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An Aggregative Model of Optimal Economic Growth for Investments with Time Lag

A certain amount of the investments has an instantaneous participation in the increase of the means of production, while the remainder undergoes a certain delay

D. Cass analyzes in [1], [2] an aggregative model of optimal economic growth meant to maximize the population's welfare in a finite time interval. The welfare is expressed by the functional relation

$$W = \int_0^T U[c(t)]e^{-\sigma t}\,dt \tag{1}$$

where

$[t_0, T]$ = the time interval for which the planning is drawn up
$c(t)$ = the consumption per inhabitant at moment t
σ = the ratio of updating
and $U(c)$ = the efficiency function of consumation, with the properties:

$$U'(c) > 0,\ U''(c) < 0, \text{ for any } c > 0 \tag{2}$$

$$\lim_{c\to 0} U'(c) = \infty,\ \lim_{c\to\infty} U'(c) = 0$$

The economic background on which D. Cass places his computation is, in short, the following: the output is described as a production function with constant return to scale (homogeneous, of first degree that can be written under the form

$$y = f(k) \tag{3}$$

having the properties

$$f(k) > 0,\ f'(k) > 0,\ f''(k) < 0,\ \text{for any } k > 0 \tag{4}$$

$$\lim_{k \to 0} f'(k) = \infty,\ \lim_{k \to \infty} f'(k) = 0$$

where

k = the ratio between the means of production and the manpower (both of them expressed as values)
and y = the value of output per inhabitant.
If we denote by $z(t)$ the investment per inhabitant made at moment t, then

$$y(t) = c(t) + z(t) \tag{5}$$

$$z(t) \geqq 0\ \ c(t) > 0.$$

By denoting by μ the wear degree of production means and by n the proportion of population's growth, the time evolution of production means is given by relation

$$\dot{k}(t) = z(t) - \lambda k(t),\ \ \lambda = n + \mu.$$

The initial amount of production means is known:

$$k(0) = k_0,$$

and the final amount cannot be as small as possible, since k^T is prescribed

$$k(T) \geqq k^T \geqq 0. \tag{6}$$

By using the Pontryagin's principle of maximum, D. Cass finds out that an optimal trajectory $k(t)$, $c(t)$, $z(t)$ shall satisfy the above relations as well as

$$\dot{q}(t) = (\sigma + \lambda)\ q(t) - U'[c(t)]\ f'[k(t)]$$

$$q(T) \geqq 0 \text{ and } q(T) = 0 \text{ when } k(T) > k^T$$

$$-U'[c(t)] + q(t) \leqq 0 \text{ the equality occurring when } z(t) > 0.$$

He proves that these conditions are also sufficient for achieving the optimality of a trajectory.

Here, we admit the possibility of an investment lag and find the required optimality conditions to be similar to the above ones. Likewise, we have to prove the sufficiency of the conditions found for a trajectory optimality.

1. THE MODEL

The resulting output is partly distributed to the consumption and partly to the investment fund, according to relation (5). The investments participate in the growth of the production means in the following way: p per cent at once and $(1 - p)$ per cent with a lag $\tau(t) > 0$. We shall admit that $\tau(t)$

is derivable in the interval $[t_0, T]$. Consider $\beta(t) = t - \tau(t)$. It is natural to assume $\beta(t)$ to be an increasing function

$$\dot{\beta}(t) > 0$$

hence, inversive on $[t_0, T]$, and consider $\gamma(t)$ to be its inverse defined in $[\beta(t_0), \beta(T)]$

$$\beta\ (\gamma(t)) = t$$

From the manner in which investments participate in the growth of production means there follows

$$\dot{k}(t) = pz(t) + (1 - p)\ z(\beta(t)) - \lambda k(t) \tag{7}$$

which becomes, taking into acconnt (3) and (5),

$$\dot{k}(t) = p(f[k(t)] - c(t)) + (1 - p)\ (f[k(\beta(t))] - c(\beta(t))) - \lambda k(t) \tag{8}$$

Owing to the investment lag, the time period $[t_0 - \tau(t_0), t_0]$ will bear an influence on the economy behaviour in the interval $[t_0, T]$. As the body drawing up the planning is aware of the economy behaviour prior to moment t_0, this means that

$$k(t_0 + s) = \varphi(s),\ c(t_0 + s) = \psi(s);\ s \in [-\tau(t_0), 0] \tag{9}$$

are given.

The functional relation to be optimized is still (1), and the final amount of means of production shall satisfy (6). To solve such a problem, certain results of the optimal checking for systems with time lag are required. To this end we further present the results obtained by A. Halanay [3].

2. AN OPTIMAL CHECKING FOR SYSTEMS WITH VARIABLE TIME LAG

Consider a system of the form

$$\dot{x}(t) = f(t,\ x(t),\ x(t - \tau_1(t)), \ldots,\ x(t - \tau_k(t)),\ u(t),\ u(t - \tau_1(t)), \ldots,$$
$$u(t - \tau_k(t),\ b)$$

$$0 < \tau_i(t) \leqq \tau_{i+1}(t) - \alpha,\ \alpha > 0,\ \dot{\tau}_i(t) < 1$$

$$x(t_0 + \sigma) = \varphi(\sigma, b),\ u(t_0 + \sigma) = \eta(\sigma, b);\ \sigma \in [-\tau_k(t_0), 0),\ x(T) = \omega(b)$$

and the functional relations

$$I_\gamma(u, b) = g_\gamma(b) + \int_{t_0}^{T} L_\gamma(t,\ x(t),\ x(t - \tau_1(t)), \ldots,\ x(t - \tau_k(t)),\ u(t), \ldots,$$
$$u(t - \tau_k(t)),\ b)$$
$$\gamma = 0, 1, \ldots, m$$

We denote by $\beta_i(t) = t - \tau_i(t)$ and by γ_i the inverse of function β_i. The checking couple (u, b) is assumed to be admissible if u is continuous on

portions in $[t_0 - \tau_k(t_0),\ T]$, $u(t) \in M$ for $t \in [t_0 - \tau_k(t_0),\ T)$, where M is a given set, $u(t_0 + \sigma) = \eta(\sigma, b)$, $b \in B$ where B is a given open set and the suitable solution to the system, defined by $x(t_0 + \sigma) = \varphi(\sigma, b)$ satisfies $x(\mathrm{T}) = \omega(b)$, $I_\gamma \leqq 0$ for $1 \leqq \gamma \leqq m'$, $I_\gamma = 0$ for $m' < \gamma \leqq m$. The couple is optimal if it is admissible and minimizes I_0 in the class of admissible couples.

Let $(\tilde{u}, \tilde{b})$ be an optimal couple and $\tilde{x}$ the solution corresponding to the optimal couple. We write:

$$A(t) = \frac{\partial f}{\partial x}(t, \tilde{x}(t), \tilde{x}(\beta_1(t)), \ldots, \tilde{x}(\beta_k(t)), \tilde{u}(t), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$

$$B_j(t) = \frac{\partial f}{\partial y_i}(t, \tilde{x}(t), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$

$$c_\gamma(t) = \frac{\partial L_\gamma}{\partial x}(t, \tilde{x}(t), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$

$$d_{\gamma j}(t) = \frac{\partial L_\gamma}{\partial y_j}(t, \tilde{x}(t), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$

There is valid the proposition:

If $(\tilde{u}, \tilde{b})$ is optimal, then there are the multipliers $\lambda_0, \lambda_1, \ldots, \lambda_{m+n}$ (n being the system size) with the properties:

(a) $\lambda_0^2 + \ldots + \lambda_{m+n}^2 \neq 0$, $\lambda_\gamma \geqq 0$ for $0 \leqq \gamma \leqq m'$, $\lambda_\gamma = 0$ if $I_\gamma(\tilde{u}, \tilde{b}) < 0$

(b) Let $\tilde{\psi}$ be the solution to the system

$$\dot{\psi}(t) = -\psi(t)A(t) - \sum_j \psi(\gamma_j(t))B_j(\gamma_j(t))\dot{\gamma}_j(t) + \sum_{\alpha=0}^{m} \lambda_\alpha [c_\alpha(t) + \sum d_{\alpha j}(\gamma_j(t))\dot{\gamma}_j(t)]$$

defined by the condition $\psi(t) = 0$ for $t > T$, $\psi_j(T) = -\lambda_{m+j}$ and let

$$H(t, x, y_1, \ldots, y_k, u, v_1, \ldots, v_k, b) = \tilde{\psi}(t) f(t, x, y_1, \ldots, y_k, u, v_1, \ldots, v_k, b) - \sum_\alpha \lambda_\alpha L_\alpha(t, x, y_1, \ldots, y_k, u, v_1, \ldots, v_k, b).$$

Then, for all $u \in M$ and for all $t \in [t_0, T]$ where $\tilde{u}$ is continuous, the following inequality occurs (the principle of maximum):

$$H(t, \tilde{x}(t), \tilde{x}(\beta_1(t)), \ldots, \tilde{x}(\beta_k(t)), u, \tilde{u}(\beta_1(t)), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$
$$+ H(\gamma_1(t), \tilde{x}(\gamma_1(t)), \tilde{x}(t), \ldots, \tilde{x}(\beta_k(\gamma_1(t))), \ldots, \tilde{x}(t), \tilde{u}(\gamma_1(t)), u, \ldots, \tilde{u}(\beta_k(\gamma_1(t)), \tilde{b})\dot{\gamma}_1(t)$$
$$+ \ldots + H(\gamma_k(t), \tilde{x}(\gamma_k(t)), \tilde{x}(\beta_1(\gamma_k(t))), \ldots, \tilde{x}(t), \tilde{u}(\gamma_k(t)), \tilde{u}(\beta_1(\gamma_k(t))), \ldots, u, \tilde{b})\dot{\gamma}_k(t)$$
$$\leqq H(t, \tilde{x}(t), \tilde{x}(\beta_1(t)), \ldots, \tilde{x}(\beta_k(t)), \tilde{u}(t), \tilde{u}(\beta_1(t)), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})$$
$$+ H(\gamma_1(t), \tilde{x}(\gamma_1(t)), \tilde{x}(t), \ldots, \tilde{x}(\beta_k(\gamma_1(t)), \tilde{u}(\gamma_1(t)), \tilde{u}(t), \ldots, \tilde{u}(\beta_k(\gamma_1(t))), \tilde{b})\dot{\gamma}_1(t)$$
$$+ \ldots + H(\gamma_k(t), \tilde{x}(\gamma_k(t)), \tilde{x}(\beta_1(\gamma_k(t))), \ldots, \tilde{x}(t), \tilde{u}(\gamma_k(t)), \tilde{u}(\beta_1(\gamma_k(t))), \ldots, \tilde{u}(t)\ \tilde{b})\dot{\gamma}_k(t).$$

(c) The transversality conditions occur:

$$\sum_{\alpha=0}^{m} \lambda_\alpha \frac{\partial g_\alpha}{\partial b}(\tilde{b}) - \tilde{\psi}(t_0)\frac{\partial \varphi(t_0, \tilde{b})}{\partial b} + \tilde{\psi}(T)\frac{\partial \omega(\tilde{b})}{\partial b} - \sum_j \int_{t_0}^{\gamma j(t_0)} \left[\tilde{\psi}(s) B_j(s) + \right.$$

$$\left. + \sum_{\alpha=0}^{m} \lambda_\alpha d_{kj}(s)\right] \frac{\partial \varphi(\beta_j(s), \tilde{b})}{\partial b}\, ds - \int_{t_0}^{T} \frac{\partial H(t\, \tilde{x}(t), \ldots, \tilde{u}(\beta_k(t)), \tilde{b})}{\partial b}\, dt = 0$$

3. OPTIMAL TRAJECTORIES

Observing the terminology used so far, we shall call an admissible consumption the function continuous on portions $c(t) > 0$, satisfying the second relation in (9), for which the solution to equation (8) defined by the first relation in (9) satisfies (6). The consumption will be called optimal if it maximizes the functional relation (1) in the class of admissible consumptions.

By writing

$$u(t) = \frac{c(t)}{f[k(t)]} \qquad (0 < u(t) \leqq 1)$$

$$k(T) = b$$

we bring the problem in §1 to the standard form

$$\dot{k}(t) = pf[k(t)]\,(1 - u(t)) + (1 - p)\, f[k(\beta(t))]\,(1 - u(\beta(t))) - \lambda k(t) \qquad (8')$$

$$k(t_0 + s) = \varphi(s),\ u(t_0 + s) = \frac{\psi(s)}{f[\varphi(s)]}\ ;\quad s \in [-\tau(t_0), 0] \qquad (9')$$

$$I_0 = \int_{t_0}^{T} - U[u(t) f(k(t))]\, e^{-\sigma t}\, dt \qquad (1')$$

$$I_1 = k^T - b \leqq 0$$

$$k(T) = b \qquad (6')$$

An admissible couple is matched by an admissible consumption, and an optimal couple is matched by an optimal consumption.

Using the results obtained so far, it follows that:

If $\tilde{u}(t)$, $\tilde{b}$ is an optimal couple, there exist multipliers λ_0, λ_1, λ_2, not all of them nought, so that

(a) $\lambda_0 \geqq 0$, $\lambda_1 \geqq 0$ and $\lambda_1 = 0$ when $k^T - \tilde{b} < 0$

(b) let $\tilde{Q}(t)$ be the solution to equation

$$\dot{Q}(t) = \lambda \tilde{Q}(t) - p f'[\tilde{k}(t)]\,(1 - \tilde{u}(t))\, Q(t) - (1 - p)\, f'[\tilde{k}(t)]\,(1 -$$

$$- \tilde{u}(t)) Q(\gamma(t))\, \dot{\gamma}(t) - \lambda_0 U'[\tilde{u}(t) f(\tilde{k}(t))]\, \tilde{u}(t)\, f'[\tilde{k}(t)]\, e^{-\sigma t} \qquad (10)$$

defined by

$$Q(T) = -\gamma_2 \text{ and } Q(t) \equiv 0 \text{ for } t > T; \tag{11}$$

then, for any $u \in (0, 1]$ and all $t \in [t_0, T]$ where $\tilde{u}(t)$ is continuous, there occurs

$$\begin{aligned}
&\tilde{Q}(t)\,[pf[\tilde{k}(t)]\,(1-u) + (1-p)\,f[\tilde{k}(\beta(t))]\,(1-\tilde{u}\beta(t))) - \lambda k(\tilde{t})] + \\
&+ \lambda_0 U[uf(\tilde{k}(t))]\,e^{-\sigma t} + \tilde{Q}(\gamma(t))\,[pf[\tilde{k}(\gamma(t)))]\,(1-\tilde{u}(\gamma(t))) + \\
&+ (1-p)\,f[\tilde{k}(t)]\,(1-u) - \lambda\tilde{k}\,(\gamma(t))]\dot{\gamma}(t) + \lambda_0 U\,[\tilde{u}(\gamma(t))\,f(\tilde{k}(\gamma(t)))]\,e^{-\sigma'}\dot{\gamma}(t) \\
&\leqq \tilde{Q}(t)\,[pf[\tilde{k}(t)]\,(1-\tilde{u}(t)) + (1-p)\,f[\tilde{k}(\beta(t))]\,(1-\tilde{u}(\beta(t))) - \lambda\tilde{k}(tt)] + \\
&+ \lambda_0 U[\tilde{u}(t)\,f(\tilde{k}(t))]\,e^{-\sigma t} + \tilde{Q}(\gamma(t))\,[pf[\tilde{k}(\gamma(t)))]\,(1-\tilde{u}(\gamma(t))) + \\
&+ (1-p)\,f[\tilde{k}(t)]\,(1-\tilde{u}(t)) - \lambda\tilde{k}(\gamma(t))] + \lambda_0 U[\tilde{u}(\gamma(t))\,f(\tilde{k}(\gamma(t)))]\,e^{-\sigma\gamma(t)}\dot{\gamma}(t)
\end{aligned}$$

or

$$\begin{aligned}
&-[p\tilde{Q}(t) + (1-p)\,\tilde{Q}(\gamma(t))\,\dot{\gamma}(t)]f[\tilde{k}(t)]\,u + \lambda_0 U[uf(\tilde{k}(t))]\,e^{-\sigma t} \leqq -[p\tilde{Q}(t) + \\
&+ (1-p)\,\tilde{Q}(\gamma(t)\,\dot{\gamma}(t)]\,f[\tilde{k}(t)]\,\tilde{u}(t) + \lambda_0 U[\tilde{u}(t)\,f(\tilde{k}(t))]\,e^{-\sigma t}
\end{aligned} \tag{12}$$

where $\tilde{k}(t)$ is the solution corresponding to the optimal couple;

(c) the transversality condition

$$-\lambda_0 + \tilde{Q}(T) = 0 \tag{13}$$

occurs, or, taking account of (11),

$$\lambda_1 = -\lambda_2 \tag{14}$$

Remark. If $\lambda_0 = 0$, relation (12) is reduced to

$$[p\tilde{Q}(t) + (1-p)\,\tilde{Q}(\gamma(t))\,\dot{\gamma}(t)]\,(\tilde{u}(t) - u) \leqq 0; \tag{12'}$$

when $\tilde{u}(t) = 1$, there follows

$$p\tilde{Q}(t) + (1-p)\,\tilde{Q}(\gamma(t))\,\dot{\gamma}(t) \leqq 0;$$

when $\tilde{u}(t) < 1$, there follows

$$p\tilde{Q}(t) + (1-p)\tilde{Q}(\gamma(t))\,\dot{\gamma}(t) = 0.$$

In both cases

$$(1-\tilde{u}(t))\,[p\tilde{Q}(t) + (1-p)\,\tilde{Q}(\gamma(t))\,\dot{\gamma}(t)] = 0$$

and the (10) is reduced to

$$\dot{Q}(t) = \lambda\,Q(t),\ Q(T) = -\lambda_2$$

having as a solution

$$\tilde{Q}(t) = -\lambda_2 e^{\lambda(t-T)}$$

by which (12′) becomes

$$-\lambda_2 [p e^{\lambda(t-T)} + (1-p)\, e^{\lambda(\gamma(t)-T)} \dot{\gamma}(t)]\, (\tilde{u}(t) - u) \leqq 0$$

or, using (14),

$$\lambda_1 (\tilde{u}(t) - u) \leqq 0.$$

Since this relation is true for any $u \in (0, 1]$ and all $t \in [t_0, T]$, there follows that either $\lambda_1 = 0$, $\tilde{u}(t) = 0$. Both situations are inacceptable and, consequently, the assumption $\lambda_0 = 0$ is absurd.

We write

$$\mathcal{H}(t, u) = -p\tilde{Q}(t)\, f[\tilde{k}(t)]\, u - (1-p)\, \tilde{Q}(\gamma(t)\, f[\tilde{k}(t)]\, \dot{\gamma}(t)\, u + \\ + U[uf(\tilde{k}(t))]\, e^{-\sigma t}$$

Obviously

$$\frac{d\mathcal{H}}{du} = -[p\tilde{Q}(t) + (1-p)\, \tilde{Q}(t)\, \dot{\gamma}(t)]\, f[\tilde{k}(t)] + U'[uf(\tilde{k}(t))]\, f(\tilde{k}(t))\, e^{-\sigma t}$$

$$\frac{d^2\mathcal{H}}{du^2} = U''[uf(\tilde{k}(t))]\, [f(\tilde{k}(t))]^2\, e^{-\sigma t} < 0$$

and (12) is written

$$\mathcal{H}(t, u) \leqq \mathcal{H}(t, \tilde{u}(t)).$$

Hence, when $\tilde{u}(t) = 1$,

$$\frac{d\mathcal{H}(t, \tilde{u}(t))}{du} \geqq 0$$

and when $\tilde{u}(t) < 1$

$$\frac{d\mathcal{H}(t, \tilde{u}(t))}{du} = 0$$

or

$$-p\, \tilde{Q}(t) - (1-p)\, \tilde{Q}(\gamma(t))\, \dot{\gamma}(t) + U'[\tilde{c}(t)]\, e^{-\sigma t} \geqq 0, \tag{15}$$

the equality occurring when

$$z(t) > 0.$$

Then (10), (11) and (13) become

$$\dot{Q}(t) = \lambda\, Q(t) - U'[\tilde{c}(t)] f'[\tilde{k}(t)]\, e^{-\sigma t} \tag{10′}$$

$$Q(T) = -\lambda_2,\ Q(t) \equiv 0 \text{ for } t > T \tag{11′}$$

$$Q(T) = \lambda_1 \geqq 0. \tag{13′}$$

To conclude, (8), (9), (10′), (11′), (13′) and (15) are the required optimality conditions.

In the following, it is proved that these conditions are also sufficient for optimality. We denote by index 1 the trajectory satisfying the above stated conditions and by index 2 the trajectory satisfying only (8), (9), (10'), (10') and (11'). We point out that

$$\delta W = \int_{t_0}^{T} [U[c_1(t)] - U[c_2(t)]]\, e^{-\sigma t} dt > 0. \tag{16}$$

We can write

$$\delta W = \int_{t_0}^{T} [U[c_1(t)] - U[c_2(t)] - U'[c_1(t)]\ (c_1(t) - c_2(t)) +$$

$$+ U'[c_1(t)]\,(c_1(t) - c_2(t))]\, e^{-\sigma t} dt + \int_{t_0}^{T} Q_1(t)[p z_1(t) +$$

$$+ 1 - p)\, z_2\,(\beta(t)) - \lambda k_1(t) - \dot{k}_1(t)]\, dt - \int_{t_0}^{T} Q_1(t)\ [p z_2(t) +$$

$$+ (1 - p)\ z_2\,(\beta(t)) - \lambda k_2(t) - \dot{k}_2(t)]\, dt$$

the last two integrals being nought after (7). Taking into account (2)

$$\delta W > \int_{t_0}^{T} U'[c_1(t)]\, e^{-\sigma t}(c_1(t) - c_2(t))\, dt + \int_{t_0}^{T} p Q_1(t)\ [(z_1(t)) - z_2(t) dt +$$

$$+ \int_{t_0}^{T} (1 - p)\ Q_1(t)\ (z_1(\beta(t) - z_2(\beta(t)))\ dt - \int_{t_0}^{T} \lambda Q_1(t)\ (k_1(t) - k_2(t))\ dt -$$

$$- \int_{t_0}^{T} Q_1(t)\ (\dot{k}_1(t) - \dot{k}_2(t))\ dt$$

In the third integral we change a variable, never forgetting that $Q_1(t) \equiv 0$ for $t > T$. We use also (5)

$$\delta W > \int_{t_0}^{T} - U'\,[c_1(t)] e^{-\sigma t}(z_1(t) - z_2(t))\ dt + \int_{t_0}^{T} U'\,[c_1(t)] e^{-\sigma t}(f(k_1(t)) -$$

$$- f(k_2(t)))\, dt + \int_{t_0}^{T} p\, Q_1(t)\ \ (z_1(t) - z_2(t)))\ dt + \int_{\beta(t_0)}^{\beta(T)} (1 - p)\, Q_1(\gamma(t))\, (z_1(t) -$$

$$- z_2(t))\, \dot{\gamma}(t)\, dt - \int_{t_0}^{T} Q_1(t)\ \ (\dot{k}_1(t) - \dot{k}_2(t))\, dt$$

Since in the interval $[\beta(t_0), t_0]$.

$$z_1(t) \equiv z_2(t)$$

it follows, taking into account (4),

$$\delta W > \int_{t_0}^{T} [- U'[c_1(t)]\, e^{-\sigma t} + p\, Q_1(t) + (1 - p)\, Q_1\,(\gamma(t))\, \dot{\gamma}(t)]\ \ (z_1(t) - z_2(t)\, dt +$$

$$+ \int_{t_0}^{T} U'\ [c_1(t)]\, f'\ [k_1(t)]\ \ e^{-\sigma t}(k_1(t) - k_2(t))\ \ dt - \int_{t_0}^{T} \lambda Q_1(t)\ (k_1(t) - k_2\ (t))\, dt -$$

$$- \int_{t_0}^{T} Q_1(t)\ (\dot{k}_1(t) - \dot{k}_2(t)\ \ dt$$

After (15) the first integral is non-negative, so that on integrating by parts we find

$$\delta W \mid > \int_{t_0}^{T} U' \left[c_1 \ (t)\right] f' \left[k_1(t)\right] e^{-\sigma t} - \lambda Q_1(t) + \dot{Q}_1(t)\right] (k_1(t) - k_2(t))\, dt -$$

$$- \lambda_1 \ (k_1(T) - k_2(T)) \ = - \lambda_1 \ (k_1(T) - k_2(T)) \geqq 0$$

which proves (16).

Let us remark, as a conclusion, that, when $p = 1$, results are being obtained from [1] and [2], and when $p = 0$, the case arises that the investments are wholly lagged.

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UN MODELE AGGREGATIF DE CROISSANCE ECONOMIQUE OPTIMALE POUR LES INVESTISSEMENTS A ARGUMENT RETARDE

Pour obtenir un modèle aggrégatif de croissance économique optimale, ayant pour but la maximisation du bien-être social pendant un certain laps de temps, on prend pour point de départ l'hypothèse que les investissements contribuent à l'accroissement des moyens de production tant de façon instantanée, qu'avec un certain retard.

Utilisant les derniers résultats de A. Halanay concernant le contrôle optimal à argument retardé, on donne une caractérisation complète des trajectoires de croissance optimale.

АГРЕГАТИВНАЯ МОДЕЛЬ ОПТИМАЛЬНОГО ЭКОНОМИЧЕСКОГО РОСТА ДЛЯ ВЛОЖЕНИЙ С ЗАПОЗДАВШИМ АРГУМЕНТОМ

В работе описывается агрегативная модель оптимального экономического роста, целью которой является максимизация социального благосостояния за определенный промежуток времени.

Автор исходит из гипотезы, что вложения участвуют в росте средств производства как одновременно, так и с определенным опозданием.

Используя результаты, полученные А. Халанай, касающиеся оптимального контроля с запоздавшим аргументом, автор дает полную характеристику траекторий оптимального роста.