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Models of Growth for the National Economy

Planning is a vital function of the socialist economic system and the State Plan is the most sophisticated social control.

Among the potential methods and techniques contributing to the improvements of the planning methods designed for the national economy a prominent place is held by the theory of economic growth models.

This rather recent theory is currently considered as one of the major chapters of economic sciences. By calling upon the newest chapters of mathematics and a refined technique of economic analysis, the economic growth theory and models have a double purpose: they are endowed with an operational role, on the one hand, and they supply an explanation for a set of phenomena related to the economic growth, on the other hand. Through the solution of various types of growth models, the optimum structure of the national economy can be found, as well as the optimum way of developing the various sectors, the optimum annual ratio of the accumulation to the consumption fund, and the like. Furthermore, through analyzing the various types of growth models, a number of major elements can be explained, such as the role played by the technical progress in the national economy development, the impact of a higher working productivity as well as of an increased allocation of funds to the working process on the national economy development.

In terms of the time period they refer to, the economic growth models can be classified in short-, medium- and long-term models.

Short-term models apply to a time interval varying from one month up to three years. Short-term models can be classified in aggregative and non-aggregative models. The most representative among the non-aggregative models are those based on the interbranch balance. Among the short-term models a prominent part is played by the prediction models, serving as a basis for finding the population's demand for various goods and services, the evolution of money turnover, the demand for various goods in the world

market, and the like. Owing to the short time for which these models are designed, they can be generally linearized.

Medium-term models generally apply to periods varying from three to five years. These models are also for the most part of the static or dynamic model type of the interbranch balance.

The long-term models, mostly applying to the entire national economy, are aimed at finding the optimal development policy for time limits of ten to twenty years ahead.

In order to determine the optimal national economy development plan, it is necessary to establish several starting points. In the first place, we must know the production funds and manpower volume in the initial year, as well as their dynamics; then, the production functions which emphasize the functional dependence prevailing between the volume of production funds and of manpower on the one hand and the final social product, on the other, must also be established. Finally, it is necessary to set up a criterion for what is considered to be an optimum.

In terms of aggregation, we can distinguish one-sector, two-sector and multi-sector long-term models.

When speaking about a one-sector growth model, the whole national economy is viewed as a single aggregate. Within these models, the control parameter whose optimal value is to be calculated is the *accumulation rate* which reflects one of the most important ratios in the national economy. It has to be noted that a careful analysis of the phenomena occurring at the national economy level emphasizes the fact that the correlation between the accumulation and the consumption funds actually involves the whole economic machinery.

Several types of one-sector growth models have been described up to now in the literature on the subject. One of the most representative is the model analyzed by the econometrist David Cass, from the Cowles Foundation for Research in Economics [1]. Almost all the models analyzed by various authors are continuous and autonomous models, which means that the "time" factor is never explicitly involved in the model.

In our opinion, this simplification may highly reduce the practical consequence of results. In a long-term model some time-dependent factors such as the technical progress, working productivity increase or a change in the physical and moral wear degree of fixed funds are highly significant.

At the Centre of Economic Computation and Economic Cybernetics a number of one-sector and two-sector models with discrete variables have been designed and analyzed. Among the benefits resulting from these models we can mention that the effect exerted by the time factor on the production function could be introduced, a time varying wear rate of production funds could be considered, a.s.o. Moreover, we consider that even the reflection of the economic phenomenon itself can be more accurately achieved in a discrete model rather than in a continuous one.

In the following paragraphs we shall consider the following one-sector discrete model:

$$P_t = F_t(K_t,\ L_t,) \tag{1}$$

$$C_t + I_t = P_t \tag{2}$$

$$K_{t+1} = K_t - \lambda_t K_t + I_t \tag{3}$$

$$max.\ U_T = \sum_{t=0}^{T-1} \alpha^t U_t(C_t) \tag{4}$$

where we have used the following notations:

K_t = the volume of production funds during time t;
L_t = the volume of manpower during time t;
P_t = the ultimate social product during time t, which is a function of the volume of production funds and of the volume of manpower;
C_t = the consumption volume during time t, to which the non-productive accumulations are added;
I_t = the volume of gross investments during time t;
$F_t(.)$ = the production function, a homogeneous concave function of the first degree; and
$U_t(.)$ = the function to be maximized, a concave function.

We shall assume that manpower L is an exogeneous variable, with an exponential increase. Remembering that we assumed the production function to be homogeneous of the first degree, we can change the notation, by "per capita" values. In this case, by agreeing to designate by small letters the per capita values, the model (1) — (4) becomes:

$$P_t = f_t(k_t) \tag{1'}$$

$$c_t + i_t = p \tag{2'}$$

$$k_{t+1} = (1 - \mu_t)k_t + i_t \tag{3'}$$

$$max.\ U_T = \sum_{t=0}^{T-1} \alpha^t u_t(c_t) \tag{4'}$$

We denote by U_t the control parameter of the model, which in this case is the gross accumulation rate, i.e. $u_t = \frac{i_t}{p_t}$, and the model becomes:

$$k_{t+1} = (1 - \mu_t)k_t + u_t f_t(k_t)$$

$$max.\ U_T = \sum_{t=0}^{T-1} \alpha^t u_t[(1 - u_t)p_t]$$

In order to find the optimal solution, we applied Pontryagin's maximum principle for discrete systems and found an effective computation algorithm as well as a number of qualitative properties.

* * *

The economic growth models are not merely endowed with a purely operational significance of finding the optimal values of the various control parameters; they also provide the economic analysis with valuable information. As will be seen later, the analysis allows the setting forth of several important correlations and aspects regarding the efficiency of production funds, the sensitivity of the optimal policy as related to the alteration of the optimum criteria or of the ultimate values exogeneously prescribed to the various parameters, a.s.o.

Many of the model's properties are connected to dual variables, which in economic terms are interpreted in the same way as the dual variables in linear programming. It is well known, however, that these variables provide a great many useful information for the work of economic analysis. In our case, the dual variable q_n satisfies the following equations:

$$q_{t-1} = \alpha[1 - \lambda + f'(\tilde{k}_t)]\, q_t - \alpha f'(\tilde{k}_t)\,(1 - \tilde{u}_t)\,[q_t - u'(\tilde{c}_t)]$$

From the economic viewpoint, q_n is the "shadow cost", i.e. the "efficiency cost" of fixed funds.

We have the following interesting property of the model:

$$q_t = u^t(c_t)$$

if $0 < \tilde{u}_n < 1$. In other words, if for the gross accumulation rate we omit the ordinary values 0 per cent and 100 per cent, at each time interval t, the "efficiency cost" of funds for the optimal solution is equal to the increase of the model's efficiency function, when the consumption fund is increased by one unit.

As to the sensitivity of the optimal value of the function to be maximized in relation to the original value k_0 and to the final value k_T of work endowment with production funds, we have the following relationships:

$$\frac{\partial U_T}{\partial k_0} = u'_{t_0}(c_0)\,[1 - \mu + f'_{t_0}(k_0)] > 0$$

$$\frac{\partial U_T}{\partial k_T} = -\alpha^{N-1} q_T = -\alpha^{N-1} U'(\tilde{C}_T) < 0$$

if

$$0 < u_0 < 1 \text{ and } 0 < u_{N-1} < 1.$$

The first equation shows that the greater the original work endowment with funds, the higher the value of the function to be optimized. The second equation shows that the higher the required final value of fund endowment, the lower the value of the function to be optimized.

Furthermore, the qualitative survey of the model also shows that the final endowment of work which is reached after the period of T years is a function of the original accumulation rate, namely an increasing function. In other words, the higher the original accumulation rate, the longer the time endowing work; this will allow the economic level, for the period immediately following the lapse of T years, to be high enough to provide a steady increase of the population's welfare during the next period.

For illustration purposes we show the evolution set forth by the above model, in which we assumed the production function to be $f(k_t) = 21\ k_t^{0.3}$, the production fund wear to be 5 per cent per annum, while the functional to be optimized is logarithmic.

As can be seen from Table I, it was assumed that the original work endowment (k) is 900. According to the production function considered here, this allows a production of 161.6. If we take the original gross accumulation rate to be 35.1 per cent, it follows that a quantity of 105.1 is allocated to consumption, while gross investments receive 56.5.

It can be noted that each year the optimal rate is located around 34 — 36 per cent. It can be proved that any other solution, except that given in Table I, is worse as compared to the established efficiency function.

Table I

Year	Size of funds per capita (k_t)	Final social product per capita (p_t)	Gross investments per capita (i_t)	Consumption volume per capita (c_t)	Optimal gross accumulation rate
1	900	161.6	56.5	105.1	35.1 %
2	911.6	156.6	51.2	104.5	32.7 %
3	917.2	162.5	56.8	105.7	35.0 %
4	928.2	161.7	55.6	106.1	34.4 %
5	937.4	163.3	56.9	106.4	34.9 %
6	947.5	164.1	57.5	106.6	35.0 %
7	957.6	164.6	57.8	106.8	35.13 %
8	967.6	165.1	58.0	107.1	35.18 %
9	977.3	166.4	59.2	107.2	35.54 %
10	987.6	166.2	58.8	107.4	35.38 %
11	997.0	166.7	59.2	107.5	35.48 %
12	1,006.3	167.2	59.7	107.5	35.67 %
13	1,015.6	167.6	60.1	107.5	35.84 %
14	1,024,8	168.0	60.5	107.5	36.02 %
15	1,034.1	168.5	61.1	107.4	36.26 %

Moreover, it is possible to prove that if in the basic year we would have had a work endowment of 1,000 and the accumulation rate would have been 30 per cent, the optimal rate would have been 30 per cent each year. In other words, we would have had a "stationary" optimal solution. This fact easily results also from the following considerations: the production corresponding to an endowment of 1,000 is 166.8; if we accept a 30 per cent rate, it follows that a quantity of 116.8 will be assigned to consumption, and a quantity of 50 to gross investments. This quantity covers only fixed funds wear plus the manpower endowment newly resulting for the next year. Hence, the volume of fixed funds per capita during the next year is also 1,000 and this situation will go on forever.

In the above model, important factors have been overlooked, namely the technical progress and work productivity increase, which cause that an ever increasing production volume be yearly obtained with the same work endowment. If in the production function the technical progress element and the work productivity increase are also included the evolution is faster. However, even under these circumstances, the reaching of a high level in

economy involves a high accumulation rate. For illustration purposes, we shall consider two numerical examples.

Table II shows the evolution of economy over a period of seven years assuming the production function to be:

$$p_t = 20\ (1.05)^t k_t^{0.3};$$

Table II

Year	Size of funds per capita (k_t)	Final social product per capita (p_t)	Gross investments per capita (i_t)	Consumption volume per capita (c_t)	Optimal gross accumulation rate
1	1,000	332.0	132	200	39.0 %
2	1,082	358.2	135.3	223.2	38.7 %
3	1,163.2	394.8	144.8	250	36.7 %
4	1,249.9	429.8	149.0	280.8	36.47 %
5	1,336.4	466.6	152.1	314.5	32.6 %
6	1,421.7	505.3	150,7	354.6	27.8 %
7	1,501.4	545.2	145.2	400	26.6 %

as can be seen, the annual production increase, as a result of technical progress and the increase of work productivity, was supposed to be 5 per cent.

Starting with a work endowment of 1,000 and an original accumulation rate of 39 per cent, the production volume is 332, out of which 200 are assigned to consumption. After seven years have elapsed, the level of work endowment reaches 1,501.4 and the optimal rate reaches 26.6 per cent.

Under these circumstances, the production level increases from 332 to 545.2, while the consumption level increases from 200 to 400.

The evolution of economy is described in Table III, for the case where the production function is:

$$P_t = 20\,(1.05)^t k_t^{0.3};$$

Table III

Year	Size of funds per capita (k_t)	Final social product per capita (p_t)	Gross investments per capita (i_t)	Consumption volume per capita (c_t)	Optimal gross accumulation rate
1	800	156	31	125	20 %
2	791	162.2	36.1	126.1	22.3 %
3	788.6	171.3	43.6	127.7	25.4 %
4	792.7	180	50.3	129.7	28.0 %
5	803.4	189.8	57.6	132.2	30.3 %
6	820.8	200.6	65.5	135.1	32.6 %
7	845.3	211.0	72.7	138.3	34.5 %

while the level of the original work endowmen t is 800.

If one starts with a low accumulation rate, i.e. 20 per cent, the mathematical computation shows that this must steadily increase, so that after seven years it reaches 35 per cent.

At the end of the period the level of fund endowment reaches 845.3, the production level 211 and the consumption level reaches 138.

As can be seen from these two examples, if at the beginning of the period one starts with a lower gross acumulation rate, over time has to be increased, in order to optimize the production level; conversely, if one starts with a very high gross accumulation rate, in the long run the accumulation already achieved permits the reduction of this rate.

* * *

Another particularly important control parameter is the ratio of sector I (production of means of production) to sector II (production of consumer goods). As the practical knowledge of countries all over the world has proved, the priority development, under specific historical circumstances, of the production of means of production is the essential factor allowing the reaching within a comparatively short time of a high economic level.

The continuous extension of production under socialism does not require a mere priority development of sector I, but the setting up of consistent rates instead, which by taking into account the objective circumstances prevailing each time, allows a better use to be made of the natural resources and the uninterrupted increase of social work productivity. Consequently, a problem having a particular practical significance is the quantitative ascertainment, for each stage, of the optimal ratio of the production of means of production development and the development of consumer goods production.

We shall now consider the following two-sector model:

$$k^1_{t+1} = (1 - \lambda^1_t)\, k^1_t + u_t f(k^1_t)$$

$$k^2_{t+1} = (1 - \lambda^2_t)\, k^2_t + (1 - u_t)\, f(k^2_t)$$

$$max.\ U_T = \Sigma \alpha_n U(k^2_t)$$

where k^1_t and k^2_t stand for work endowment with funds in sector I and II, respectively, the remaining notations being similar to those considered in the one-sector model.

Assuming the original values k^1_0 and k^2_0 as known, as well as the final values which are to be reached by the two indexes at the end of the T year period, by applying Pontryagin's maximum principle it is possible to find the annual optimal value of the control parameter u_t, which in the present case is the weight of investments to be made for sector I.

As an example, Table IV shows the progress of parameters belonging to a two-sector model, where the production function is $f(k_t) = 21\, k_t^{0.3}$, the fund wear in both sectors is 5 per cent, and the functional to be optimized is logaritmic.

It can be seen that we assumed the technical endowment of work to be in the first year 500 in sector I, and 300 in sector II. The original weight of investments in sector I was considered to be 62.5%; consequently, out of sector I production volume of 135.49, 86.71 units were assigned to sector I and 48.78 units to sector II. As can be seen in Table IV, in order to optimize the suggested ratio, the investement weight in sector I must constantly in-

Table IV

Year	Volume of production funds in sector I $k^1_{n+1} = 0.95k^1_n + 21\,k^{0.3}_n\,U_n$	Volume of production funds in sector II $k^2_{n+1} = 0.95\,k^2_n + 21\,k^{0.3}_n\,(1 - U_n)$	Production of sector I $21\,k^{0.3}_n$	Optimal weight of gross investments in sector I within the total gross investments (control parameter) U_n	Size of gross investments in sector I	Size of gross investments in sector II	Weight of production funds in sector I
1	500	300	135.49	64.0%	86.71	48.78	62.5%
2	561.7	333.7	140.32	64.5%	90.51	49.81	62.7%
3	624.13	367.04	144.81	65.5%	94.85	49.96	62.9%
4	687.77	398.64	149.08	66.2%	99.17	49.91	63.3%
5	752.47	429.1	153.88	67.0%	103.09	50.79	63.7%
6	817.93	458.4	157.03	67.75%	106.4	50.63	64.09%
7	883.43	486.14	160.7	66.1%	106.2	54.5	64.5%
8	945.49	516.3	246.05	72.0%	118.1	45.95	64.7%
9	1,016.33	536.44	167.6	70.0%	117.32	50.28	65.4%
10	1,082.85	559.9	170.83	69.5%	118.72	52.11	65.9%
11	1,147.44	584.0	173.81	73.5%	127.75	46.06	66.3%
12	1,217.82	600.87	176.97	72.5%	128.3	48.67	66.06%
13	1,285.23	619.49	179.84	73.5%	132.19	57.65	67.47%
14	1,353.16	636.18	182.63	74.0%	135.14	47.49	68.02%
15	1,420.65	651.85	185.3	74.5%	135.05	50.25	68.54%
16	1,487.67	666.51	187.9	76.5%	143.74	44.16	69.06%

crease, reaching 69 per cent during the 16th year. The endowment with funds reaches in the last year the values 1,487.67 in sector I and 666.51 in sector II, respectively. In other words, while in the first year the ratio of work endowment in sector I and II was 64 per cent, it attains 76.5 per cent in the last year.

Besides their purely operational role, the two-sector models also play a prominent part in the economic analysis, supplying it with valuable information. Much of the information supplied is related also, in this case, to the dual variables belonging to the model, which are given by the following system of equations:

$$\Psi^1_{t-1} = [(1 - \lambda^1_t + u_n f'(k^1_t)]\,\Psi^1_t + (1 - u_n) f'(k^6_t)\Psi^2_t$$

$$\Psi^2_{t-1} = (1 - \lambda^2_t)\,\Psi^2_t + \alpha^t u'(k^2_t)$$

If the weight of investments in sector I has not the ordinary values 0 or 1, the application of Pontryagin's principle leads to the result $\Psi^1_t = \Psi^2_t$. Moreover, this value is:

$$\Psi^1_t = \Psi = \frac{\alpha^t u'(k^2_t)}{\lambda^2_t - \lambda^1_t + f'(k^1_t)}$$

If the wear rate in both sectors is the same, i.e. $\lambda^1_t = \lambda^2_t$, and $\alpha = 1$, the value of the dual variable is:

$$\Psi^1_t = \Psi^2_t = \frac{u'(k^2_t)}{f'(k^1_t)}$$

In other words, the value of the dual variable is equal to the ratio of the efficiency function variation and the production function variation.

Both in the one-sector model and in the two-sector model a special part in the economic analysis is played by the so-called "stationary solutions". For the two-sector model, the stationary solution k^{1*}/k^{2*} and u^* is given by the following relationships:

$$u^* = \frac{\lambda k^*}{f(k^*} \quad k^{2*} = \frac{f(k^1) - \lambda k^{1*}}{\lambda}$$

where k^{1*} is the solution of equation $f'(k^1) = \lambda$.

For the above-mentioned numerical case, the stationary solution has the following values:

$$k^{1*} = 1{,}000,\ k^{2*} = 2{,}336,\ u^* = 30 \text{ per cent.}$$

* * *

In the preceding paragraphs we referred only to problems related to one-sector and two-sector economic growth models, leaving aside the multi-sector models. We shall now refer to some problems regarding the establishment of the optimum criterion, a key problem for any growth model.

Socialist construction has shown in practice that the objective criterion of social work assignment is given by the social demands, expressed through

the tasks set before social production. There is a great number of possible ways to meet the social demand, hence it is necessary to choose among them, the way which is most appropriate to accomodate possibilities and demands. This alternative is called optimal and corresponds to an *economic optimum* of development.

The economic optimum is characterized by several parameters among which a special significance is bestowed upon the optimum criterion.

The opinions prevailing in this respect can be divided into two groups:

(1) at the national economy level the optimum criterion is the national income, and

(2) at the national economy level the optimum criterion is the consumption.

Referring to the first group of opinions, we consider that the problem regarding the economic optimum is not properly stated. A model whose functional is the national income must solve the development problem so that the national income be a maximum. Some supporters of this opinion think that the maximization of the national income should be achieved for the last year of the contemplated time limit, while others think that it should be cumulatively achieved for the entire period.

The economic analysis of this problem shows that a maximal national income can be achieved only if during the preceding year, or during all preceding years, the whole national income has been accumulated. In other words, if the accumulation rate has been 100 per cent. By adopting any other accumulation rate, a decreased national income will obviously result.

Nevertheless, it is impossible to achieve a 100 per cent accumulation rate, as it is necessary to provide for at least the minimum consumption requirements in order to recover the workforce. In ascertaining these requirements, the consumption rate and implicitly the accumulation rate are extracted from the model; the problem is solved, hence the model is objectless.

Similarly, it could also be argued in response to the opinion according to which the optimum criterion should be the consumption during the last year of the period taken into account. It is then required to point out a development path, so that during the last year the highest possible consumption be secured. The problem is reduced to the above mentioned one, as in order to get a maximum consumption during the last year, it is necessary for the national income to be a maximum during the last but one year. Consequently, we are under the same circumstances as in the preceding case.

In answering the problem regarding the optimum criterion, it is necessary to start with the purpose aimed at by the socialist production: the systematic increase of the people's welfare. This means that the entire economic development is aimed at providing for the ever increasing requirements of the members of society. In Romania, the process of building up a modern economy is continued at a high pace; this modern economy is based on a powerful industry and on an advanced agriculture, on successfully utilizing the breakthroughs achieved by the contemporary technical and scientific revolution and making the best possible use of national resources, on the intensive utilization of the overall production potential, on a high social work productivity.

At the national economy level, the objective of the model describing the economic development must be precisely the (individual and social) consumption, expressed as a consumption fund, yearly cumulated over the whole time period contemplated. In order to do this, it is necessary to determine each year either the accumulation rate or the consumption rate (the weight of the consumption fund within the national income).

The accumulation rate is a first order control parameter to which the solution of the major overall economic development problems are directly related. Such problems are: the setting up of a powerful technical and material production basis, the higher utilization of material and labour resources, bridging the existing gap as compared to developed countries, providing a high level of welfare and civilization for the population as a whole. It is then possible to assume that the high accumulation rate is one of the defining features for the economic development and this rate must be the control parameter of an economic growth model.

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MODELES DE CROISSANCE DE L'ECONOMIE NATIONALE

Les auteurs expriment l'opinion que l'utilisation des variables discrètes à la formation des modèles de croissance économique est plus adéquate, présentant des avantages dans le sens de la possibilité d'introduire le facteur temps, d'utiliser un taux variable d'usage des fonds fixes, etc.

On présente un modèle de croissance économique à un seul secteur et un autre à deux secteurs. On fait l'analyse des paramètres de commande du modèle: le taux de l'accumulation et le rapport entre les secteurs I et II. Dans la dernière partie on fait quelques observations concernant le critère d'optimum.

МОДЕЛИ РОСТА НАРОДНОГО ХОЗЯЙСТВА

Авторы высказывают мнение о том, что применение дискретных переменных в моделях экономического роста является гораздо более целесообразным, так как оно обеспечивает выгоды в отношении возможности включения фактора времени, использования переменной нормы износа основных фондов и т.д.

Приводятся односекторная и двусекторная модели; даются примеры и приводится толкование результатов. Рассматриваются параметры управления модели — норма накопления и соотношение между подразделениями I и II. В заключение приводятся некоторые замечания о критерии оптимума.