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Recent Research in the Theory of Boolean Matrices

The problem treated in the present paper involves the development of an algorithm for the computation of matrix $\mathbf{A}^{p-1} = \mathbf{A}^p = \ldots$, *of a single line or of a submatrix of matrix* $\mathbf{A}^{p-1}$, *where* $\mathbf{A}$ *is a Boolean matrix with* 1 *on the main diagonal and the method is extended to the case that the Boolean matrix* $\mathbf{A}$ *has* $a_{\alpha\alpha} \neq 1$, *or matrix* $\mathbf{A}$ *represents the distances or the direct capacities between the nodes of a transport network.*

The theoretical bases of Boolean matrices have been pointed out in the works of Lunts [13], [14], Aranovich [1] and were subsequently revised by Hohn and Schissler [9].

A detailed study of Boolean matrices with a view to synthesize multiterminals with contacts and relays, as well as to reduce the number of contacts in the combinational switching circuits has been accomplished by P. Constantinescu in [5], [6].

Since matricial operations of the same kind are involved in the theory of graphs and in the theory of transport and communication networks as well [3], [8], [10], [11], [17], [21], some algebraic structures more general than Boolean algebras, for which the operations with matrices have the same properties, have been developed by C. Benzaken [2], G. Moisil [16], L. Nolin [17], G. Povarov [18], M. Yoeli [27], etc.

It is a well-known fact that a combinational circuit with contacts and relays having p nodes can be described by a Boolean matrix $A = \{a_{\alpha\beta}\}_{\alpha,\beta=1,\ldots,p}$ where $a_{\alpha\beta}$ are Boolean expressions containing a certain number of variables and $a_{\alpha\alpha} = 1$.

If we denote by $+$ and $\times$ the logical disjunction and conjunction, respectively, and if we use the following equality:

$$\mathbf{A} \times \mathbf{B} = \left\{ \sum_{k=1}^{p} a_{\alpha k} b_{k\beta} \right\}_{\alpha,\beta=1,\ldots,p},$$

for the definition of the product of two Boolean matrices **A** and **B** of order p, then matrix $\mathbf{A}^{p-1} = \mathbf{A}^p = \ldots$ represents the matrix of total conductivities of a circuit with contacts and relays, if **A** is the matrix of direct conductivities between the nodes of this circuit.

While the results on Boolean matrices given by Lunts are now considered as classical and are reported in most studies regarding the algebraic methods of analysis of combinational finite automata, the development of algorithms allowing to obtain matrix $\mathbf{A}^{p-1}$, or only some elements of matrix $\mathbf{A}^{p-1}$, involving a least possible amount of computations (for instance, the solution is more rapid than by the methods which form the sequence $\{A^{2^n}\}_{n \in N}$) and lending themselves to the utilization of a computer, is of outstanding interest.

In the present paper, we describe a method designed for the computation of total conductivities of a circuit with p nodes, requiring a number of computations of order p^3 and efficiently utilizing the storage capacity of a computer, as it assumes the storing of a p-order Boolean matrix only, during the whole set of computations.

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If two Boolean matrices, **A** and **B** of order p are given, the partial order relation $\mathbf{A} \leqslant \mathbf{B}$ is defined if $a_{\alpha\beta} \leqslant b_{\alpha\beta}$ for any $\alpha, \beta = 1, \ldots, p$ and it is verified that $\mathbf{A} \leqslant \mathbf{B}$ and $\mathbf{C} \leqslant \mathbf{D}$ imply $\mathbf{A} \times \mathbf{C} \leqslant \mathbf{B} \times \mathbf{D}$.

If $a_{\alpha\alpha} = 1$, then $\mathbf{A} \leqslant \mathbf{A}^2$ and as a result relation $\mathbf{A} \leqslant \mathbf{A}^2 \leqslant \mathbf{A}^3 \leqslant \ldots$ $\ldots \leqslant \mathbf{A}^{p-1} = \mathbf{A}^p = \ldots$ is obtained, where matrix $\{\mathbf{A}^{p-1}\}_{\alpha\alpha} = 1$ and

$$\{\mathbf{A}^{p-1}\}_{\alpha\beta} = a_{\alpha\beta} + \sum_{r=1}^{p-2} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_r \in \{1, 2, \ldots, p\} \setminus \{\alpha, \beta\} \\ \gamma_1 \neq \gamma_2; \ldots; \gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1} a_{\gamma_1\gamma_2} \ldots a_{\gamma_r\beta}.$$

On the set of Boolean matrices $\mathbf{A} = \{a_{\alpha\beta}\}_{\alpha, \beta = 1, \ldots, p}$ with $a_{\alpha\alpha} = 1$, we shall define the matrix operators $T_{\mu\nu}$ as follows:

$$T_{\mu\nu}(\mathbf{A}) = \mathbf{B} \text{ if } b_{\mu\nu} = \sum_{\rho=1}^{p} a_{\mu\rho} a_{\rho\nu} = a_{\mu\nu} + \sum_{\substack{\rho=1 \\ \rho \neq \mu, \nu}}^{p} a_{\mu\rho} a_{\rho\nu} = \{\mathbf{A}^2\}_{\mu\nu} \text{ and } b_{\alpha\beta} =$$

$= a_{\alpha\beta}$ for any $(\alpha, \beta) \neq (\mu, \nu)$ and operator $T = \prod_{\mu, \nu=1}^{p} T_{\mu\nu} = \prod_{\substack{\mu, \nu=1 \\ \mu \neq \nu}}^{p} T_{\mu\nu}$ depending on the order of constituent operators $T_{\mu\nu}$.

Since $\mathbf{A} \leqslant T_{\mu\nu}(\mathbf{A}) \leqslant \mathbf{A}^2$, it follows that $\mathbf{A}^{p-1} \leqslant (T_{\mu\nu}(\mathbf{A}))^{p-1} \leqslant (\mathbf{A}^2)^{p-1} =$ $= \mathbf{A}^{p-1}$, hence $(T_{\mu\nu}(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$.

By iterating this property, relation $(T(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$ is derived for any operator T. It may be proved inductively that $(T^n(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$.

Indeed, $(T^n(\mathbf{A}))^{p-1} = (T(T^{n-1}(\mathbf{A})))^{p-1} = (T^{n-1}(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$.

A property of operators T is $T^n(\mathbf{A}) \geqslant \mathbf{A}^{2^n}$ for any $n \in N$ (see [22]).

In order to compute matrix $\mathbf{A}^{p-1}$ by using operators T, it is sufficient to find an index q, for which $T^q(\mathbf{A}) = T^{q+1}(\mathbf{A})$.

In this case $T^1(\mathbf{A}) = \mathbf{A}^{p-1}$. In the following paragraphs it will be shown that the order of operators $T_{\mu\nu}$ is essential when considering the amount of necessary computations for matrix $\mathbf{A}^{p-1}$ by using operators T.

The sequence of operators $T_{\mu\nu}$ in the product *defining operator* T, induces a transitive relation in the set of pairs (α_i, β_i) with $\alpha_i, \beta_i \in \{1, 2, ..., p\}$ and $\alpha_i \neq \beta_i$, so that $(\alpha_k, \beta_k) \subset_T (\alpha_l, \beta_l)$ if operator $T_{\alpha_k\beta_k}$ is preceding operator $T_{\alpha_l\beta_l}$, that is if

$$T = T_{\alpha_1\beta_1} \dots T_{\alpha_{k-1}\beta_{k-1}} T_{\alpha_k\beta_k} \dots T_{\alpha_l\beta_l} T_{\alpha_{l+1}\beta_{l+1}} \dots T_{\alpha_{p^2-1}\beta_{p^2-p}}.$$

Proposition 1. Let U be an operator verifiying conditions

$$(\alpha, \beta) \subset_U (\gamma, \beta); \quad (\beta, \alpha) \subset_U (\beta, \gamma)$$

for any $\alpha, \beta, \gamma \in \{1, 2, ..., p\}$, $\alpha \neq \beta \neq \gamma$ and $\gamma < \alpha$. (1)

If $\delta_1, \delta_2, ..., \delta_q \in \{1, 2, ..., p\}$, $\delta_i \neq \delta_j$ and $\delta_i < min(\alpha, \beta)$ for any $1 \leqslant i \leqslant q$, then $\{U(\mathbf{A})\}_{\alpha\beta} \geqslant \{U(\mathbf{A})\}_{\alpha\delta_1} \{U(\mathbf{A})\}_{\delta_1\delta_2} \dots \{U(\mathbf{A})\}_{\delta_q 3}$.

Proof. Considering $\delta_{j_1} = max(\delta_1, \delta_2, ..., \delta_q)$, we infer $(\alpha, \beta) \subset_U (\alpha, \delta_{j_1})$, $(\alpha, \beta) \subset_U (\delta_{j_1}, \beta)$ and hence $\{U(\mathbf{A})\}_{\alpha\beta} \geqslant \{U(\mathbf{A})\}_{\alpha\delta j_1} \{U(\mathbf{A})\}_{\delta j_1\beta}$. If $j_1 > 1$ we consider $max(\delta_1, ..., \delta_{j_1-1})$ and if $j_1 < q$ then $max(\delta_{j_1+1}, ..., \delta_q)$ must be taken into account as well.

By applying this procedure, after a number of steps equal to or less than q, we obtain all pairs of adjacent numbers of the ordered set $\{\alpha, \delta_1, \delta_2, ..., \delta_q, \beta\}$, that is $\{U(\mathbf{A})\}_{\alpha\beta} \geqslant \{U(\mathbf{A})\}_{\alpha\delta j_1} \{U(\mathbf{A})\}_{\delta_{j_1}\beta} \geqslant \dots \geqslant \{U(\mathbf{A})\}_{\alpha\delta_1} \{U(\mathbf{A})\}_{\delta_1\delta_2} \dots \{U(\mathbf{A})\}_{\delta_q\beta}$. q.e.d.

If a rooted tree Γ_0 is defined, with the pair (α, β) as a root and $\Gamma_0\{(\delta_i, \delta_j)\} = \varnothing$ if $j = i + 1$, while in the reverse case $\Gamma_0\{(\delta_i, \delta_j)\} = \{(\delta_i, \delta_k), (\delta_k, \delta_j)\}$ where $\delta_k = max(\delta_{i+1}, ..., \delta_{j-1})$ and $\delta_0 = \alpha$, $\delta_{q+1} = \beta$, then each path from vertex (α, β) to a final vertex of the tree, contains only the pairs constituting a chain with respect to the partial order relation $\subset_U$, the first element of the chain being the pair (α, β) and its last element being the final vertex of this path.

If operator V verifies conditions:

$$[(\gamma, \beta) \subset_V (\alpha, \beta); \quad (\beta, \gamma) \subset_V (\beta, \alpha)]$$

for any $\alpha, \beta, \gamma \in \{1, 2, ..., p\}$, $\alpha \neq \beta \neq \gamma$ and $\gamma < \alpha$, (2)

then it may be similarly shown, by substituting minimum for maximum, that $\{V(\mathbf{A})\}_{\alpha\beta} \geqslant \{V(\mathbf{A})\}_{\alpha\varepsilon_1} \{V(\mathbf{A})\}_{\varepsilon_1\varepsilon_2} \dots \{V(\mathbf{A})\}_{\varepsilon_s\beta}$ for $\varepsilon_1, \varepsilon_2, ..., \varepsilon_s \in \{1, 2, ..., p\}$, $\varepsilon_i \neq \varepsilon_j$ şi $\varepsilon_i > max(\alpha, \beta)$ for any $i, j = 1, ..., s$.

Theorem 1. If operators U and V verify conditions (1) and (2), respectively, then $VU(\mathbf{A}) = UV(\mathbf{A}) = \mathbf{A}^{p-1}$ for any Boolean matrix $\mathbf{A}$ with $a_{\alpha\alpha} = 1$.

This theorem is proved in a number of stages. Consider the mutually different numbers $\gamma_0 = \alpha, \gamma_1, \gamma_2, ..., \gamma_r, \gamma_{r+1} = \beta \in \{1, 2, ..., p\}$, and possessing the following properties:

$min(\alpha, \beta) < min(\gamma_1, \gamma_2, ..., \gamma_r)$ and $max(\gamma_1, \gamma_2, ..., \gamma_r) < max(\alpha, \beta)$.

Only the following situations are possible in this case:

a) $min\,(\alpha, \beta) < \gamma_1 < \gamma_2 < \ldots < \gamma_k < \gamma_{k+1} < \ldots < \gamma_r < max\,(\alpha, \beta)$ or

b) there exist indexes $i, j \in \{1, 2, .., r\}$ for which $\gamma < min\,(\gamma_{i-1}, \gamma_{1+})$, and $\gamma_j > max\,(\gamma_{j-1}, \gamma_{j+1})$.

On the basis of this observation it can be inductively proved (with respect to r) that $\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_1}\{U(A)\}_{\gamma_1\gamma_2} \ldots \{U(A)\}_{\gamma_r\beta}$.

For $r = 1$, $\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_1}\{VU(A)\}_{\gamma_1\beta} \geqslant \{U(A)\}_{\alpha\gamma_1}\{U(A)\}_{\gamma_1\beta}$ since $(\alpha, \beta) \subset_V (\gamma_1, \beta)$.

In case a) $\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_1}\{VU(A)\}_{\gamma_1\beta} \geqslant \{U(A)\}_{\alpha\gamma_1}\{U(A)\}_{\gamma_1\gamma_2}$ $\{VU(A)\}_{\gamma_2\beta} \geqslant \ldots \geqslant \{U(A)\}_{\alpha\gamma_1} \ldots \{U(A)\}_{\gamma_{r-1}\gamma_r}\{VU(A)\}_{\gamma_r\beta} \geqslant \{U(A)\}_{\alpha\gamma_1} \ldots \{U(A)\}_{\gamma_r\beta}$.

In case b), using the inductive assumption,

$$\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_1} \ldots \{U(A)\}_{\gamma_{i-1}\gamma_{i+1}} \ldots \{U(A)\}_{\gamma_r\beta}$$

is obtained.

However, $\{U(A)\}_{\gamma_{i-1}\gamma_{i+1}} \geqslant \{U(A)\}_{\gamma_{i-1}\gamma_i}\{U(A)\}_{\gamma_i\gamma_{i+1}}$ since $\gamma_i < min\,(\gamma_{i-1}, \gamma_{i+1})$ and as a result, $(\gamma_{i-1}, \gamma_{i+1}) \subset_U (\gamma_{i-1}, \gamma_i)$ and $(\gamma_{i-1}, \gamma_{i+1}) \subset_U (\gamma_i, \gamma_{i+1})$, consequently, $\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_1} \ldots \{U(A)\}_{\gamma_r\beta}$ has been obtained.

Consider now $min\,(\alpha, \beta) < min\,(\gamma_1, \gamma_2, \ldots, \gamma_r)$. If $\gamma_{m_1}, \gamma_{m_2}, \ldots, \gamma_{m_s}$ are all numbers of set $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ with the property that $\gamma_{mi} > max\,(\alpha, \beta)$ and $m_i < m_{i+1}$, by taking into account proposition 1 and the previous result, we obtain:

$$\{VU(A)\}_{\alpha\beta} \geqslant \{V(U(A))\}_{\alpha\gamma_{m_1}}\{V(U(A))\}_{\gamma_{m_1}\gamma_{m_2}} \ldots \{V(U(A))\}_{\gamma_{m_s}\beta} \geqslant$$

$$\geqslant \{U(A)\}_{\alpha\gamma_1}\{U(A)\}_{\gamma_1\gamma_2} \ldots \{U(A)\}_{\gamma_r\beta}.$$

We now consider the numbers $\gamma_1, \gamma_2, \ldots, \gamma_r$ with $\gamma_i \neq \alpha, \beta$ and $\gamma_i \neq \gamma_j$ for any $i, j \in \{1, 2, \ldots, r\}$ and $i \neq j$. We denote by $\gamma_{p_1}, \gamma_{p_2}, \ldots \gamma_{p_t}$ all numbers of set $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ with the property that $\gamma_{p_i} > min\,(\alpha, \beta)$ and $p_i < p_{i+1}$ It follows then

$$\{VU(A)\}_{\alpha\beta} \geqslant \{U(A)\}_{\alpha\gamma_{p_1}}\{U(A)\}_{\gamma_{p_1}\gamma_{p_2}} \ldots \{U(A)\}_{\gamma_{p_t}\beta} \geqslant$$

$$\geqslant \{U(A)\}_{\alpha\gamma_1}\{U(A)\}_{\gamma_1\gamma_2} \ldots \{U(A)\}_{\gamma_r\beta} \geqslant a_{\alpha\gamma_1}a_{\gamma_1\gamma_2} \ldots a_{\gamma_r\beta}$$ since for any $p_{j+1} > p_j + 1$, relation:

$$max\,(\gamma_{p_j+1}, \ldots \gamma_{p_{j+1}-1}) < min\,(\alpha, \beta) < min\,(\gamma_{p_j}, \gamma_{p_{j+1}})$$

holds, hence

$$\{U(A)\}_{\gamma_{p_j}\gamma_{p_{j+1}}} \geqslant \{U(A)\}_{\gamma_{p_j}\gamma_{p_j+1}} \ldots \{U(A)\}_{\gamma_{p_{j+1}-1}\gamma_{p_{j+1}}}.$$

Consequently,

$$\{VU(A)\}_{\alpha\beta} \geqslant a_{\alpha\beta} + \sum_{r=1}^{p-2} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_r \in \{1, 2, \ldots, p\} \setminus \{\alpha, \beta\} \\ \gamma_1 \neq \gamma_2, \ldots, \gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2} \ldots a_{\gamma_r\beta} = \{A^{p-1}\}_{\alpha\beta}.$$

However, $\{VU(\mathbf{A})\}_{\alpha\beta} \leqslant \{(VU(\mathbf{A}))^{p-1}\}_{\alpha\beta} = \{(U(\mathbf{A}))^{p-1}\}_{\alpha\beta} = \{\mathbf{A}^{p-1}\}_{\alpha\beta}$, hence $\{VU(\mathbf{A})\}_{\alpha\beta} = \{\mathbf{A}^{p-1}\}_{\alpha\beta}$ for any $\alpha, \beta \in \{1, 2, ..., p\}$.

The proof of equality $\{UV(\mathbf{A})\}_{\alpha\beta} = \{\mathbf{A}^{p-1}\}_{\alpha\beta}$ is derived from the above mentioned proof through the duality defined as follows: $U \leftrightarrow V$; $max \leftrightarrow min$; " $< $" $\leftrightarrow$ " $>$" q.e.d.

It follows that operators U and V verify the following equalities $UV = VU = UVU = VUV$.

In the specific case of computing matrix $\mathbf{A}^{p-1}$, for instance, by using operator VU, in the expression of operator U we may consider only operators $T_{\mu\nu}$ for which $a_{\mu\nu} \neq 1$, while in the expression of operator V, only operators $T_{\mu\nu}$ for which $\{U(\mathbf{A})\}_{\mu\nu} \neq 1$ are to be considered.

If operator T satisfies conditions (1), it is obvious that operator T^*, obtained from T by inverting the order of all operators $T_{\mu\nu}$ constituting it, satisfies conditions (2) and in particular we find $TT^*(\mathbf{A}) = T^*T(\mathbf{A}) = \mathbf{A}^{p-1}$. The following operators T satisfy conditions (1):

$$T_1 = T_{pp-1}T_{pp-2} \dots T_{p2}T_{p1} \dots T_{2p} \dots T_{23}T_{21}T_{1p} \dots T_{13}T_{12}$$

$$T_2 = T_{p-1p}T_{p-2p} \dots T_{1p} \dots T_{p2}T_{p-12} \dots T_{32}T_{12}T_{p1} \dots T_{31}T_{21}$$

$$T_3 = T_{pp-1}T_{p-1p} \dots T_{p2} \dots T_{42}T_{32}T_{2p} \dots T_{24}T_{23}T_{p1} \dots T_{31}T_{21}T_{1p} \dots T_{13}T_{12}$$

$$T_4 = T_{pp-1}T_{p-1p} \dots T_{p2}T_{2p} \dots T_{32}T_{23} \dots T_{p1}T_{1p} \dots T_{31}T_{13}T_{21}T_{12}.$$

An outline for the computation of the matrix of total conductivities in the specific case of utilizing operator $T_1^*T_1$ is shown in Fig. 1.

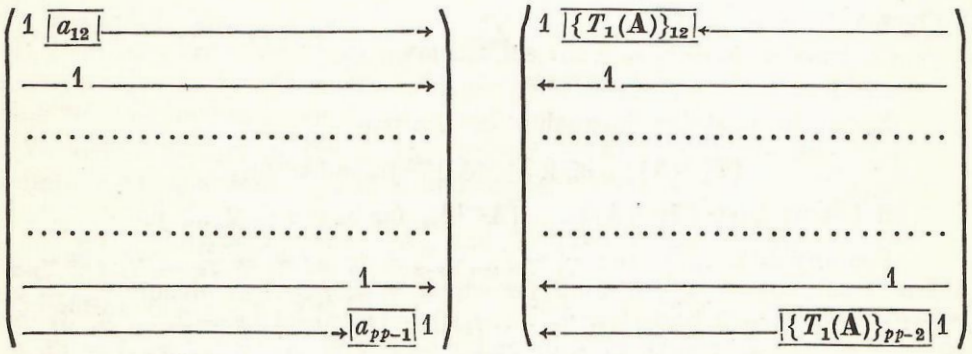

Figure 1.

In the case of symmetric Boolean matrices (corresponding to circuits without valve elements) operator $T_1 = S_{p-1f}S_{p-2p}S_{p-2p-1} \dots S_{2p} \dots S_{23}S_{1p} \dots S_{13}S_{12}$ where operators $S_{\mu\nu}$ are defined as follows:

$$S_{\mu\nu}(\mathbf{A}) = \mathbf{B} \text{ if } b_{\mu\nu} = b_{\nu\mu} = \{\mathbf{A}^2\}_{\mu\nu} = \{\mathbf{A}^2\}_{\nu\mu} \text{ and } b_{\alpha\beta} = a_{\alpha\beta}$$

for any $(\alpha, \beta) \neq (\mu, \nu); (\nu, \mu)$.

Further, we shall describe an algorithm for the computation of a sole row of matrix $\mathbf{A}^{p-1}$ by using a matrix „cascade" multiplication which requires for the majority of computed elements, $p-3$ iterations at most.

According to the above definition, $T_{1\alpha}(\mathbf{A}) = \mathbf{B}$ if

$$b_{1\alpha} = \sum_{\beta=1}^{p} a_{1\beta}a_{\beta\alpha} = a_{1\alpha} + \sum_{\substack{\beta=2 \\ \beta\neq\alpha}}^{p} a_{13}a_{\beta\alpha} \text{ and } b_{\mu\nu} = a_{\mu\nu}$$

for any $(\mu, \nu) \neq (1, \alpha)$.

We define operator $T_1 = T_{1p}T_{1p-1} \dots T_{12}$. It follows that $(T_1(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$ since $(T_{1\alpha}(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$ for any $\alpha = 2, \dots, p$ and $(T_1^n(\mathbf{A}))^{p-1} = (T_1(T_1^{n-1}(\mathbf{A})))^{p-1} = (T_1^{n-1}(\mathbf{A}))^{p-1} = \dots = \mathbf{A}^{p-1}$ for any natural number n.

It may be inductively proved that $\{T_1^n(\mathbf{A})\}_{1\alpha} \geqslant \{\mathbf{A}^{n+1}\}_{1\alpha}$ for any $\alpha = 1, \dots, p$ and there is an index $r \leqslant p-2$ for which $T_1^r(\mathbf{A}) = T_1^{r+1}(\mathbf{A}) = \dots$ thus obtaining $\{T_1^r(\mathbf{A})\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}$ (see [23], [25]).

Theorem 2. There is relation $\{T_1^{p-2}(\mathbf{A})\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}$ for $\alpha = 2, \dots, p$ and in addition $\{T_1^{p-3}(\mathbf{A})\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}$ if $\alpha \geqslant 4$.

Proof. Consider conjunction $a_{1\gamma_1}a_{\gamma_1\gamma_2} \dots a_{\gamma_r\alpha}$ with the indexes $\gamma_1, \gamma_2, \dots, \gamma_r \in \in \{1, 2, \dots, p\} \setminus \{1, \alpha\}$; $\gamma_1 \neq \gamma_2$; $\gamma_1 \neq \gamma_3$; $\dots$; $\gamma_{r-1} \neq \gamma_r$ and $r \leqslant p-2$.

It follows that

$$\{T_1^{p-2}(\mathbf{A})\}_{1\alpha} = \{T_1(T_1^{p-3}(\mathbf{A}))\}_{1\alpha} \geqslant \{T_1^{p-3}(\mathbf{A})\}_{1\gamma_r}a_{\gamma_r\alpha} \geqslant$$

$$\geqslant \{T_1^{p-4}(\mathbf{A})\}_{1\gamma_{r-1}}a_{\gamma_{r-1}\gamma_r}a_{\gamma_r\alpha} \geqslant \dots \geqslant \{T_1^{p-q}(\mathbf{A})\}_{1\gamma_{r-q+3}}a_{\gamma_{r-q+3}\gamma_{r-q+4}} \dots a_{\gamma_r\alpha} \geqslant \dots \geqslant$$

$$\geqslant \{T_1(\mathbf{A})\}_{1\gamma_2}a_{\gamma_2\gamma_3} \dots a_{\gamma_r\alpha} \geqslant a_{1\gamma_1}a_{\gamma_1\gamma_2} \dots a_{\gamma_r\alpha}.$$

In concluding

$$\{T_1^{p-2}(\mathbf{A})\}_{1\alpha} \geqslant a_{1\alpha} + \sum_{r=1}^{p-2} \sum_{\substack{k_1, k_2, \dots, k_r \in \{1, 2, \dots, p\} \setminus \{1, \alpha\} \\ k_1 \neq k_2, \dots, k_{r-1} \neq k_r}} a_{1k_1}a_{k_1k_2} \dots a_{k_r\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}.$$

Since the contrary inequality is also true,

$$\{T_1^{p-2}(\mathbf{A})\}_{1\alpha} \leqslant \{(T_1^{p-2}(\mathbf{A}))^{p-1}\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha},$$

it follows that $\{T_1^{p-2}(\mathbf{A})\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}$ for any $\alpha = 2, \dots, p$.

For any $= 2, \dots, p$ and $\gamma_1, \gamma_2, \dots, \gamma_{p-2} \neq 1, \alpha$; $\gamma_1 \neq \gamma_2, \dots, \gamma_{p-3} \neq \gamma_{p-2}$, there is an index $2 \leqslant j \leqslant p-2$ for which $\gamma_j < \gamma_{j+1}$ if we denote $\gamma_{p-1} = \alpha$, except cases $\alpha = 2$ and $(\gamma_1, \gamma_2, \dots, \gamma_{p-2}) = (p, p-1, p-2, \dots, 4, 3)$ or $(p-1, p, p-2, p-3, \dots, 4, 3)$ and $\alpha = 3$ and $(\gamma_1, \gamma_2, \dots, \gamma_{p-2}) = (2, p, p-1, p-2, \dots, 5, 4)$.

We obtain that

$$\{T_1^{p-3}(\mathbf{A})\}_{1\alpha} \geqslant \{T_1^{p-4}(\mathbf{A})\}_{1\gamma_{p-2}}a_{\gamma_{p-2}\alpha} \geqslant \dots \geqslant \{T_1^{j-1}(\mathbf{A})\}_{1\gamma_{j+1}} \prod_{i=j+1}^{p-3} a_{\gamma_i\gamma_{i+1}}a_{\gamma_{p-2}\alpha} \geqslant$$

$$\geqslant \{T_1^{j-1}(\mathbf{A})\}_{1\gamma_j} \prod_{i=j}^{p-3} a_{\gamma_i\gamma_{i+1}}a_{\gamma_{p-2}\alpha} \geqslant \dots \geqslant \{T_1(\mathbf{A})\}_{1\gamma_2} \prod_{i=2}^{p-3} a_{\gamma_i\gamma_{i+1}}a_{\gamma_{p-2}\alpha} \geqslant$$

$$\geqslant a_{1\gamma_1} \prod_{i=1}^{p-3} a_{\gamma_i\gamma_{i+1}}a_{\gamma_{p-2}\alpha}.$$

Since by using a similar proof it is obtained that $\{T_1^{p-3}(\mathbf{A})\}_{1\alpha}$ contains all conjunctions $a_{1\gamma_1}a_{\gamma_1\gamma_2}\dots a_{\gamma_r\alpha}$ for any $r \leqslant p-3$, it results that $\{T_1^{p-3}(\mathbf{A})\}_{1\alpha} \geqslant \{\mathbf{A}^{p-1}\}_{1\alpha}$ for any $\alpha = 4, \dots, p$.

However, $\{T_1^{p-3}(\mathbf{A})\}_{1\alpha} \leqslant \{(T_1^{p-3}(\mathbf{A}))^{p-1}\}_{1\alpha} = \{\mathbf{A}^{p-1}\}_{1\alpha}$ for any $\alpha = 2, \dots, p$, hence, $\{\mathbf{A}^{p-1}\}_{1\alpha} = \{T_1^{p-3}(\mathbf{A})\}_{1\alpha}$ for $4 \leqslant \alpha \leqslant p$ while $\{\mathbf{A}^{p-1}\}_{12}$ and $\{\mathbf{A}^{p-1}\}_{13}$ are obtained by adding the lacking conjunctions, that is

$$\{\mathbf{A}^{p-1}\}_{12} = \{T_1^{p-3}(\mathbf{A})\}_{12} + a_{1p}\prod_{i=3}^{p} a_{ii-1} + a_{1p-1}a_{p-1p}a_{pp-2}\prod_{i=3}^{p-2} a_{ii-1}$$

and

$$\{A^{p-1}\}_{13} = \{T_1^{p-3}(\mathbf{A})\}_{13} + a_{12}a_{2p}\prod_{i=4}^{p} a_{ii-1}, \quad \text{q.e.d.}$$

In order to determine the transitive closure of a finite graph, **B.** Roy suggested in [19], [20] a matricial algorithm utilizing operators similar to the operators applied in the elimination of a variable from the characteristic function, suggested by A. G. Lunts. The algorithm has been resumed by Warshall in [26] and is operating on Boolean matrices constituted exclusively of zero and one.

The matricial algorithm of B. Roy requires a number of computations of order p^3 and can be easily extended [30] to the computation of matrix $\mathbf{A}^{p-1}$, if the elements of the main diagonal in the Boolean matrix **A** are equal to 1.

Operators $T_{\mu\nu}^{(\rho)}$ with $\mu, \nu, \rho \in \{1, 2, \dots, p\}$ defined on the set of matrices of order p, having the elements of the main diagonal equal to 1, are introduced as follows: $T_{\mu\nu}^{(\rho)}(\mathbf{A}) = \mathbf{B}$ if $b_{\mu\nu} = a_{\mu\nu} + a_{\mu\rho}a_{\rho\nu}$ and $b_{\alpha\beta} = a_{\alpha\beta}$ for any $(\alpha, \beta) \neq (\mu, \nu)$.

It is to be mentioned that operators $T_{\mu\nu}^{(\rho)}$ have also been studied by P. Constantinescu in [5], [6] on the occasion of investigating some problems referring to the simplification of the circuits with contacts.

We deduce that $\mathbf{A} \leqslant T_{\mu\nu}^{(\rho)} \leqslant \mathbf{A}^2$, hence $(T_{\mu\nu}^{(\rho)}(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$. By using the definition of operators $T_{\mu\nu}^{(\rho)}$, it is immediately obtained:

$$T_{\mu_1\nu_1}^{(\rho_1)}(T_{\mu_2\nu_2}^{(\rho_2)}(\mathbf{A})) = T_{\mu_2\nu_2}^{(\rho_2)}(T_{\mu_1\nu_1}^{(\rho_1)}(\mathbf{A})) \text{ and } T_{\mu\nu}^{(\rho)}(T_{\mu\nu}^{(\rho)}(\mathbf{A})) = T_{\mu\nu}^{(\rho)}(\mathbf{A}).$$

Operators $U_\rho = \prod_{\mu,\nu=1}^{p} T_{\mu\nu}^{(\rho)} = \prod_{\substack{\mu,\nu=1 \\ \mu\neq\nu,\ \mu\neq\rho,\ \nu\neq\rho}}^{p} T_{\mu\nu}^{(\rho)}$ as well as operator $U = \prod_{\rho=1}^{p} U_\rho$ are defined.

In defining operator U, the order in which operators U_ρ are taken does not matter, since the product of these operators is commutative:

$$\{U_{\rho_1}U_{\rho_2}(\mathbf{A})\}_{\alpha\beta} = a_{\alpha\beta} + a_{\alpha\rho_2}a_{\rho_2\beta} + (a_{\alpha\rho_1} + a_{\alpha\rho_2}a_{\rho_2\rho_1})(a_{\rho_1\beta} + a_{\rho_1\rho_2}a_{\rho_2\beta}) =$$

$$= a_{\alpha\beta} + a_{\alpha\rho_1}a_{\rho_1\beta} + a_{\alpha\rho_2}a_{\rho_2\beta} + a_{\alpha\rho_1}a_{\rho_1\rho_2}a_{\rho_2\beta} + a_{\alpha\rho_2}a_{\rho_2\rho_1}a_{\rho_1\beta} = a_{\alpha\rho_1}a_{\rho_1\beta} +$$

$$+ (a_{\alpha\rho_2} + a_{\alpha\rho_1}a_{\rho_1\rho_2})(a_{\rho_2\beta} + a_{\rho_2\rho_1}a_{\rho_1\beta}) = \{U_{\rho_2}U_{\rho_1}(\mathbf{A})\}_{\alpha\beta}.$$

As $(T^{(\rho)}_{\mu\nu}(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$, by iteration we obtain that $(U_\varrho(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$ and $(U(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$.

Theorem 3. $U(\mathbf{A}) = \mathbf{A}^{p-1}$ for any Boolen matrix A with $a_{\alpha\alpha} = 1$.

Proof. Commutativity and idempotency of the product of operators $T^{(\rho)}_{\mu\nu}$ induce the commutativity and idempotency of the product of operators U_ϱ, hence $U^2(\mathbf{A}) = U(\mathbf{A})$.

By applying operator U to this equality, we obtain $U(\mathbf{A}) = U^2(\mathbf{A}) = \ldots$

It can be inductively proved with respect to s, that:

$$\{U_{k_s} U_{k_{s-1}} \ldots U_{k_1}(\mathbf{A})\}_{\alpha\beta} \geqslant a_{\alpha\beta} + \sum_{\gamma \in \{k_1, k_2, \ldots, k_s\} \setminus \{\alpha, \beta\}} a_{\alpha\gamma} a_{\gamma\beta}$$

and coresequently, $U(\mathbf{A}) \geqslant \mathbf{A}^2$, which determines $U^2(\mathbf{A}) \geqslant \mathbf{A}^4; \ldots; U^n(\mathbf{A}) \geqslant \mathbf{A}^{2^n}$ for any $n \in N$.

If, however, r is a natural number, $r \geqslant \left[\frac{\log(p-1)}{\log 2}\right] + 1$, then $U^r(\mathbf{A}) \geqslant \mathbf{A}^{2^r} \geqslant \mathbf{A}^{p-1}$. The opposite inequality: $U^r(\mathbf{A}) \leqslant (U^r(\mathbf{A}))^{p-1} = (U(\mathbf{A}))^{p-1} = \mathbf{A}^{p-1}$, is also true, hence $U(\mathbf{A}) = U^2(\mathbf{A}) = \ldots U^r(\mathbf{A}) = \mathbf{A}^{p-1}$ q.e.d.

A. G. Lunts has introduced the operators for the elimination of a variable from the characteristic function of a multiterminal, transforming a matrix $\mathbf{A}$ of order p, in a matrix $\mathbf{B}$ of order $p-1$. If we denote by (Ex_p) the elimination operator of node x_p, then $\mathbf{B} = (Ex_p)\,\mathbf{A}$ if $b_{\alpha\beta} = a_{\alpha\beta} + a_{\alpha p} a_{p\beta}$ for any $\alpha, \beta = 1, \ldots, p-1$.

It is proved that $\{((Ex_p)\mathbf{A})^{p-2}\}_{\alpha\beta} = \{\mathbf{A}^{p-1}\}_{\alpha\beta}$ for any $\alpha, \beta = 1, \ldots, p-1$.

We shall denote by $(Ex_{k+1}, \ldots, x_p)$ the product of operators (Ex_{k+1}), $(Ex_{k+2}), \ldots, (Ex_p)$ that is $(Ex_{k+1}, \ldots, x_p)\mathbf{A} = (Ex_{k+1})(Ex_{k+2}) \ldots (Ex_p)\mathbf{A}$.

Proposition 2. For any $\alpha, \beta = 1, \ldots, k$, the following relation holds:

$$\{(Ex_{k+1}, \ldots, x_p)\mathbf{A}\}_{\alpha\beta} = a_{\alpha\beta} + \sum_{r=1,\ldots,p-k} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_r \in \{k+1, \ldots, p\} \\ \gamma_1 \neq \gamma_2; \ldots; \gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1} a_{\gamma_1\gamma_2} \ldots a_{\gamma_r\beta}$$

The proof is made inductively with respect to $p-k$. If $p-k = 1$, the definition of the operator required for the elimination of a node is obtained.

If the above relation is true, then for any $\alpha, \beta = 1, \ldots, k-1$ we obtain:

$$\{(Ex_k, \ldots, x_p)\mathbf{A}\}_{\alpha\beta} = \{(Ex_{k+1}, \ldots, x_p)\mathbf{A}\}_{\alpha\beta} + \{(Ex_{k+1}, \ldots, x_p)\mathbf{A}\}_{\alpha k} \{(Ex_{k+1}, \ldots, x_p)\mathbf{A}\}_{k\beta} =$$

$$= a_{\alpha\beta} + \sum_{r=1,\ldots,p-k} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_r \in \{k+1, \ldots, p\} \\ \gamma_1 \neq \gamma_2, \ldots, \gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1} a_{\gamma_1\gamma_2} \ldots a_{\gamma_r\beta} +$$

$$+ \left(a_{\alpha k} + \sum_{s=1,\ldots,p-k} \sum_{\substack{\delta_1, \delta_2, \ldots, \delta_s \in \{k+1, \ldots, p\} \\ \delta_1 \neq \delta_2, \ldots, \delta_{s-1} \neq \delta_s}} a_{\alpha\delta_1} a_{\delta_1\delta_2} \ldots a_{\delta_s k} \right) \left(a_{k\beta} + \right.$$

$$+\sum_{t=1,\ldots,p-k}\ \sum_{\substack{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_t\in\{k+1,\ldots,p\}\\ \varepsilon_1\neq\varepsilon_2;\ldots;\varepsilon_{t-1}\neq\varepsilon_t}} a_{k\varepsilon_1}a_{\varepsilon_1\varepsilon_2}\ldots a_{\varepsilon_t\beta}\Bigg) = a_{\alpha\beta} +$$

$$+\sum_r\sum_{\gamma_1,\ldots,\gamma_r} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta} + a_{\alpha k}a_{k\beta} + \sum_t\sum_{\varepsilon_1,\ldots,\varepsilon_t} a_{\alpha k}a_{k\varepsilon_1}\ldots a_{\varepsilon_t\beta} +$$

$$+\sum_s\sum_{\delta_1,\ldots,\delta_s} a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_s k}a_{k\beta} +$$

$$+\left(\sum_s\sum_{\delta_1,\ldots,\delta_s} a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_s k}\right)\left(\sum_t\sum_{\varepsilon_1,\ldots,\varepsilon_t} a_{k\varepsilon_1}a_{\varepsilon_1\varepsilon_2}\ldots a_{\varepsilon_t\beta}\right)$$

where in order to simplify the writing in the last sum, we have assumed the addition indices as covering the same sets of values, this assumption holding in the following as well.

However,

$$\sum_r\sum_{\gamma_1,\ldots,\gamma_r} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta} + \left(\sum_s\sum_{\delta_1,\ldots,\delta_s} a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_s k}\right)\left(\sum_t\sum_{\varepsilon_1,\ldots,\varepsilon_t} a_{k\varepsilon_1}a_{\varepsilon_1\varepsilon_2}\ldots a_{\varepsilon_t\beta}\right) =$$

$$=\sum_r\sum_{\gamma_1,\ldots,\gamma_r} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta} +$$

$$+\sum_{\substack{s,t=1,\ldots,p-k\\ s+t\leqslant p-k}}\ \sum_{\substack{\delta_1,\ldots,\delta_s,\varepsilon_1,\ldots,\varepsilon_t\in\{k+1,\ldots,p\}\\ \delta_1\neq\delta_2;\ldots;\delta_1\neq\varepsilon_t;\ldots;\varepsilon_{t-1}\neq\varepsilon_t}} a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_s k}a_{k\varepsilon_1}a_{\varepsilon_1\varepsilon_2}\ldots a_{\varepsilon_t\beta}.$$

Indeed, if there are two indexes $\delta_i = \varepsilon_j$ (for instance, when $s + t > p - k$), then by using the absorbtion law, we obtain:

$$a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_{i-1}\delta_i}a_{\delta_i\varepsilon_{j+1}}\ldots a_{\varepsilon_t\beta} + a_{\alpha\delta_1}\ldots a_{\delta_i\delta_{i+1}}a_{\delta_s k}a_{k\varepsilon_1}\ldots a_{\varepsilon_{j-1}\varepsilon_j}a_{\varepsilon_j\varepsilon_{j+1}}\ldots a_{\varepsilon_t\beta} =$$

$$= a_{\alpha\delta_1}a_{\delta_1\delta_2}\ldots a_{\delta_{i-1}\delta_i}a_{\delta_i\varepsilon_{j+1}}\ldots a_{\varepsilon_t\beta},$$

the last term being contained in the sum

$$\sum_r\sum_{\gamma_1,\ldots,\gamma_r} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta}.$$

We have thus proved the above proposition.

If a Boolean matrix $\mathbf{A} = \{a_{\alpha\beta}\}_{\alpha,\,\beta=1,\ldots,p}$ is given with $a_{\alpha\alpha} = 1$, we shall delimit the following submatrices, for an index $2 \leqslant k \leqslant p-1$:

$$\mathbf{A}_{(k,\,k)} = \{a_{\alpha\beta}\}_{\alpha,\,\beta=1,\ldots,k};\quad \mathbf{B}_{(k,\,p-k)} = \{a_{\alpha\beta}\}_{\substack{\alpha=1,\ldots,k\\ \beta=k+1,\ldots,p}};$$

$$\mathbf{C}_{(p-k,\,p-k)} = \{a_{\alpha\beta}\}_{\alpha,\,\beta=k+1,\ldots,p} \text{ and } \mathbf{D}_{(p-k,\,k)} = \{a_{\alpha\beta}\}_{\substack{\alpha=k+1,\ldots,p\\ \beta=1,\ldots,k}}$$

F. Brown [4] has generalized the method given by Lunts for the elimination of a node, by building matrix $\mathbf{A}_{(k,\,k)} + (\mathbf{B}_{(k,\,p-k)} \times \mathbf{C}^{p-k-1}_{(p-k,\,p-k)} \times \mathbf{D}_{(p-k,\,k)})$ if $\{\mathbf{A} + \mathbf{B}\}_{\alpha\beta} = a_{\alpha\beta} + b_{\alpha\beta}$ is defined.

Theorem 4. The following relation holds:

$$(Ex_{k+1}, \ldots, x_p)\mathbf{A} = \mathbf{A}_{(k,\,k)} + (\mathbf{B}_{(k,\,p-k)} \times \mathbf{C}^{p-k-1}_{(p-k,\,p-k)} \times \mathbf{D}_{(p-k,\,k)}).$$

Prof. Consider element

$$\{\mathbf{A}_{(k,\,k)} + (\mathbf{B}_{(k,\,p-k)} \times \mathbf{C}^{p-k-1}_{(p-k,\,p-k)} \times \mathbf{D}_{(p-k,\,k)})\}_{\alpha\beta} = a_{\alpha\beta} +$$

$$+ \sum_{i,\,j=k+1,\,\ldots,\,p} a_{\alpha i}\{\mathbf{C}^{p-k-1}_{(p-k,\,p-k)})\}_{ij}a_{j\beta} = a_{\alpha\beta} + \sum_{i,\,j=k+1,\,\ldots,\,p} a_{\alpha i}\Bigg(a_{ij} +$$

$$+ \sum_{r=1,\,\ldots,\,p-k-2} \sum_{\substack{\gamma_1,\,\gamma_2,\,\ldots,\,\gamma_r \in \{k+1,\,\ldots,\,p\} \\ \gamma_1 \neq \gamma_2;\ldots;\,\gamma_{r-1} \neq \gamma_r}} a_{i\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r j}\Bigg) a_{j\beta} = a_{\alpha\beta} +$$

$$+ \sum_{i=k+1,\,\ldots,\,p} a_{\alpha i}a_{i\beta} + \sum_{\substack{i,\,j=k+1,\,\ldots,\,p \\ i\neq j}} a_{\alpha i}a_{ij}a_{j\beta} + \sum_{\substack{i,\,j=k+1,\,\ldots,\,p \\ i\neq j}} \sum_{r} \sum_{\gamma_1,\gamma_2,\,\ldots,\,\gamma_r} a_{\alpha}\,a_{i\gamma_1}\ldots a_{\gamma_r j}a_{j\beta},$$

as $a_{ii} = 1$.

If $i = \gamma_s$, then term $a_{\alpha i}a_{i\gamma_1}\ldots a_{\gamma_{s-1}i}a_{i\gamma_{s+1}}\ldots a_{\gamma_r j}a_{j\beta}$ is absorbed by term $a_{\alpha i}a_{i\gamma_{s+1}}\ldots a_{\gamma_r j}a_{j\beta}$ also belonging to the sum

$$\sum_{\substack{i,\,j=k+1,\,\ldots,\,p \\ i\neq j}} \sum_{r} \sum_{\gamma_1,\,\ldots,\,\gamma_r} a_{\alpha i}a_{i\gamma_1}\ldots a_{\gamma_r j}a_{j\beta} \quad \text{or to sum} \quad \sum_{\substack{i,\,j=k+1,\,\ldots,\,p \\ i\neq j}} a_{\alpha i}a_{ij}a_{j\beta}$$

(in the case of $s = r$).

An analogous conclusion is obtained when $j = \gamma_t$, hence we may consider

$i, j \neq \gamma_1, \gamma_2, \ldots, \gamma_r$ that is $\{\mathbf{A}_{(k,\,k)} + (\mathbf{B}_{(k,\,p-k)} \times \mathbf{C}^{p-k-1}_{(p-k,\,p-k)} \times \mathbf{D}_{(p-k,\,k)})\}_{\alpha\beta} =$

$$= a_{\alpha\beta} + \sum_{r=1,\,\ldots,\,p-k} \sum_{\substack{\gamma_1,\,\gamma_2,\,,\ldots,\,\gamma_r \in \{k+1,\,\ldots,\,p\} \\ \gamma_1 \neq \gamma_2,\,\ldots,\,\gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta}$$

for any $\alpha, \beta = 1, \ldots, k$ and by taking into account proposition 2, the theorem is proved.

In order to compute matrix $\mathbf{C}^{p-k-1}_{(p-k,\,p-k)}$, operators T or operator U involved in Roy's algorithm may also be used.

If the Boolean matrix $\mathbf{A}$ does not contain element 1 on the main diagonal, the property: $\sum_{n=1}^{\infty} \mathbf{A}^n = \sum_{n=1}^{p-1} \mathbf{A}^n$ is valid.

By denoting $\mathbf{A}^* = \sum_{n=1}^{p-1} \mathbf{A}^n$, then

$$\{\mathbf{A}^*\}_{\alpha\beta} = a_{\alpha\beta} + \sum_{r=1}^{p-2} \sum_{\substack{\gamma_1\gamma_2,\,\ldots,\,\gamma_r \in \{1,\,2,\,\ldots,\,p\} \setminus \{\alpha,\,\beta\} \\ \gamma_1 \neq \gamma_2,\,\ldots,\,\gamma_{r-1} \neq \gamma_r}} a_{\alpha\gamma_1}a_{\gamma_1\gamma_2}\ldots a_{\gamma_r\beta}$$

for any $\alpha, \beta = 1, \ldots, p$ and we obtain $\mathbf{A} \leqslant \mathbf{A}^*$.

In the case of $a_{\alpha\alpha} = 1$, for any $\alpha = 1,..., p$ it is obvious that $\mathbf{A}^* = \mathbf{A}^{p-1}$.

Matrix $\mathbf{A}^*$ is involved, for instance, in the computation of the transitive closure of a finite graph and, consequently, may be utilized in the determination of the Hamiltonian paths of a graph, by using the algorithm of Foulkes. In this case, matrix $\mathbf{A}$ is the incidence matrix associated to the graph and its constitutive elements are zero and one only [20]. Matrix $\mathbf{A}^*$ is also involved in determining the loops of a graph representing an electronic circuit [7].

For any Boolean matrix $\mathbf{A}$, we define $T_{\mu\nu}(\mathbf{A}) = \mathbf{B}$ if $b_{\mu\nu} = \{\mathbf{A} + \mathbf{A}^2\}_{\mu\nu} = a_{\mu\nu} + \sum_{\substack{\rho=1 \\ \rho\neq\mu,\,\nu}}^{p} a_{\mu\rho}a_{\rho\nu}$ and $b_{\alpha\beta} = a_{\alpha\beta}$ for any $(\alpha, \beta) \neq (\mu, \nu)$.

Since $\mathbf{A} \leqslant T_{\mu\nu}(\mathbf{A}) \leqslant \mathbf{A} + \mathbf{A}^2$, we obtain $\mathbf{A}^* \leqslant (T_{\mu\nu}(\mathbf{A}))^* \leqslant (\mathbf{A} + \mathbf{A}^2)^* = \mathbf{A}^*$, hence $(T_{\mu\nu}(\mathbf{A}))^* = \mathbf{A}^*$ and the methods of computation obtained in the above paragraphs for matrix $\mathbf{A}^{p-1}$ are transposed for matrix $\mathbf{A}^*$ on condition of also considering operators $T_{\mu\nu}$ with $\mu = \nu$ or operators $T_{\mu\nu}^{(\rho)}$ with $\mu = \nu$ or $\mu = \rho$ or $\nu = \rho$. In the case of computing the first row of matrix $\mathbf{A}^*$, operator $T_1 = T_{1p}T_{1p-1} \ldots T_{11}$ will be used.

The matricial procedures described in the present paper for Boolean matrices are applied in the same form in the theory of transport networks, for the determination of minimal distances between the nodes of the network (in which case $(+, \times, 1)$ are replaced by the usual addition, minimum and zero, respectively) of for the maximal transport capacity determination between the nodes or the network (in which case $(+, \times, 1)$ are replaced by (min, max, ∞)), as well as for matrices with elements belonging to more general algebriac structures, such as those suggested by the authors mentioned in the introductory part of this paper.

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RECHERCHES RECENTES CONCERNANT LA THEORIE DES MATRICES BOOLEENNES

Partat des résultats obtenus par Luntz sur les matrices dont les éléments sont des fonctions booléennes qui ont sur leur diagonale principale l'élément 1, l'article expose de nouveaux résultats concernant les algorithmes de calcul de la matrice $A^{p-1} = A^p = \ldots$, d'une seule ligne ou d'une sous matrice de la matrice A^{p-1}.

L'auteur présente une méthode de multiplication matricielle „en cascade" qui permet, grâce à l'utilisation d'une certain ordre préférentiel des opérateurs matriciels introduits, de déterminer en deux étapes la matrice A^{p-1}.

Il est intéressant de constater que B. Farbey, A. Land et J. Murchland ont proposé de manière indépendente une méthode analogue dans les termes des matrice de distance et que cette méthode a été généralisée par T.C.Hu.

En partant de la même idée de la multiplication en cascade, l'auteur présente un algorithme pour la détermination d'une seule ligne de la matrice $\mathbf{A}^{p-1}$, nécessitant tout au plus $p-3$ itérations.

Cet algorithme, tranposé aux termes des matrices de distance, fournit une méthode permettant de déterminer les distances minimales entre les sommets d'un réseau, plus vite que l'algorithme de Bellman-Kalaba, à condition de prendre pour la solution initiale de l'équation fonctionnelle respective des valeurs égales aux distances directes entre les sommets du réseau.

L'auteur présente ensuite l'algorithme matriciel de B. Roy, proposé également par S. Warshall, qui exige un nombre réduit de calculs.

Pour déterminer une sous-matrice de la matrice $\mathbf{A}^{p-1}$, l'auteur présente les méthodes d'élimination des sommets de A. Luntz et de F. Brown et démontre que la méthode de F. Brown peut être obtenue par l'application répétée de l'algorithme de l'élimination d'un sommet de Luntz.

On indique en conclusion que ces méthodes peuvent être utilisées pour detérminer la matrice $\mathbf{A} = \sum_{p=1}^{\infty} \mathbf{A}^k = \sum_{k=1}^{p=1} A^k$ où A est une matrice booléenne qui ne contient plus nécesssairement sur sa diagonale principale l'élément 1.

Les résultats présentés dans cet article peuvent être utilisés, par exemple, pour déterminer la matrice de la fermeture transitive d'un graphe fini et la matrice des conductibilités totales d'un schéma à contacts; ils peuvent être transposés dans le cas des réseaux de transport en remplaçant les opérations booléennes «*ou*», «*et*», par «+», «*min*» ou «*min*», «*max*», etc.

Les algorithmes présentés sont faciles à utiliser pour un ordinateur et demandent un nombre de calculs beaucoup moindre que la méthode de formation de la suite de carrés successifs de la matrice booléenne *A*.

СОВРЕМЕННЫЕ ИССЛЕДОВАНИЯ ПО ТЕОРИИ МАТРИЦ БУЛЯ

В этой статье, в которой имеются в виду результаты Лунца в исследовании матриц, элементы которых являются функциями Буля, имеющими на главной диагонали элемент 1, показываются новые результаты, которые касаются расчетного алгоритма матрицы $\mathbf{A}^{p-1} = \mathbf{A}^p = \ldots$, одной единственной линии или одной подматрицы матрицы $\mathbf{A}^{p-1}$.

Автор демонстрирует метод матричного умножения «каскадом», благодаря которому, при применении определенного привилегированного порядка введенных матричных операций, возможно определение матрицы $\mathbf{A}^{p-1}$ в два этапа.

Интересно отметить, что аналогичный метод в слагаемых матриц дистанций был предложен независимо друг от друга Б. Фарбейем, А. Ландом, Дж. Мэрчландом и обобщен Т. К. Ху.

Используя ту же самую идею каскада, автор демонстрирует алгоритм для определения одной единственной линии матрицы $\mathbf{A}^{p-1}$, требующей самое больше $p-3$ итераций.

Переведенный в члены матриц дистанций этот алгоритм подсказывает метод, определяющий минимальные дистанции между узлами сети быстрее, чем алгоритм Бельмана-Калабы, если при решении соответствующего функционального уравнения исходить из начальных величин, равных прямым расстояниям между узлами сети.

В продолжение демонстрируется матричный алгоритм Б. Роя, предложенный также и С. Уаршалем, который требует меньшего числа расчетов.

Для определения подматрицы матрицы $\mathbf{A}^{p-1}$ даются методы исключения узлов А. Лунца и Ф. Брауна и демонстрируется, что метод Ф. Брауна достигается путем повторного применения алгоритма исключения узла А. Лунца.

В заключение показывается, что эти методы могут быть использованы для определения матрицы $\mathbf{A}=\sum_{k=1}^{\infty}\mathbf{A}^k=\sum_{k=1}^{p-1}\mathbf{A}^k$ где $\mathbf{A}$ является матрицей Буля, которая не содержит в обязательном порядке элемент 1 на главной диагонали.

Результаты, демонстрируемые в этой статье, могут быть использованы, например, при определении матрицы транзитивного закрытия законченного графа или матрицы полных проводимостей схемы с контактами и можно приложить к случаям транспортных сетей, заменив операции Буля «*или*» «*и*» через «$+$», «*min*» или «*min*», «*max*» и т.д.

Представленные алгоритмы легко применимы на электронном вычислении и требуют меньшего числа расчетов, чем метод образования цепи последовательных квадратов матрицы $\mathbf{A}$.