partially or totally superimposed in the Gantt graph; furthermore, if one or both activities are uncritical and their superposition in the graph cannot be fully removed by sliding.

7° If after applying rule 6°, it follows that for one (or more) activity, all $\eta_{i'j}$ corresponding to the respective activity (or activities) and to the n' actual performers, amount to ∞, the procedure is as follows:

— the first matrix M is looked for a performer j, intended for that activity x_j for which $\eta_{ij} = \min\limits_{i} (\eta_{ij})$; the operator thus selected cannot perform another activity during the period in which he achieves activity x_i.

— the value t_j corresponding to the operator thus found out is recalculated:

$$t_j = t_j + (d_i - t_j);$$

— operator j is recorded in matrix M arrived at in the respective step, if this operator was not already included in its composition;

— after its recording, if operator j was not previously included in the matrix M composition of the respective step, the η_{ij} indexes corresponding to operator j are established, as well as the rest of existing activities x_i in the M matrix composition of that step; this is achieved by means of disjunctions $\mathcal{D}_i$ and the corrections indicated by rule 6°.

By designating with m and n the number of columns and lines, respectively, of the new matrix, the following iteration is achieved, through the repeated application of the rules 1° to 7°.

The problem may be considered as being fully solved, defining the last iteration, when all activities have been assigned to actual operators only. As a result, the time availabilities of the actual operators are corrected, by adding to the initial time availability of each performer the necessary increases resulting from the application of rule 7° to each iteration.

A COMPUTATION EXAMPLE

Let the following graph $G = (X, \Gamma)$ represent the performance of a specific task.

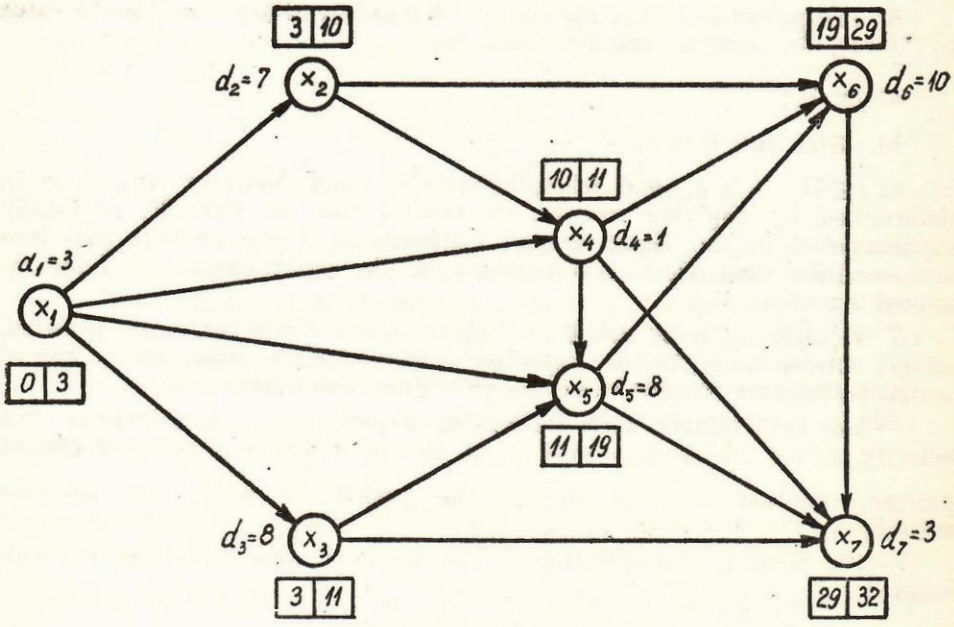

Fig. 1

The critical path is computed only through moments ID and TD. We obtain $D = 32$, that is the total performing time of the task.

We consider that the task will be achieved by a number of $n = 4$ operators, namely A, B, C and D, each of them disposing during the time interval D of the following available times:

$$t_A = 10; \quad t_B = 15; \quad t_C = 10; \quad t_D = 9.$$

By taking into account the qualitative performances which each of the above mentioned operators is able to achieve, the following disjunctions will be associated to each activity:

$$\mathcal{D}_1 = A \vee D \vee B; \qquad \mathcal{D}_5 = B \vee D \vee A \vee C;$$

$$\mathcal{D}_2 = C \vee A; \qquad \mathcal{D}_6 = C \vee B \vee A;$$

$$\mathcal{D}_3 = B \vee C \vee D \vee A; \qquad \mathcal{D}_7 = A \vee D;$$

$$\mathcal{D}_4 = D \vee B \vee A;$$

It may be seen that a number of $\sum_{k=1}^{4} A_4^k = 64$ possible disjunctions can be formed with the four operators.

By checking the condition: $\max_i (d_i)_j \leqslant t_j$ for each operator j, we obtain:

$$\text{for } A\text{: } (d_{jA})_{max} = 10 = t_A = 10;$$

$$\text{for } B\text{: } (d_{iB})_{max} = 10 < t_B = 15;$$

$$\text{for } C\text{: } (d_{iC})_{max} = 8 < t_C = 10;$$

$$\text{for } D\text{: } (d_{iD})_{max} = 8 < t_D = 9.$$

We also check the existence condition of the problem:

$$\sum_j (d_{ij})_{max} = 36 < \sum_j t_j = 44 > \sum_i d_i = 40.$$

The priority matrix $\mathrm{M} = \{\eta_{ij}\}$ is then drawn up, by keeping in mind that for $j \notin \mathcal{D}_i \Rightarrow \eta_{ij} = \infty$

M =

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
A	1	2	4	3	3	3	1
B	3	∞	1	2	1	2	∞
C	∞	1	2	∞	4	1	∞
D	2	∞	3	1	2	∞	2

Rule **1°** is applied and $m - n = 3$ new lines are included, corresponding to the hypothetical operators E_1, E_2 and E_3, for whom the values η_{iE} are computed, according to rule **2°** and using relationship (**6**).

We obtain the matrix:

$M_1 =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
A	1	1	4	3	3	3	1
B	3	∞	1	2	1	2	∞
C	∞	1	2	∞	4	1	∞
D	2	∞	3	1	2	∞	2
E_1	4	3	5	4	5	4	3
E_2	5	4	6	5	6	5	4
E_3	6	5	7	6	7	6	5

Rule **3°** is applied for the first iteration, by using the Hungarian algorithm. The following solutions x_{ij}^0 are obtained:

$$\boxed{x_{1A},\ x_{3B},\ x_{6C},\ x_{4D}},\ x_{55},\ x_{26},\ x_{77}$$

The values t are recalculated according to rule **4°**:

$$t_A = 10 - 3 = 7;$$

$$t_B = 15 - 8 = 7;$$

$$t_C = 10 - 10 = 0;$$

$$t_D = 9 - 1 = 8.$$

A new matrix M is drawn up, for the distribution of activities, x_2, x_5 and x_7 among the actual performers A, B and D.

In this new matrix, we take into account that:

$$j \notin \mathcal{D}_i \Rightarrow \eta_{ij} = \infty.$$

The performance of each of activities x_2, x_5 and x_7 must not be superimposed in time with the performance of x_1.

The same is true for activity x_5 versus activity x_3.
The same is true for activities, x_5 and x versus activity x_4.
We obtain the matrix:

$M =$

	x_2	x_5	x_7
A	2	3	1
B	∞	1	∞
D	∞	2	2

As in this matrix $d_5 > t_j$ for $j = A$ and B, the priority indexes: $\eta_{5A} = \eta_{5B} = \infty$ will be accordingly modified. This leads to the elimination of operator B from the new matrix M:

$M =$

	x_2	x_5	x_7
A	2	∞	1
D	∞	2	2

The new matrix M_1 is built up according to rules **1°** and **2°**:

$M_1 =$

	x_2	x_5	x_7
A	∞	2	1
D	2	∞	2
E_1	3	3	3

Rule **3°** is applied for the second iteration, by using the same algorithm. The following new solutions x_{ij}^0 are obtained:

$$\boxed{x_{7A},\ x_{5D}}\ ,\quad x_{21}$$

The values t_j are then computed as follows:

$$t_A = 7 - 3 = 4$$

$$t_B = 7$$

$$t_D = 8 - 8 = 0.$$

The third iteration is achieved by composing a new matrix M for the distribution of activity x_2 among the operators A and B, by taking into account the condition:

$$j \notin \mathcal{D}_i \Rightarrow \eta_{ij} = \infty$$

and checking up on the Gantt graph that the performance of activity x_2 is not superimposed in time, over the performance of activity x_1, x_3 or x_7.

Matrix:

M =

	x_2
A	2
B	∞

is obtained.

As $d_2 > t_A$, it follows:

M =

	x_2
A	∞
B	∞

The application of rule 7° leads to the new solution x_{ij}^0:

x_{2C}

with the corresponding increase: $\Delta_C = 7$.

The following optimal distribution of activities, according to operators, is derived:

x_{1A}, x_{2C}, x_{3B}, x_{4D}, x_{5D}, x_{6C}, x_{7A}

leading to:

$$P_{Max} = [MIN] \sum_i \sum_j \eta_{ij} = 8.$$

Furthermore, it follows that the available time of operator C will have to increase from 10 to 17, meaning that 6 units will be used from t_A, 8 units from t_B, while t_D will be integrally used.

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UN PROBLEME D'OPTION ASSOCIE A UN GRAPHE

L'auteur présente le modèle mathématique et l'algorithme pour la solution d'un problème qui consiste en la répartition optimum de quelques exécuteurs pour les activités d'un travail représenté par un graphe; cette répartition permet de réaliser un travail de la meilleure qualité.

Le travail en question peut être la rédaction d'un projet, l'étude d'un sujet de recherche, un travail à caractère industriel ou une construction (les exécuteurs peuvent être des hommes ou des machines).

Afin d'obtenir la répartition optimum on utilise les disjonctions exclusives associées à chaque activité, où sont indiquées aussi bien le sous-ensemble d'exécuteurs, qui peuvent le mener à bien, que les différences de performances entre ceux-ci, différences matérialisées par l'ordre dans lequel les exécuteurs sont inscrits dans la disjonction.

ПРОБЛЕМА ВЫБОРА, ПРИОБЩЕННАЯ К ГРАФУ

Демонстрируется математическая модель и алгоритм решения проблемы, состоящей из оптимального распределения исполнителей по определенным видам работ и представляемой в виду графа. Путем такого распределения достигается осуществление работ в максимальных качественных условиях.

Работа может означать составление проекта, разработку исследовательской темы или выполнение работы промышленного или строительного характера (исполнителями могут быть люди или машины).

Для достижения оптимального распределения операции производятся исключающими подразделениями, приобщенными к каждому виду работ, в которых указывается как подмножество исполнителей, могущих выполнить эти работы, так и разницы результатов их работы материализованные порядком записи исполнителей в подразделении. Качество работы не зависит как и от самой стоимости исполнителей, так и от оптимального соотношения между их возможностями и требованиями каждого вида работ.