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A Model and Computer Program for Numerical Simulation of Certain Problems with Boundary Conditions

The numerical solving of Fourier-type problems with a computer results in a high economic efficiency.

Consider an open, bounded field of the tridimensional Euclidian space, and $\Sigma = \mathrm{Fr}\ \Omega$; it is required to be found a function φ defined on $\bar{\Omega} = \Omega \cup \Sigma$ so that

$$\Delta\varphi = 0 \quad \text{in } \Omega \tag{1}$$

$$(a\varphi + bD_n\varphi)|_\Sigma = c \tag{2}$$

where (1) is Laplace's equation;

D_n is derivation operator according to the direction of the normal $\bar{n}$ against Σ;

a, b, c are given functions, continuous on Σ. Equations (1) and (2) are the so-called Fourier's problem [8].

1. THE SIMULATING MODEL

The solving of boundary conditions problems of the type (1) and (2) is most often performed through simulating techniques and proceedings based on the so-called network method [2]. This is an approximation method replacing the partial derivative equations by finite differences equations, so that the boundary conditions problems of the type (1) and (2) are replaced by so-called "discrete" problems [5].

In order to pass from problems (1) and (2) to a "discrete" problem, a network of triorthogonal lines with steps $h_s > 0$ according to the three directions $0_{x_s}(s = 1, 2, 3,)$ of a triorthogonal Cartesian system is applied to field

$\Omega \subset E^3$; the set of points M, called nodes, in which these lines intersect, make up a so-called discretizing network $\mathcal{R}_h$, defined by

$$\mathcal{R}_h = \{M \mid M = (m_1 h_1,\ m_2,\ h_2,\ m_3 h_3),\ m_s \in Nh,\ > 0;\ s = 1,\ 2,\ 3\}, \quad (3)$$

where N is the set of natural numbers. In the particular case $h_s = h$ $(s = 1, 2, 3)$ the network has equal steps and is cube-shaped.

The essential idea of discretizing equations (1) and (2) consists in the replacement of the partial derivatives operator Δ (Laplace operator) by finite differences operator Δ_h which is to be applied to a function Φ defined on $\mathcal{R}_h$; this operator approximates function φ in (1) and (2), defined on Ω. By writing

$$\Omega_h = \{M \mid M \in \mathcal{R}_h \subset \Omega\}, \quad (4)$$

$$\Sigma_h = \{M \mid M \in \mathcal{R}_h \cap \Sigma'\}, \quad (5)$$

$$\bar{\Omega}_h = \Omega_h \cup \Sigma_h, \quad (6)$$

—where Σ' is a polygonal boundary drawn up only on the sides of network $\mathcal{R}_h$ which "best" approximates $\Sigma = \mathrm{Fr}\ \Omega$ — the "discrete" boundary conditions problems relating to Ω_h are stated like this:

Find function Φ defined on $\bar{\Omega}_h$ verifying the system

$$\Delta_h \Phi = 0 \text{ in } \bar{\Omega}_h \quad (7)$$

$$(\alpha\Phi + \beta D_{nh}\Phi) \mid_{\Sigma h} = \gamma \quad (8)$$

where α, β, γ are given on Σ_h and follow from the functions a, b and c, respectively, according to condition (2) by approximating the boundary Σ by Σ', and D_{nh} is the (difference) numerical derivation operator on the direction of the normal $\bar{n}$ to Σ'. The system (7), (8) represents the so-called Fourier's "discrete" problem.

Since set $\bar{\Omega}_h$ is a finite set, function Φ will follow from (7), (8) under the form of a vector

$$\{\Phi(M) \mid M \in \bar{\Omega}_h\}, \quad (9)$$

having a finite number of components, equal to the number n of nodes $M \in \bar{\Omega}_h$. Thus, the solution to problem (1), (2) is expressed by a table with numerical values (9) approximating the values determined by function φ in the nodes of the discretizing network $\mathcal{R}_h$ contained in $\bar{\Omega}$.

In order that system (7), (8) can be compatible, the discretizing operator Δ_h must be found in such a way that the approximation theorem [5] should be satisfied. In agreement with this theorem

$$h \to 0 \Rightarrow \Phi(M_i) \to \varphi(M_0), \quad (10)$$

where M_i are nodes situated at distance h from a central node $M_0 (M_i,\ M_0 \in \bar{\Omega}_h$ $i = 1, 2, \ldots, q$, where q is the number of nodes on which the discretizing is performed).

1.1. *Approximating the Laplace's equation (1) by finite differences equations.* In order to find the discretizing operator Δ_h in (7), we consider a network $\mathcal{R}_h$ with equal steps ($h_s = h$, $s = 1, 2,3$) and — for each node $M = (m_s h) \in \Omega_h$, $s = 1, 2, 3$) — we write the set of neighbouring nodes $\mathcal{R}_q \subset \mathcal{R}_h$:

$$\mathcal{R}_q = \{M \mid M = [(m_s + p_s)\, h];\ m_s \in N;\ p_s \in \{-1, 0, 1\};\ s = 1, 2, 3\} \quad (11)$$

called *characteristic structure* of network $\mathcal{R}_h$ [9]. $\mathcal{R}_q$ has $q + 1 = 27$ nodes; we can, however, consider also characteristic structures with less than 27 nodes as, for instance, in E': $q + 1 = 3$ nodes, in E^2 : $q + 1 = 5$ or 9 nodes, and in E^3: $q + q + 1 = 7$ or 19 nodes, which are subset of $\mathcal{R}_q$ in (11).

As shown in [9], by discretizing equation $\Delta\varphi\ (M) = 0$ for each node $M \in \Omega_h$ on the only basis of the q nodes neighbouring M, that is on the characteristic structure $\mathcal{R}_q$, finite differences equations with $q + 1$ terms can be obtained, of the form

$$\sum_{m_1, m_2, m_3 \in N;\ p_1, p_2, p_3, = -1, 0, 1} \sigma_{(m_1+p_1,\ m_2+p_2,\ m_3+p_3)}\ \Phi_{(m_1+p_1,\ m_2+p_2,\ m_3+p_3)} = 0; \quad (12)$$

where indices m_1, m_2, m_3 stand for node $M = (m_1 h,\ m_2 h,\ m_3 h) \in \Omega_h$, and $\sigma_{(m_s + p_s)} (s = 1, 2,3)$ are certain coefficients, real numbers, which are determined from the condition that equation (12) be an admissible approximation of Laplace's equation (1), with a certain approximation error order $0\ (h^k)$, that is:

$$\Delta\varphi(m_1 h,\ m_2 h,\ m_3 h) = \sum_{p_1, p_2, p_3 = -1, 0, 1} \sigma_{(m_1+p_1,\ m_2+p_2,\ m_3+p}\ \Phi_{(m_1,+p_1,\ m_2+p_2,\ m_3+p_3} + \\ + 0\,(h^k). \quad (13)$$

We can find coefficients $\sigma_{(m_s + p_s)} (s = 1,\ 2,\ 3)$ by development function φ in a Taylor series in points $M \in \mathcal{R}_q$, defined by (11). If $\varphi \in \mathcal{C}^{k+1}\ (\Omega)$, by developing in Taylor series, there follows for each m_1, m_2, $m_3 \in N$:

$$\varphi_{(m_1+p_1),\ m_2+p_2,\ m_3+p_3)} = \varphi_{(m_1,\ m_2,\ m_3)} + h\,(p_1 T_{100} + p_2 T_{010} + \\ + p_3 T_{001}) + \sum_{s=2}^{k-1} h^s \sum_{s_1+s_2+s_3=s} p_1'^{s_1} p_2'^{s_2} p_3'^{s_3} T_{s_1 s_2 s_3} + 0(h^k), \quad (14)$$

$$(p_1, p_2, p_3 \in \{-1, 0, 1\};\ p_1', p_2', p_3' \in \{-1, 1\})$$

where notation

$$T_{s_1 s_2 s_3} = \frac{1}{s_1!\, s_3!\, s_2!}\ D^s_{x_1^{s_1} x_2^{s_2} x_3^{s_3}}\ \varphi_{m_1,\ m_2,\ m_3)}.$$

$$(s = s_1 + s_2 + s_3)$$

was used.

By multiplying equations (14) by $\sigma_{(m_s+p_s)}$ ($p_s = -1, 0, 1$; $s = 1, 2, 3$) and adding them member by member, after regrouping the terms in the right side and taking out values $h^s T_{s_1 s_2 s_3}$ as a common factor, there follows:

$$\sum_{p_1, p_2, p_3 = -1, 0, 1} \sigma_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)}\ \varphi_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)} =$$

$$\sum_{s=0}^{k-1} h^s \sum_{s_1+s_2+s_3} T_{s_1 s_2 s_3} \sum_{p,\, p_1, p_2, p_3 \in \{-1, 0, 1\}} \sigma_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)} + 0(h^k),$$

$$m_1,\ m_2,\ m_3 \in N \tag{15}$$

Since φ is a harmonic function, i.e. satisfies equations $\Delta\varphi = 0$, $\Delta^2\varphi = 0$, $\Delta^3\varphi = 0, \ldots$, then, in order that

$$\sum_{p_1, p_2, p_3, = -1, 0, 1} \sigma_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)}\ \varphi_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)}$$

approximates the Laplace operator with an error size order $0(h^k)$, so that this is of the form (12), we must find the coefficients $\sigma_{(m_s+p_s)}$ ($p_s = -1, 0, 1$; $s = 1, 2, 3$) in the right-side of equation (15) so that this is of the form:

$$\sum_{p_1, p_2, p_3 = -1, 0, 1} \sigma_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)}\ \varphi_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)} =$$

$$= h^2 a\Delta\varphi + h^4 b\Delta^2\varphi + h^6 c\Delta^3\varphi + h^8 d\Delta^4\varphi + \ldots + 0(h^k),\ m_1, m_2, m_3 \in N \tag{16}$$

where $a, b, c, d, \ldots$ are certain arbitrary constants, strictly positive.

Taking into account that

$$\Delta\varphi = D^2_{x_1^2}\varphi + D^2_{x_2^2}\varphi + D^2_{x_3^2}\varphi = 2(T_{200} + T_{020} + T_{002}),$$

$$\Delta^3\varphi = \Delta\Delta\varphi = 24\ (T_{400} + T_{040} + T_{004}) + 8(T_{220} + T_{202} + T_{022}),$$

and carrying out the approximation $0(h^k) = 0$ and $\Phi \cong \varphi$, from condition (16) we get the following expressions for equation (12), approximating by finite differences Laplace's equation:

— on the plane, for $q + 1 = 5$ nodes

$$\Phi_{(m_1-1,\, m_2)} + \Phi_{(m_1+1,\, m_2)} + \Phi_{(m_1,\, m_2-1)} + \Phi_{(m_1,\, m_2+1)} - 4\Phi_{(m_1,\, m_2)} = 0$$

$$m_1,\ m_2 \in N; \tag{17.1}$$

the order of magnitude of the approximation error being $0(h^4)$. In [9] we have made clear that on 5 nodes in the plane an approximation order better than $0(h^4)$ is not possible;

— on the plane, for $q + 1 = 9$ nodes:

$$4[\Phi_{(m_1-1,\, m_2)} + \Phi_{(m_1+1,\, m_2)} + \Phi_{(m_1,\, m_2-1)} + \Phi_{(m_1,\, m_2+1)}] + \Phi_{(m_1-1,\, m_2-1)} +$$
$$+ \Phi_{(m_1+1,\, m_2-1)} + \Phi_{(m_1-1,\, m_2+1)} + \Phi_{(m_1+1,\, m_2+1)} - 20\Phi_{(m_1,\, m_2,\, m_3)} = 0$$

$$m_1,\ m_2 \in N \tag{17.2}$$

whose order of magnitude of the approximation error is $0(h^8)$;

— in the tridimensional space, for $q + 1 = 7$ nodes:

$$\Phi_{(m_1-1,\, m_2,\, m_3)} + \Phi_{(m_1+1,\, m_2,\, m_3)} + \Phi_{(m_1,\, m_2-1,\, m_3)} + \Phi_{(m_1,\, m_2+1,\, m_3)} +$$
$$+ \Phi_{(m_1,\, m_2,\, m_3-1)} + \Phi_{(m_1,\, m_2,\, m_3+1)} - 6\Phi_{(m_1,\, m_2,\, m_3)} = 0, \tag{17.3}$$

$$m_1,\ m_2,\ m_3 \in N$$

whose order of magnitude of the approximation error is $0(h^4)$;

— in the tridimensional space, for $q+1=19$ nodes:

$$2[\Phi_{(m_1-1,\,m_2,\,m_3)}+\Phi_{(m_1+1,\,m_2,\,m_3)}+\Phi_{(m_1,\,m_2-1,\,m_3)}+\Phi_{(m_1,\,m_2+1,\,m_3)}+$$
$$+\Phi_{(m_1,\,m_2,\,m_3-1)}+\Phi_{(m_1,\,m_2,\,m_3+1)}]+\Phi_{(m_1,\,m_2-1,\,m_3-1)}+\Phi_{(m_1,\,m_2+1,\,m_3-1)}+$$
$$+\Phi_{(m_1,\,m_2-1,\,m_3+1)}+\Phi_{(m_1,\,m_2+1,\,m_3+1)}+\Phi_{(m_1-1,\,m_2,\,m_3-1)}+\Phi_{(m_1+1,\,m_2,\,m_3-1)}+$$
$$+\Phi_{(m_1-1,\,m_2,\,m_3+1)}+\Phi_{(m_1+1,\,m_2,\,m_3+1)}+\Phi_{(m_1-1,\,m_2-1,\,m_3)}+\Phi_{(m_1+1,\,m_2-1,\,m_3)}+$$
$$+\Phi_{(m_1-1,\,m_2+1,\,m_3)}+\Phi_{(m_1+1,\,m_2+1,\,m_3)}-24\Phi_{(m_1,\,m_2,\,m_3)}=0;\ m_1,m_2,m_3\in N \quad (17.4)$$

whose order of magnitude of the approximation error is $0(h^6)$;

— in the tridimensional space, for $q+1=27$ nodes:

$$24[\Phi_{(m_1-1,\,m_2,\,m_3)}+\Phi_{(m_1+1,\,m_2,\,m_2)}+\Phi_{(m_1,\,m_2-1,\,m_3)}+\Phi_{(m_1,\,m_2+1,\,m_3)}+$$
$$+\Phi_{(m_1,\,m_2,\,m_3-1)}+\Phi_{(m1,\,m_2,m_3+1)}]+8[\Phi_{m_1,\,m_2-1,\,m_3-1)}+\Phi_{(m_1,\,m_2+1,\,m_3-1)}+$$
$$+\Phi_{(m_1,\,m_2-1,\,m_3+1)}+\Phi_{(m_1,\,m_2+1,\,m_3+1)}+\Phi_{(m_1-1,\,m_2,\,m_3-1)},+\Phi_{(m_1+1,\,m_2,\,m_3-1)}+$$
$$+\Phi_{(m_1-1,\,m_2\,m_3+1)}+\Phi_{(m_1+1,\,m_2,\,m_3+1)}+\Phi_{(m_1-1,\,m_2-1,\,m_3)}+\Phi_{(m_1+1,\,m_2-1,\,m_3)}+$$
$$+\Phi_{(m_1-1,\,m_2+1,\,m_3)}+\Phi_{(m_1+1,\,m_2+1,\,m_3)}]+\Phi_{(m_1-1,\,m_2-1,\,m_3-1)}+\Phi_{(m_1+1,\,m_2-1,\,m_3-1)}+$$
$$+\Phi_{(m_2-1,\,m_2+1,\,m_3-1)}+\Phi_{(m_1+1,\,m_2+1,\,m_3-1)}+\Phi_{(m_1-1,\,m_2-1,\,m_3+1)}+$$
$$+\Phi_{(m_1+1,\,m_2-1,\,m_3+1)}+\Phi_{(m_1-1,\,m_2+1,\,m_3+1)}+\Phi_{(m_1+1,\,m_2+1,\,m_3+1)}-$$
$$-248\Phi_{(m_1,\,m_2,\,m_3)}=0;\ m_1,\ m_2,\ m_3\in N \quad (17.5)$$

whose order of magnitude of the approximation error is $0(h^6)$.

1.2. *Approximating the boundary conditions.* We discretize equation (2), stating the conditions on $\Sigma = \mathrm{Fr}\,\Omega$, taking into account the concrete problem considered. In principle, we shall proceed like this:

a) Boundary Σ, whatever, is approximated by a polygonal boundary Σ' drawn up only on the network lines of $\mathcal{R}_h$, being made up only by the sides intersections on Σ, according to the example in Fig. 1.

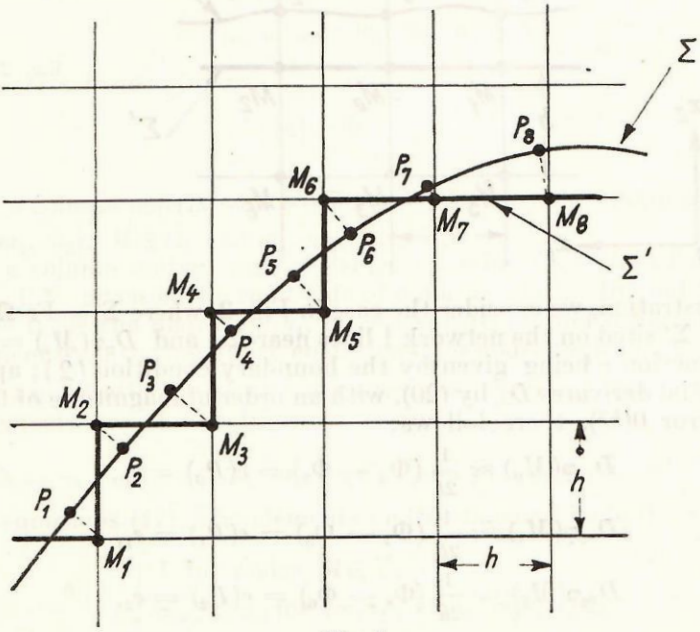

Fig. 1

b) Term $\varphi|_\Sigma = \frac{c}{a} = d$ in condition (2), occurring in Dérichelet problems, is replaced by $\Phi|_{\Sigma_h} = \frac{\gamma}{\alpha}$ from condition (8) in the following way: for each node $M \in \Sigma'$ near Σ, we choose a point $P \in \Sigma$ so that $|MP| < h$ is minimum (Fig. 1) and then consider

$$\Phi(Mi) = d(P_i),\ M_i \in \Sigma',\ P_i \in \Sigma \text{ şi } h > |M_iP_i| = \text{min. } i \in N, \quad (18)$$

function d being given by the boundary condition (2)

c) The term $D_n\varphi|_\Sigma = \frac{c}{b} = e$, from condition (2), which occurs in the case of Neumann's problems, is replaced by $D_{nh}\Phi|_{\Sigma_h} = \frac{\gamma}{\beta}$ from condition (8) obtained by:

$$D_n\varphi(M_i) = D_{x_j}\varphi(M_i) \approx e(P_i),\ M_i \in \Sigma',\ P_i \in \Sigma,\ i \in N,\ j \in \{1, 2, 3\}, \quad (19)$$

that is, by approximating the normal $\bar{n}$ at Σ through one of directions 0_{x_j}, $j \in \{1, 2, 3\}$ normal at the polygonal boundary Σ'; then, the derivatives D_{x_j}, $j = 1, 2, 3$ are approximated by finite differences with the formula:

$$D_{x_j} = \frac{\mu_j}{h_j}\left(\delta_j - \frac{1}{6}\delta_j^3 + \frac{1}{30}\delta_j^5 - \frac{1}{400}\delta_j^7 + \frac{1}{360}\delta_j^9 - \ldots\right), \quad (20)$$

where δ_j is the central difference operator and μ_j the mean operator [6].

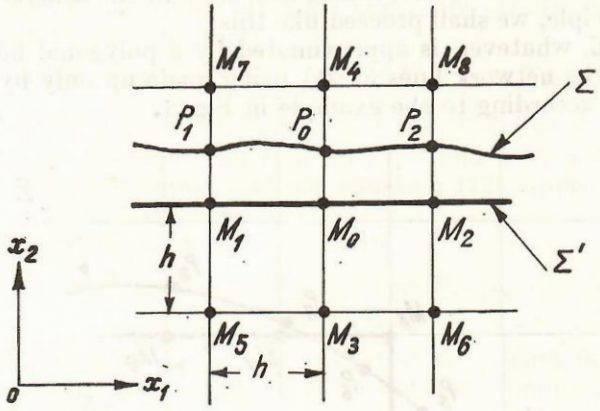

Fig. 2

For illustration, we consider the case in Fig. 2, where $\Sigma = \text{Fr } \Omega$ is approximated by Σ' sited on the network 1 lines near Σ, and $D_n\varphi(M_i) = D_{x_2}\varphi(M_i) = e(P_i)$, function e being given by the boundary condition (2); approximating the partial derivates D_{x_2} by (20), with an order of magnitude of the approximation error $0(h^3)$, there follows:

$$D_{x_2}\varphi(M_0) \approx \frac{1}{2h}(\Phi_4 - \Phi_3) = e(P_0) = e_0,$$

$$D_{x_2}\varphi(M_1) \approx \frac{1}{2h}(\Phi_7 - \Phi_5) = e(P_1) = e_1,$$

$$D_{x_2}\varphi(M_2) \approx \frac{1}{2h}(\Phi_8 - \Phi_6) = e(P_2) = e_2,$$

where from, for nodes $M \in \mathcal{R}_h \subset \bar{\Omega}_h$ there follows:

$$\Phi_4 = 2he_0 + \Phi_3,\ \Phi_7 = 2he_1 + \Phi_5 \text{ and } \Phi_8 2he_2 + \Phi_3 \tag{21}$$

By using the more general notation

$$M_0 = [m_1h_1, m_2h],\ M_1 = [(m_1 - 1)\ h,\ m_2\ h)],\ M_2 = [(m_1 + 1)h,\ m_2h)], \ldots,$$

$$M_8 = [(m_1 + 1)h,\ (m_2 + 1)h]$$

and introducing the values (21) in (17.2),there follows:

$$5\Phi_{m_1, m_2} - \Phi_{(m_1-1, m_2)} - \Phi_{(m_1+1, m_2)} - 2\Phi_{(m_1, m_2-1)} - \frac{1}{2}\Phi_{(m_1-1, m_2-1)} -$$

$$-\frac{1}{2}\Phi_{(m_1+1, m_2-1)} = h\left[2e_{(m_1, m_2)} + \frac{1}{2}e_{(m_1-1, m_2)} + \right.$$

$$\left. + \frac{1}{2}e(m_1 + 1,\ m_2)\right];\ m_1,\ m_2 \in N \tag{22}$$

which is also of the form (12), being used for approximating the Laplace operator in Neumann's problem for the nodes $M \in \Sigma'$.

1.3. *Solving the system* (7), (8). On the basis of the discretizing equations (17), (18) and (22), the system (7), (8) is of the form:

$$\sum_{p_1, p_2, p_3 = -1, 0, 1} \sigma_{(m_1+p_1, m_2+p_2, m_3+p_3)} \cdot \Phi_{(m_1+p_1, m_2+p_2, m_3+p_3)} =$$

$$= f_{(m_1, m_2, m_3)}\ ;\ m_1,\ m_2,\ m_3 \in N \tag{23.a}$$

or, in a matrix form:

$$[\sigma] \cdot [\Phi] = [f] \tag{23.b}$$

where

$[\Phi]$ is a column matrix with n elements, which are the unknowns $\Phi(M) = \Phi(m_1, m_2, m_3)$, $M \in \bar{\Omega}_h$ and $m_1, m_2, m_3 \in N$;

$[f]$ is a column matrix with n elements having the values $f(m_1, m_2, m_3)$, $m_1, m_2, m_3 \in N$, given in the right-side of equations (17), (18) and (22), that is $f_{(m_1, m_2, m_3)} = 0$ for the nodes $(m_1h_1,\ m_2\ h_2,\ m_3h_3) \in \Omega_h$ and different from zero (in agreement with the boundary conditions) for nodes $(m_1h_1,\ m_2h_2,\ m_3h_3) \in \Sigma_h$;

$[\sigma]$ is a quadratic matrix of order n (n being the number of nodes $M \in \bar{\Omega}_h$), whose elements are the coefficients

$$\sigma_{(m_1+p_1, m_2+p_2, m_3+p_3)},\ m_1,\ m_2,\ m_3 \in N \text{ and } p_1,\ p_2,\ p_3 = -1,\ 0,\ 1,$$

from the equations (17). The elements on the diagonal have the values:

$$\sigma_{ii} = \begin{cases} 1 \text{ for nodes } M \in \Sigma_h \\ \sigma_{(m_1, m_2, m_3)} \text{ for } (m_1h_1\ ,\ m_2h_2,\ m_3h_3) \in \Omega_h \end{cases} \tag{24}$$

the other elements being

$$\sigma_{ij} = \begin{cases} 0 \text{ for } M \in \Sigma_h \\ \sigma_{(m_1+p_1,\, m_2+p_2,\, m_3+p_3)} \; (m_s h_s) \in \Omega_h, \; p_s \in \{-1, 0, 1\} \end{cases} \tag{25}$$

$s = 1, 2, 3$ and for at least one s $p \neq 0$

On each line and each column of $[\sigma]$ there are $1 \div q + 1$ elements different from zero ($q + 1$ is the number of nodes of the characteristic structure on which discretizing is carried out).

Matrix $[\sigma]$ has the following peculiarities:

(*i*) It has a diagonal preponderance, that is

$$|\sigma_{ii}| \geqslant \sum_{\substack{j=1 \\ i \neq j}}^{n} |\sigma_{ij}|, \quad i = 1, 2, \ldots, n. \tag{26}$$

(*ii*) It is band-on-portions shaped, that is

$$\sigma_{ij} = 0 \quad \text{if} \quad |i - j| > \frac{1}{2} l_s \quad s \in N$$

where l_s stands for the band width on portion s (l_s is the number of lines with the same elements translated by a step, so that σ_{i+j}, $j + 1 = \sigma_{ij}$. This peculiarity makes possible to simplify the computation program.

(*iii*) It is non-singular, since the matrix, having a diagonal preponderance according to property (*i*), satisfies a theorem of Hadamar's [10].

As a result of property (*iii*), matrix $[\sigma]$ is invertible and system (7), (8) or (23) always complies with solution

$$[\sigma] = [\sigma]^{-1} \cdot [f] \tag{27}$$

Since, in order to better approximate the partial derivative equations by finite differences equations a relatively great number (usually thousands) of nodes $M \in \bar{\Omega}_h$ is chosen, system (23) has very large sizes and exact methods for solving such a system cannot be applied even with quick computing equipment. In such cases, computing methods based on successive approximations are used [1], [4], [7].

In a linear iterative process, the exact solution (27) is approximated by column vector $[\Phi^{(l+1)}] \in R^n$ found after l successive iterations which have led to the row of column vectors

$$[\Phi^{(1)}], [\Phi^{(2)}], \ldots, [\Phi^{(l)}] \in R^n, \tag{28}$$

in agreement with a recursive formula having the form

$$[\Phi^{(l+1)}] = [A_l] [\Phi^{(l)}] + [B_l] [f], \tag{29}$$

where matrices $[B_1], \ldots, [B_l]$ and $[A_1], \ldots, [A_l]$ must be chosen so that row (28) tends towards the exact solution (27), that is the convergence condition

$$\lim_{l \to \infty} [\Phi^{(l)}] = [\Phi] \tag{30}$$

must be satisfied.

Matrices $[B_1], ..., [B_l]$ are independent of $[f]$ and are previously chosen, after which matrices $[A_1], ..., [A_l]$ are found by means of relation

$$[A_l] = [I] - [B_l][\sigma] \tag{31}$$

where $[I]$ is the unit matrix.

If choosing

$$[B_1] = ... = [B_l] = [B], \text{ it follows } [A_1] = ... = [A_l] = [A], \tag{32}$$

and the iterative process is called stationary; the convergence condition (30) is satisfied when

$$\lim_{l \to \infty} [A]^l = 0 \tag{33}$$

The approximation error for an iteration l is a column vector $[e^{(l)}] \in R^n$ determined by the differences

$$[e^{(l)}] = [\Phi^{(l)}] - [\Phi] \tag{34}$$

In [1, 7] it is pointed out that if $\| \quad \|_v$ is an Euclidian norm on R^n and $\| \quad \|_m$ is a norm on the matrices algebra compatible with $\| \quad \|_v$, then

$$\frac{\| e^{(l+1)} \|_v}{\| e^{(1)} \|_v} \leqslant (\| A \|_m)^l \tag{35}$$

where $\| e^{(l+1)} \|_v$ is the Euclidian norm of error vector from iteration $(l+1)$, while $\| e^{(1)} \|_v$ is the same norm for the error at the first iteration. The ratio in (35) expresses the convergence speed, of the stationary iterative process; the less $\| A \|_m$, the quicker is the convergence of an iterative process.

For a stationary iterative approximation of solution in system (23), this system can be written

$$\Phi_{(m_1, m_2, m_3)} = \frac{f(m_1, m_2, m_3)}{\sigma_{(m_1, m_2, m_3)}} - \frac{1}{\sigma_{(m_1, m_2, m_3)}} \sum_{p_1, p_2, p_3 = -1, 0, 1} \sigma_{(m_1+p_1, m_2+p_2, m_3+p_3)} \times$$

$$p_s \neq 0 \text{ for at least one } s = 1, 2, 3$$

$$\Phi_{(m_1+p_1, m_2+p_2, m_3+p_3)}; \quad m_1, m_2, m_3 \in N \tag{36.a}$$

Comparing (36.a) to (29) there follows the shape of matrices $[B]$, which is

$$[B_1] = ... = [B_l] = [B] = \begin{vmatrix} \frac{1}{\sigma_{11}} & & 0 \\ & \frac{1}{\sigma_{22}} & \\ 0 & & \frac{1}{\sigma_{nn}} \end{vmatrix} \tag{37}$$

that is a diagonal matrix with elements $b_{ii} = \frac{1}{\sigma_{ii}}$, where σ_{ii} are given in (24).

By introducing (37) in (31), there follow matrices $[A]$:

$$[A_1] = \ldots = [A_l] = [A] = \begin{vmatrix} 0 & \frac{\sigma_{12}}{\sigma_{11}} & \cdots & \frac{\sigma_{1n}}{\sigma_{11}} \\ \frac{\sigma_{21}}{\sigma_{22}} & 0 & \ddots & \vdots \\ \frac{\sigma_{n1}}{\sigma_{nn}} & \frac{\sigma_{n2}}{\sigma_{nn}} & \cdots & 0 \end{vmatrix} \tag{38}$$

that is a quadratic matrix of order n with the elements on the diagonal equal to zero ($a_{ii} = 0$), while the other elements are $a_{ij} = \frac{\sigma_{ij}}{\sigma_{ii}}$, $i \neq j$ and $i, j = 1, \ldots, n$, in agreement with (25).

This means that an approximation $l + 1$ of the exact solution (27) is found through the process:

$$[\Phi^{(l+1)}] = [A]\,[\Phi^{(l)}] + [B][f], \tag{36.b}$$

where matrices $[A]$ and $[B]$ are given by (37) and, respectively, by (38).

Using the suggested discretizing models (17), (18) and (22), the iterative process (36.b) is always convergent since, in any case, according to (26), (38) and to a theorem of Ostrovski's [10], it follows that the proper values of matrix $[A]$ satisfy inequality $|\lambda_k| < 1$; $k = 1, \ldots, n$, which is, as stated in [1], the necessary and sufficient condition that the boundary condition (30) be satisfied.

To appreciate the convergence speed, we refer to (35) and consider a norm of matrix $[A]$ compatible with the Euclidian norm of column vector of error $\| e \|_v = \sqrt{\sum_{i=1}^{n} | e_i |^2}$. Such a norm of $[A]$ is the spherical norm defined by $\| A \|_m = \sqrt{\sum_{i,j=1}^{n} | aij |^2}$ which is always compatible with the above norm $\| e \|_v$, satisfying condition $\| Ae \|_v \leqslant \| A \|_m \| e \|_v$.

Then, in case of a problem (7), (8) on any domain $\overline{\Omega}_h$ with m nodes $M \in \Omega_h$ the norm $\| A \|_v$ is:

— for the model with $q + 1 = 5$ nodes (on the plane) $\| A_5 \|_m = \left(m_4 \frac{1}{16}\right)^{\frac{1}{2}} = k \times 0.25$;

— for $q + 1 = 9$ nodes (on the plane)

$$\| A_9 \|_m = \left[m\left(4\,\frac{16}{400} + 4\,\frac{1}{400}\right)\right]^{\frac{1}{2}} = k \cdot 0.17$$

— for $q + 1 = 7$ nodes (in the tridimensional space)

$$\| A_7 \|_m = \left(m6\,\frac{1}{36}\right)^{\frac{1}{2}} = k \times 0.166 \ldots;$$

— for $q + 1 = 19$ nodes (in the tridimensional space)

$$\| A_{19} \|_m = \left[m \left(6 \frac{4}{576} + 12 \frac{1}{576} \right) \right]^{\frac{1}{2}} = k \times 0.0666 \ldots ;$$

— and m for $q + 1 = 27$ (in the tridimensional space)

$$\| A_{27} \|_m = \left[m \left(6 \frac{576}{61,504} + 12 \frac{64}{61\,508} + 8 \frac{1}{[61,508} \right) \right]^{\frac{1}{2}} = k \times 0.0688$$

Hence, it follows that the quickest convergence speed is achieved when the discretizing models on $q + 1 = 9$ nodes (on the plane) and on $q + 1 = 19$ nodes (in the tridimensional space) are used. Depending on the problem size, the time required for computation shall also be estimated because, with respect to the 9-node model, for each iteration should be performed $4m$ additions and m products more than in the 5-node model, and with respect to the 19-node model should be performed 12. m additions and products more than in the 7-node model, which cannot be counterbalanced — in certain cases — by diminishing the number of iterations.

2. THE COMPUTATION ALGORITHM

The replacement of the boundary conditions problems (1,) (2), by "discrete" problems (7), (8) leads to systems of large size linear algebraic equations (thousands and tens of thousands of equations and unknowns), as a result of the necessity for achieving an as good approximation as possible. Because of the extremely large size of system (7), (8) or (23), the linear iterative methods — particularly those implying a linear or quadratic acceleration, as well as the relaxation and superrelaxation processes [7] require quite a great number of locations in the computer store ($2n \div 4n$ locations, besides those necessary for the program instructions).

For this reason, as a numerical simulation procedure we suggest one based on the Gauss-Seidel method (it is known that if, for (A) in (38) norm $\| A \|_m = \max_i \sum_{j=1}^{n} | aij | < 1$ then this method is more quickly convergent than a linear iterative procedure). The convergence conditions of the row of successive approximations performed in agreement with the Gauss-Seidel method, that is matrix $[\sigma]$ in (23.b) must have $| \sigma ii | > \sum_{\substack{j=1 \\ i \neq j}}^{n} | \sigma ij |$ for at least one $i = 1, 2, \ldots, n$ being always observed for the models suggested (17), (18) and (22), the main advantage of this procedure lies in the fact that it requires but a little more than n locations in the operating store (n is the number of unknowns of system (23))

The computation algorithm for solving system (23) is the following:

1° n locations are provided in the computer store, for nodes $M \in \overline{\Omega}_h$. The known values $\Phi (M)$, given by the boundary conditions, are introduced in the locations corresponding to nodes $M \in \Sigma_h$. In the remaining location-we introduce the initial values, for instance the same value $\Phi_{(M)}^{(0)} = a$, $M \in \Omega_h$s,

arbitrarily chosen, preferably from the interval bounded by the extreme values given by the boundary.

Depending on the precision required, a value E is chosen for the maximum difference admitted between the values of two successive iterations of the same components; we shall call residuum this difference. We introduce $x = 0$, as initial value, in the location where the maximum value of the residuum for an iteration is maintained.

2° We compute, with the sequence determined by $m_1, m_2, m_3 \in N$

$$\Phi^{(l+1)}_{(m_1, m_2, m_3)} = \frac{1}{\sigma_{(m_1, m_2, m_3)}} [\sigma_{(m_1-1, m_2-1, m_3-1)} \Phi^{(l+1)}_{(m_1-1, m_2-1, m_3-1)} +$$

$$+ \sigma_{(m_1-1, m_2, m_3-1)} \Phi^{(l+1)}_{(m_1-1, m_2, m_3-1)} + \sigma_{(m_1-1, m_2-1, m_3-1)}$$

$$\Phi^{(l+1)}_{(m_1-1, m_2+1, m_3-1)} + \sigma_{(m_1, m_2-1, m_3-1)} \Phi^{(l+1)}_{(m_1, m_2-1, m_3-1)} + \sigma_{(m_1, m_2, m_3-1)}$$

$$\Phi^{(l+1)}_{(m_1, m_2, m_3-1)} + \sigma_{(m_1, m_2+1, m_3-1)} \Phi^{(l+1)}_{(m_1, m_2+1, m_3-1)} + \sigma_{(m_1+1, m_2-1, m_3-1)}$$

$$\Phi^{(l+1)}_{(m_1+1, m_2-1, m_3-1)} + \sigma_{(m_1+1, m_2, m_3-1)} \Phi^{(l+1)}_{(m_1+1, m_2, m_3-1)} +$$

$$+ \sigma_{(m_1+1, m_2+1, m_3-1)} \Phi^{(l+1)}_{(m_1+1, m_2+1, m_3-1)} + \sigma_{(m_1-1, m_2-1, m_3)}$$

$$\Phi^{(l+1)}_{(m_1-1, m_2-1, m_3)} + \sigma_{(m_1-1, m_2 m_3)} \Phi^{(l+1)}_{(m_1-1, m_2, m_3)} + \sigma_{(m_1-1, m_2+1, m_3)}$$

$$\Phi^{(l+1)}_{(m_1-1, m_2+1, m_3)} + \sigma_{(m_1, m_2-1, m_3)} \Phi^{(l+1)}_{(m_1, m_2-1, m_3)} + \sigma_{(m_1, m_2+1, m_3)}$$

$$\Phi^{(l)}_{(m_1, m_2+1, m_3)} + \sigma_{(m_1+1, m_2-1, m_3)} \Phi^{(l)}_{(m_1+1, m_2-1, m_3)} + \sigma_{(m_1+1, m_2, m_3)}$$

$$\Phi^{(l)}_{(m_1+1, m_2 m_3)} + \sigma_{(m_1+1, m_2+1, m_3)} \Phi^{(l)}_{(m_1+1, m_2+1, m_3)} + \sigma_{(m_1-1, m_2-1, m_3+1)}$$

$$\Phi^{(l)}_{(m_1-1. m_2-1, m_3+1)} + \sigma_{(m_1-1, m_2, m_3+1)} \Phi^{(l)}_{(m_1-1, m_2, m_3+1)} +$$

$$+ \sigma_{(m_1-1, m_2+1, m_3+1)} \Phi^{(l)}_{(m_1-1, m_2+1, m_3+1)} + \sigma_{(m_1, m_2-1, m_3+1)} \Phi^{(l)}_{(m_1, m_2-1, m_3+1)} +$$

$$+ \sigma_{(m_1, m_2, m_3+1)} \Phi^{(l)}_{(m_1, m_2, m_3+1)} + \sigma_{(m_1, m_2+1, m_3+1)}$$

$$\Phi^{(l)}_{(m_1, m_2+1, m_3+1)} + \sigma_{(m_1+1, m_2-1, m_3+1)} \Phi^{(l)}_{(m_1+1, m_2-1, m_3+1)} +$$

$$+ \sigma_{(m_1+1, m_2, m_3+1)} \Phi^{(l)}_{(m_1+1, m_2, m_3+1)} + \sigma_{(m_1+1, m_2+1, m_3+1)}$$

$$\Phi^{(l)}_{(m_1+1, m_2+1, m_3+1)}] + \frac{F_{(m_1, m_2, m_3)}}{\sigma_{(m_1, m_2, m_3)}}; \; m_1, \; m_2, \; m_3 \in N \tag{39}$$

and the residuum

$$X' = | \Phi^{(l+1)}_{(m_1, m_2, m_3)} - \Phi^{(l)}_{(m_1, m_2, m_3)} |; \; m_1, \; m_2, \; m_3 \in N \tag{40}$$

If $X' > X$ (initially $X = 0$), then the value of X is replaced by X'. After each full iteration, x is stated again to be zero before starting the computation of the first variable in the next iteration. In this way, at the end of each

iteration we know the maximum value of the residuuum for that equation:

$$X'_{(l+1)} = \max_{m_1, m_2, m_3} (| \Phi^{(l+1)}_{(m_1, m_2, m_3)} - \Phi^{(l)}_{(m_1, m_2 m_3)} |); \quad m_1, m_2, m_3 \in N \tag{41}$$

3° At the end of each iteration the maximum value from (41) is compared to E, if

$$X'_{(l+1)} \leqslant E \tag{42}$$

then vector $\{ \Phi_{(m_1, m_2, m_3)} | \ m_1, m_2, m_3 \in N \}$ is a satisfactory approximation for the exact solution (27) and computation ceases, after which values $\Phi^{(l+1)}_{(m_1, m_2, m_3)}$ are written or, on the basis of these values, the equiscalar lines $\varphi(P) = \text{ct}$; $P \in \Omega$ are drawn with a curve-tracer coupled with the computer.

The computing program in agreement with this algorithm is drawn up taking into account the concrete elements of the problem, particularly determined by the boundary shape Σ' and the boundary conditions, according to which the number of nodes in Ω_h and in Σ'_h, their sequence and numbering are specified.

In order to illustrate the computation program and compare the simulating models suggested, we have considered a Fourier problem conceived in such a way that the exact solutions can be found analytically, which makes possible to estimate the approximations to be achieved by discretizing the partial derivatives equations. Below, we briefly present this problem.

3. A TEST PROBLEM

The following Fourier problem is given: determine function φ defined on a cube $\Phi \subset E^3$ with length edge l (Fig. 3), so that equations

$$\begin{cases} \Delta\varphi = 0 \text{ in } \Omega \\ (a\varphi + bD_n\varphi) \mid_\Sigma = c, \ \Sigma = \text{Fr } \Omega \end{cases} \tag{43}$$

$$a = \begin{cases} 1 \text{ for } x_2 = 1 \text{ and } x_2 = 10 \\ 0 \text{ for } x_1 = 1, \ x_1 = 10, \ x_3 = 1 \text{ and } x_3 = 10 \end{cases}$$

$$b = \begin{cases} 1 \text{ for } x_1 = 1, \ x_1 = 10, \ x_3 = 1 \text{ and } x_3 = 10 \\ 0 \text{ for } x_2 = 1 \text{ and } x_2 = 10 \end{cases}$$

$$c = \begin{cases} 100 \text{ for } x_2 = 1 \\ 190 \text{ for } x_2 = 10 \\ 0 \text{ for } x_1 = 1, \ x_1 = 10, \ x_3 = 1 \text{ and } x_3 = 10 \end{cases}$$

are satisfied.

Since in this case, gradient $\varphi(P)$, $P \in \overline{\Omega}$ is a uniform field of vectors with modulus $|\text{ gradient } \varphi| = \frac{190 - 100}{l}$, the exact solution to problem (43) is:

$$\varphi(P) = 100 + 10 \ (x_2 - 1) \tag{44}$$

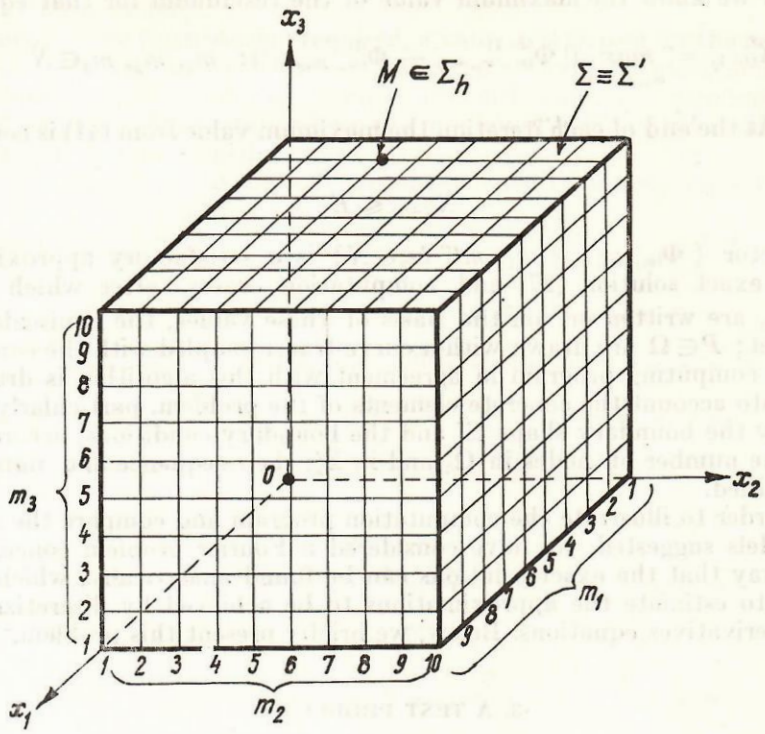

Fig. 3

In order to discretize problem (43), we consider a cubic $\mathcal{R}$ discretizing network $\left(h_1 = h_2 = h_3 = \frac{l}{9}\right)$; in this way it follows $\Sigma' \equiv \Sigma$ and $\Omega_h \equiv \Omega$, the number of nodes being 1,000 in $\overline{\Omega}_h$, 488 in Σ_h (on the cube surfaces), and 512 in Ω_h (inside the cube). Then, function φ defined on the cube $\overline{\Omega}$ is approximated through function Φ defined on the nodes $\overline{\Omega}_h$ and the previously established discretizing expressions (17), (18) and (22) are applied. To this end, the nodes $M \in \overline{\Omega}_h$ are grouped in four sub-sets:

$M \in \{x_2 = 1;\ x_2 = 10\}$, on the surfaces for which restriction $\Phi = \text{ct.}$ has been given, with 200 nodes;

$M \in \Omega_h$, inside the cube, with 512 nodes;

$M \in \{x_1 = 1;\ x_2 = 10;\ x_3 = 1;\ x_3 = 10\}$, on the surfaces where restriction $D_{nh}\,\Phi = 0$ has been given, with 256 nodes;

and the nodes on the four edges parallel with the axis Ox_2, whose number is 32.

Applying the discretizing equations on these nodes, there results a system with 1,000 algebraic linear equations, whose form depends on the model

chosen (with $q + 1 = 7$ or 19 or 27 nodes). For illustration, we write the system obtained by considering the models on 19 nodes:

$$-24\,\Phi_{(m_1,\,m_2,\,m_3)} + 2(\Phi_{(m_1-1,\,m_2\,m_3)} + \Phi_{(m_1+1,\,m_2,\,m_3)} + \Phi_{(m_1,\,m_2-1,\,m_3)} +$$
$$+ \Phi_{(m_1,\,m_2+1,\,m_3)} + \Phi_{(m_1,\,m_2,\,m_3+1)} + \Phi_{(m_1,\,m_2,\,m_3+1)}) + \Phi_{(m_1,\,m_2-1,\,m_3-1)} +$$
$$+ \Phi_{(m_1,\,m_2+1,\,m_3-1)} + \Phi_{(m_1,\,m_2-1,\,m_3+1)} + \Phi_{(m_1,\,m_2+1,\,m_3+1)} +$$
$$+ \Phi_{(m_1-1,\,m_2,\,m_3-1)} + \Phi_{(m_1+1,\,m_2,\,m_3-1)} + \Phi_{(m_1-1,\,m_2,\,m_3+1)} +$$
$$+ \Phi_{(m_1+1,\,m_2,\,m_3+1)} + \Phi_{(m_1-1,\,m_2-1,\,m_3)} + \Phi_{(m_1+1,\,m_2-1,\,m_3)} + \Phi_{(m_1-1,\,m_2+1,\,m_3)} +$$
$$+ \Phi_{(m_1+1,\,m_2+1,\,m_3)}\,;\quad (m_1,\ m_2,\ m_3 = 2,\ 3,\ \ldots\ 9) \tag{45.1}$$

(we obtain equations by applying the formula (17.4)

$$\Phi_{(m_1,\,1,\,m_3)} = 100,\ (m_1,\ m_3 = 1,\ \ldots,\ 10) \tag{45.2}$$

$$\Phi_{(m_1,\,10,\,m_3)} = 190,\ (m_1,\ m_3 = 1,\ \ldots,\ 10) \tag{45.3}$$

$$-24\,\Phi_{(m_1,\,m_2,\,10)} + 2(\Phi_{(m_1-1,\,m_2,\,10)} + \Phi_{(m_1+1,\,m_2,\,10)} + \Phi_{(m_1,\,m_2-1,\,10)} +$$
$$+ \Phi_{(m_1,\,m_2+1,\,10)}) + 4\Phi_{(m_1,\,m_2,\,9)} + \Phi_{(m_1-1,\,m_2-1,\,10)} + \Phi_{(m_1+1,\,m_2-1,\,10)} +$$
$$+ \Phi_{(m_1-1,\,m_2+1,\,10)} + \Phi_{(m_1+1,\,m_2+1,\,10)} + 2\Phi_{(m_1-1,\,m_2,\,9)} + \Phi_{(m_1+1,\,m_2,\,9)} +$$
$$+ \Phi_{(m_1,\,m_2-1,\,9)} + \Phi_{(m_1,\,m_2+1,\,9)}) = 0\,;\ (m_1,\ m_2 = 2,\ \ldots,\ 9) \tag{45.4}$$

$$-24\,\Phi_{(m_1,\,m_2,\,1)} + 2(\Phi_{(m_1-1,\,m_2,\,1)} + \Phi_{(m_1+1,\,m_2,\,1)} + \Phi_{(m_1,\,m_2-1,\,1)} +$$
$$+ \Phi_{(m_1,\,m_2+1,\,1)}) + 4\Phi_{(m_1,\,m_2,\,2)} + \Phi_{(m_1-1,\,m_2-1,\,1)} + \Phi_{(m_1+1,\,m_2-1,\,1)} +$$
$$+ \Phi_{(m_1-1,\,m_2+1,\,1)} + \Phi_{(m_1+1,\,m_2+1,\,1)} + 2(\Phi_{(m_1-1,\,m_2,\,2)} + \Phi_{(m_1+1,\,m_2,\,2)} +$$
$$+ \Phi_{(m_1,\,m_2-1,\,2)} + \Phi_{(m_1,\,m_2+1,\,2)}) = 0\,;\quad (m_1,\ m_2 = 2,\ ..,\ 9) \tag{45.5}$$

$$-24\Phi_{(1,\,m_2,\,m_3)} + 2(\Phi_{(1,\,m_2-1,\,m_3)} + \Phi_{(1,\,m_2+1,\,m_3)} + \Phi_{(1,\,m_2,\,m_3-1)} +$$
$$+ \Phi_{(1,\,m_2,\,m_3+1)}) + 4\Phi_{(2,\,m_2,\,m_3)} + \Phi_{(1,\,m_2-1,\,m_3-1)} + \Phi_{(m_1,\,m_2+1,\,m_3-1)} +$$
$$+ \Phi_{(1,\,m_2-1,\,m_3+1)} + \Phi_{(m_1,\,m_2+1,\,m_3+1)} + 2(\Phi_{(2,\,m_2-1,\,m_3)} + \Phi_{(2,\,m_2+1,\,m_3)} +$$
$$+ \Phi_{(2,\,m_2,\,m_3-1)} + \Phi_{(2,\,m_2,\,m_3+1)}) = 0\,;\quad (m_2, m_3 = 2, \ldots, 9) \tag{45.6}$$

$$-24\Phi_{(10,\,m_2\,m_3)} + 2(\Phi_{(10,\,m_2-1,\,m_3)} + \Phi_{(10,\,m_2+1,\,m_3)} +$$
$$+ \Phi_{(10,\,m_2,\,m_3-1)} + \Phi_{(10,\,m_2,\,m_3+1)}) + 4\Phi_{(9,\,m_2,\,m_3)} + \Phi_{(10,\,m_2-1,\,m_3-1)} +$$
$$+ \Phi_{(10,\,m_2+1,\,m_3-1)} + \Phi_{(10,\,m_2-1,\,m_3+1)} + \Phi_{(10,\,m_2+1,\,m_3+1)} + 2(\Phi_{(9,\,m_2-1,\,m_3)} +$$
$$+ \Phi_{(9,\,m_2+1,\,m_3)} + \Phi_{(9,\,m_2,\,m_3-1)} + \Phi_{(9,\,m_2,\,m_3+1)}) = 0\,;\ (m_2,\ m_3 = 2, \ldots, 9) \tag{45.7}$$

(we obtain equations (45.4),..., (45.7) by applying the proceeding described in 1.2.c)

$$-24\Phi_{(1,\,m_2,\,1)}+4(\Phi_{(2,\,m_2,\,1)}+\Phi_{(1,\,m_2,\,2)}+\Phi_{(2,\,m_2\,2)}+$$
$$+2(\Phi_{(1,\,m_2-1,\,1)}+\Phi_{(1,\,m_2+1,\,1)}+\Phi_{(2,\,m_2-1,\,1)}+\Phi_{(2,\,m_2+1,\,1)}+\Phi_{(1,\,m_2-1,\,2)}+$$
$$+\Phi_{(1,\,m_2+1,\,2)})=0;\ (m_2=2,\ldots,9) \tag{45.8}$$
$$-24\Phi_{(1,\,m_2,\,10)}+4(\Phi_{(2,\,m_2,\,10)}+\Phi_{(1,\,m_2,\,9)}+\Phi_{)2,\,m_2,\,9)})+$$
$$+2(\Phi_{(1,\,m_2-1,\,10)}+\Phi_{(1,\,m_2+1,\,10)}+\Phi_{(2,\,m_2-1,\,10)}+\Phi_{(2,\,m_2+1,\,10)}$$
$$+\Phi_{(1,\,m_2-1,\,9)}+\Phi_{(1,\,m_2+1,\,9)})=0;\quad (m_2,=2,\ldots 9) \tag{45.9}$$
$$-24\Phi_{(10,\,m_2,\,1)}+4(\Phi_{(9,\,m_2,\,1)}+\Phi_{(10,\,m_2,\,2)}+\Phi_{(9,\,m_2,\,2)})+$$
$$+2(\Phi_{(10,\,m_2-1,\,1)}+\Phi_{(10,\,m_2+1,\,1)}+\Phi_{(9,\,m_2-1,\,1)}+\Phi_{(9,\,m_2+1,\,1)}+$$
$$+\Phi_{(10,\,m_2-1,\,2)}+\Phi_{(10,\,m_2+1,\,2)})=0;\quad (m_2=2,\ldots,9) \tag{45.10}$$
$$-24\Phi_{(10,\,m_2,\,10)}+4(\Phi_{(9,\,m_2,\,10)}+\Phi_{(10,\,m_2,\,9)}+\Phi_{(9,\,m_2\,9)})+$$
$$+2(\Phi_{(10,\,m_2-1,\,10)}+\Phi_{(10,\,m_2+1,\,10)}+\Phi_{(9,\,m_2-1,\,10)}+\Phi_{(9,\,m_2+1,\,10)}+$$
$$+\Phi_{(10,\,m_2-1,\,9)}+\Phi_{(10,\,m_2+1,\,9)})=0;\quad (m=2,\ldots,9) \tag{45.11}$$

In agreement with (44), the exact solution of system (45) is

$$\Phi_{(m_1,\,m_2,\,m_3)}=100+10(m_2-1);\ m_1,\ m_2,\ m_3=1,2,\ldots,10 \tag{46}$$

On the basis of the computation algorithm described in section 2 (The Computation Algorithm) for solving system (45) through successive approximations, there results the computation diagram in Fig. 4.

The initial values are, according to the block

(1), $\Phi_{(m_1,\,1,\,m_3)}=100$, $\Phi_{(m_1,\,10,\,m_3)}=190$, $\Phi_{(p_1,\,p_2,\,p_3)}=150$ $(m_1,\ m_3=1,\ldots,10)$

and $(p_1,\ p_2,\ p_3=2,,\ldots,9)$, $E=0.01$ (as a maximum value admitted for the residuum), $x=0$, (for the residuum, which is initially nought) and $L=0$ (the number of iterations, which begins from zero). Through the advance block (2) we obtain the number of operation being performed, and the block for initial values (3) establishes the minimum value (m_i) and the maximum value (M_i) that can be assumed by indices m_1, m_2 and m_3 determining the coordinates of the nodes. The diagram comprised between a and b stands for the computation of the values for the nodes inside the cube (which is achieved through the formula (1) in system (45) described above); the portion c—d stands for the computation diagram of the values for the nodes on the surfaces of the cube parallel with the plane x_1Ox_2 (which is achieved through equations (5) and (4) of system (45) described above); e — f stands for the computation diagram of the values for the nodes on the cube surfaces parallel with the plane X_2OX_3 (the computation is performed by means of formulas (7) and (6) in system (45) described above), and g—h stands for the computation diagram of the values for the nodes on the cube edges (with formulas (11), (10), (9) and (8) in system (45) above described). With each iteration, the maximum residuum (written down in location x) is compared with the maximum value admitted, E, according to the conditional transfer block (4); if $x<E$, then values $\Phi^{(L)}_{(m_1,\,m_2,\,m_3)}$, $(m_1,\ m_2,\ m_3=1,\ldots,10)$ and iteration L are

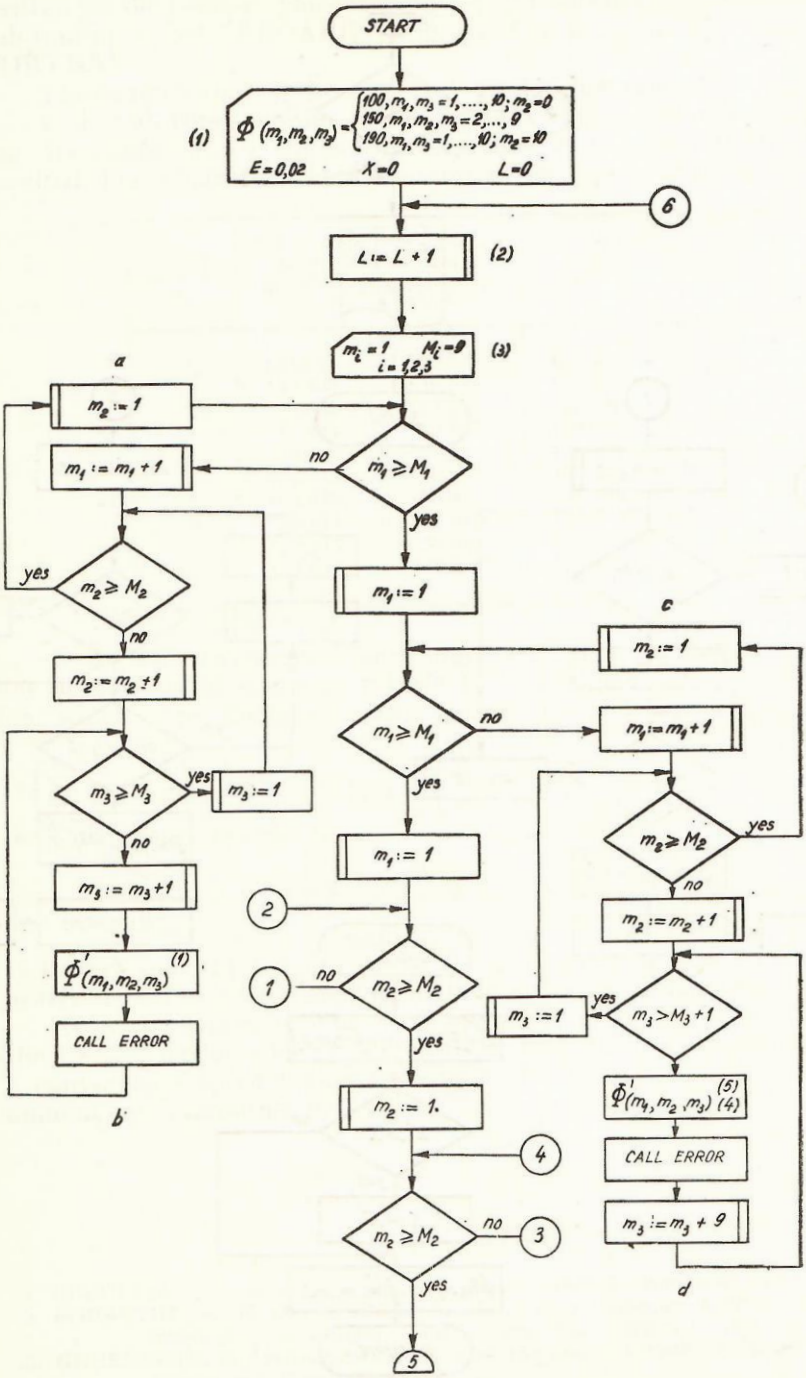
START
(1)
$\Phi(m_1,m_2,m_3)=\begin{cases}100, m_1,m_3=1,\dots,10; m_2=0\\150, m_1,m_2,m_3=2,\dots,9\\190, m_1,m_3=1,\dots,10; m_2=10\end{cases}$
E=0,02 X=0 L=0
6
L := L + 1
(2)
$m_i=1$ $M_i=9$ $i=1,2,3$
(3)
a
$m_2:=1$
$m_1:=m_1+1$
no
$m_1\geq M_1$
yes
yes
$m_2\geq M_2$
no
$m_2:=m_2+1$
$m_3\geq M_3$
yes
$m_3:=1$
no
$m_3:=m_3+1$
$\Phi'(m_1,m_2,m_3)$ (1)
CALL ERROR
b
$m_1:=1$
c
$m_2:=1$
$m_1\geq M_1$
no
$m_1:=m_1+1$
yes
$m_1:=1$
$m_2\geq M_2$
yes
no
$m_2:=m_2+1$
2
1
no
$m_2\geq M_2$
yes
$m_3:=1$
yes
$m_3>M_3+1$
$\Phi'(m_1,m_2,m_3)$ (5) (4)
CALL ERROR
$m_3:=m_3+9$
d
$m_2:=1.$
4
$m_2\geq M_2$
no
3
yes
5

Fig. 4

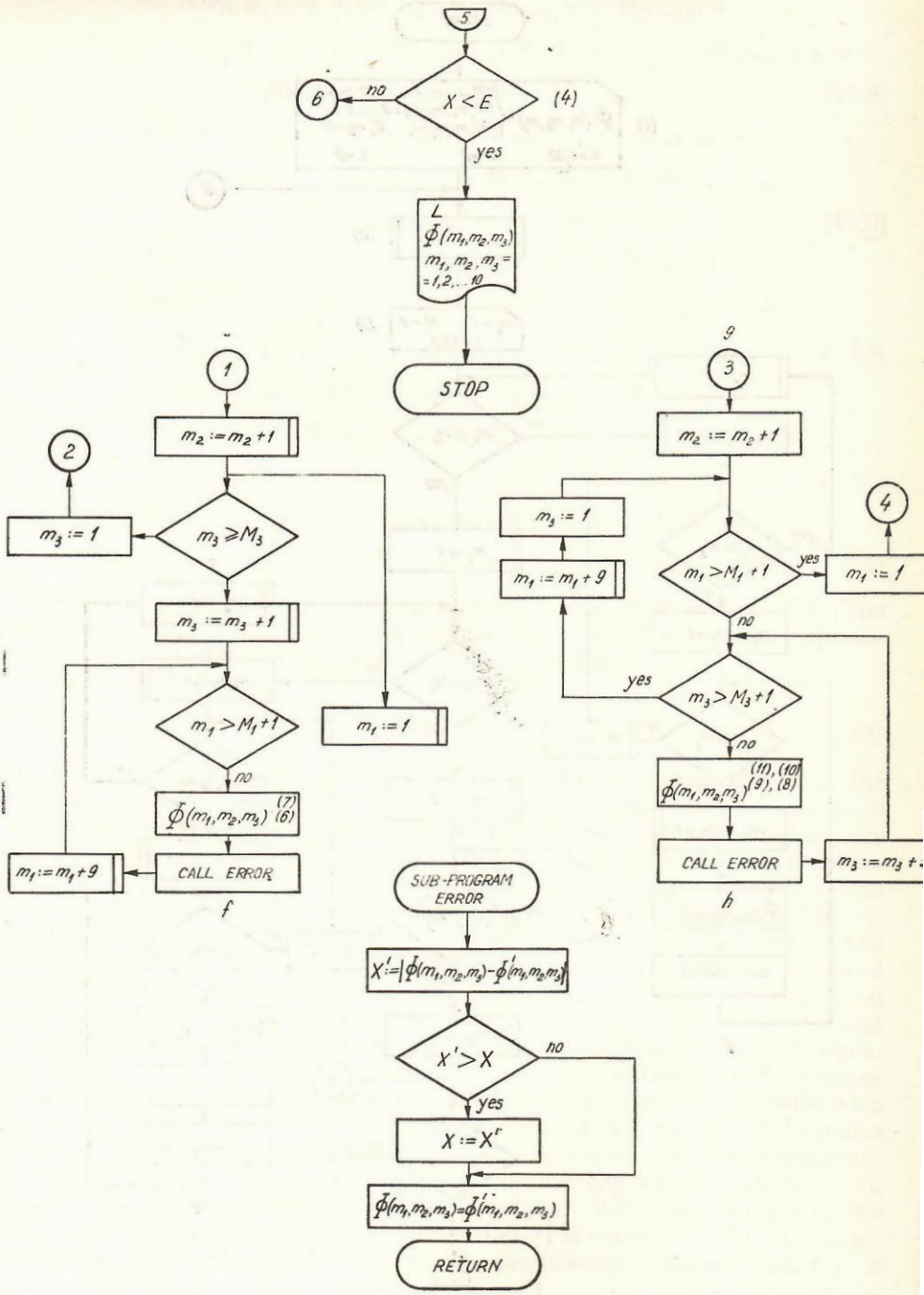
5
6
no
X < E
(4)
yes
Φ(m1,m2,m3)
m1, m2, m3 = 1,2,...10
STOP
1
m2 := m2 + 1
2
m3 := 1
m3 ≥ M3
m3 := m3 + 1
m1 > M1 + 1
m1 := 1
no
Φ(m1,m2,m3) (7) (6)
m1 := m1 + 9
CALL ERROR
f
9
3
m2 := m2 + 1
m3 := 1
m1 := m1 + 9
4
m1 > M1 + 1
yes
m1 := 1
no
yes
m3 > M3 + 1
no
Φ(m1,m2,m3) (11), (10) (9), (8)
CALL ERROR
m3 := m3 +
h
SUB-PROGRAM ERROR
X' := |Φ(m1,m2,m3) − Φ'(m1,m2,m3)|
X' > X
no
yes
X := X'
Φ(m1,m2,m3) = Φ'(m1,m2,m3)
RETURN

Fig. 5

written on the printer. The computation of residuum is performed through a sub-routine called "EROARE" (Fig. 4). The program has been written in FORTRAN.

The computation has been carried out in a few variants:

1° For all three possible models (with $q + 1 = 7$; 19 and 27). The following iterations $L_{19} = L_{27} = 102$ iterations and $L_7 = 136$ iterations have resulted. For illustration, here are some of the values of the approximations:

$m_1 = 5$ $m_2 = 5$ m_3	The 27-node model $\Phi^{(102)}_{(5, 5, m_3)}$	The 19-node model $\Phi^{(102)}_{(5, 5, m_3)}$	The 7-node model $\Phi^{(136)}_{(5, 5, m_3)}$	Exact solution $\Phi_{(5, 5, m)}$
1	139.835	139.839	139.056	
2	139.830	139.834	139.037	
3	139.832	139.835	139.055	
4	139.834	139.835	139.073	
5	139.836	139.836	139.091	140
6	139.838	139.837	139.108	
7	139.840	139.838	139.125	
8	139.841	139.840	139.142	
9	139.843	139.841	139.158	
10	139.848	139.846	139.175	

2° In order to compute the convergence speed, the following computation has been made for each model: $[\Phi^{(100)}]_{(q+1)}$ and $[\Phi^{(l)}]_{(q+1)}$, $(q = 1 + 7, 19, 27)$, for which the Euclidian error norms have been found: $\| e^{(100)} \|_{q+1} = \left(\sum_{i=1}^{800} | e_i^{(100)} |^2 \right)^{\frac{1}{2}}_{q+1}$ and $\| e_i^{(1)} \|_{q+1} = \left(\sum_{i=1}^{800} | e_i^{(1)} |^2 \right)^{\frac{1}{2}}_{q+1}$, $(q + 1 = 7,\ 19,\ 27)$ and the convergence speeds (35):

$$\frac{\| e^{(100)} \|_{27}}{\| e^{(1)} \|_{27}} = 0{,}00\,208; \quad \frac{\| e^{(100)} \|_{19}}{\| e^{(1)} \|_{19}} = 0{,}00\,205; \quad \frac{\| e^{(100)} \|_{7}}{\| e^{(1)} \|_{7}} = 0{,}01\,097$$

It follows that, by using 19-node and 27-node models, a convergence speed nearly six times higher than with the 7-node model has been obtained.

Now, to conclude: The most advisable numerical simulation model is the 19-node model, which is more precise than the 7-node model and achieves a convergence speed higher than with the 27-node model, with a smaller amount of computation.

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MODELE ET PROGRAMME DE CALCUL EN VUE DE LA SIMULATION NUMERIQUE DE CERTAINS PROBLEMES AYANT DES CONDITIONS AUX LIMITES

Le modèle de simulation et le programme présentés se rapportent à la résolution numérique, à l'aide des ordinateurs électroniques, de certains problèmes ayant des conditions aux limites Fourier, où interviennent des équations à dérivées partielles du genre élliptique.

Le programme est échafaudé en utilisant un modèle de simulation comprenant des équations rendues discrètes et ayant des dérivées partielles du genre élliptique. L'action de rendre discrètes ces équations a été faite sur un réseau à pas égaux, pour chaque noeud utilisant seulement les noeuds voisins. De cette façon, la solution du problème ayant des conditions aux limites est obtenue par approximation travaillant avec les solutions d'un système d'équations algébriques linéaires qui présentent certaines particularités de la matrice des coefficients.

En tant qu'exemples, on présente les solutions d'un problème Fourier dans l'espace, résout sur un ordinateur IBM 360/30, au moyen du programme avancé.

МОДЕЛЬ И ПРОГРАММА РАСЧЕТА ДЛЯ ЧИСЛЕННОГО СИМУЛИРОВАНИЯ ЗАДАЧ С ПРЕДЕЛЬНЫМИ УСЛОВИЯМИ

Приведенные модель и программа касаются численного решения на электронных калькуляторах задач с предельными условиями Фурье, в которых имеются уравнения с частичными производными элипсового типа.

Программа основывается на применении модели симулирования путем дискретизации уравнений с частичными производными элипсового типа. Дискретизация была произведена по сети с равными шагами путем использования для каждого узла только соседних узлов. Таким образом, решение задачи с предельными условиями находится путем приближений решениями системы алгебраических линейных уравнений, которые отмечают определенные особенности матрицы коэффициентов.

Для примера приводятся решения задачи Фурье в пространстве, разрешенной при помощи предлагаемой программы на калькуляторе I. B. M. 360/30.