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Optimization of the Manufacturing Task Assignment among Operators

Task assignment among several operators, a frequently encountered problem in economic activity, is optimally solved through linear programming.

Consider a manufacturing department in a mechanical engineering enterprise; m types of S_i machines ($i = 1, ..., m$) are available and in its annual plan the manufacturing of n types of items P_j ($j = 1, ..., n$), in amounts a_j, ($j = 1, ..., n$) is provided. Knowing the available time t_i associated to machine S_i during the year, the time t_{ij} required by machine S_i in order to make type P_j item, as well as the cost c_{ij} for making item P_j, the problem is to find the quantities x_{ij} of items P_j to be made by machine S_i in order to meet the provisions included in the production plan, while total costs remain at a minimum.

This kind of problem can be mathemtically formulated as follows:

Determine magnitudes x_{ij}, minimizing the function:

$$f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i \tag{1}$$

under the following conditions:

$$\sum_{j=1}^{n} t_{ij} x_{ij} \leqslant t_i ; \qquad i = 1, ..., m.$$

$$\sum_{i=1}^{m} x_{ij} = a_j ; \qquad j = 1, ..., n. \tag{2}$$

$$x_{ij} \geqslant 0 ; \quad i = 1, ..., m, \quad j = 1, ... n.$$

where a_j, t_i and c_{ij} are positive numbers ($j = 1, ..., n$), ($i = 1, ..., m$). We must specify that $t_{ij} \geqslant 0$

The mathematical formulation of the above-stated problem is somewhat a generalization of transportation problems. G. Dantzig [1] suggests a solving method for this kind of problems by using the similarity to transportation problems.

A solving procedure for this problem is to reduce it to a transportation problem with additional constraints. It consists of a first transformation of the inequation system (2) in an equation system:

$$\begin{aligned}
&\sum_{j=1}^{n} t_{ij} x_{ij} + x_{i,\,n+1} = t_i; \quad i = 1,\ 2, \ldots, m. \\
&\sum_{i=1}^{m} x_{ij} = a_i; \quad j = 1,\ 2, \ldots, n. \\
&x_{ij} \geqslant 0; \ i = 1,\ 2, \ldots, m; \quad j = 1,\ 2, \ldots, n. \\
&x_{i,\,n+1} \geqslant 0, \quad i = 1,\ 2, \ldots, m,
\end{aligned} \tag{3}$$

where function $f(x)$ is maintained in the same form (2), since the coefficients $c_{i,\,n+1} = 0$, for $i = 1, 2, \ldots, m$, correspond to the compensation variables $x_{i,\,n+1}$.

A second transformation is then effected:

$$x_i = \tau_i y_i, \quad i = 1,\ 2, \ldots, m, \tag{4}$$

where x_i, y_j, τ_i stand for the following matrices:

$$x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \\ x_{i,\,n+1} \end{pmatrix}; \quad y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{in} \\ y_{i,\,n+1} \end{pmatrix} \tag{5}$$

$$\tau_i = \begin{pmatrix} 1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 1 & 0 \ldots & 0 & 0 \\ 0 & 0 & 1 \ldots & 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ 0 & 0 & 0 \ldots & 1 & 0 \\ 1-t_{i1} & 1-t_{i2} & 1-t_{i3} \ldots & 1-t_{in} & 1 \end{pmatrix} \tag{5'}$$

while matrix τ_i is non-singular.

Based on transformation (4), system (3) becomes:

$$\begin{aligned}
&\text{(a)} \ \sum_{j=1}^{n+1} y_{ij} = t_i; \ i = 1, 2, \ldots, m. \\
&\text{(b)} \ \sum_{i=1}^{m} y_{ij} = a_j; \quad j = 1, 2, \ldots, n. \\
&\text{(c)} \ y_{ij} \geqslant 0; \quad i = 1,\ 2, \ldots, m, \ j = 1,\ 2, \ldots, n. \\
&\text{(d)} \ y_{i,\,n+1} - \sum_{j=1}^{n} (t_{ij} - 1)\ y_{ij} \geqslant 0; \ i = 1,\ 2, \ldots, m.
\end{aligned} \tag{6}$$

and function $f(x)$ maintains its form (2), where variables are replaced by y_{ij}, $(i = 1, 2, ..., m;\ j = 1, 2, ..., n)$, i.e.

$$g(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij}. \tag{7}$$

Thus, as a result of applying transformation (4), a transportation problem has been obtained, represented by conditions (6 a, b, c) and (7), with the additional constraint (6 d). Two cases can be distinguished in the solution of this problem.

Case I: $c_{ij} = t_{ij}$. The problem can be stated as follows:

Determine

$$min\ g(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} y_{ij} \tag{8}$$

under the following conditions

$$\text{(a)}\ \sum_{j=1}^{n+1} y_{ij} = t_i;\quad i = 1, 2, ..., m.$$

$$\text{(b)}\ \sum_{i=1}^{m} y_{ij} = a_j;\quad j = 1, 2, ..., n.$$

$$\text{(c)}\ \sum_{i=1}^{m} y_{i,n+1} = \sum_{i=1}^{m} t_i - \sum_{j=1}^{n} a_j \tag{9}$$

$$\text{(d)}\ y_{ij} \geqslant 0,\ i = 1, 2, ..., m;\ j = 1, 2, .., n.$$

$$\text{(e)}\ t_i - \sum_{j=1}^{n} t_{ij} y_{ij} \geqslant 0;\ i = 1, 2, ..., m$$

where condition (9 e) is equivalent to condition (6 d). For simplification purposes we shall write:

$$\sum_{j=1}^{n} t_{ij} y_{ij} = \varphi_i(y_i),\ i = 1, 2, ..., m, \tag{10}$$

hence, condition (9 a) will be written as:

$$t_i - \varphi_i(y_i) \geqslant 0;\ i = 1, 2, ..., m. \tag{9 e'}$$

In order to solve this problem, the method of comparative differences [2] can be used, which is encountered in the solution of transportation problems. An intial solution is found, which verifies constraints (9 b) and for which the function $g(y)$ is a minimum. This can be accomplished by using as a starting point the method of the minimum element on the column of matrix

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \ldots t_{1n} \\ t_{21} & t_{22} & t_{23} \ldots t_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ t_{m1} & t_{m2} & t_{m3} \ldots t_{mn} \end{pmatrix}.$$

Let the initial solution be:

$$Y^0 = \begin{matrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_m^0 \end{matrix}$$

The following two situations are possible:

a) $t_i - \varphi_i(y_i^0) \geqslant 0$, for any $i = 1, 2, ..., m$; solution Y^0 verifies conditions (11 e') and is an optimal solution;

b) $t_i - \varphi_i(y_i^0) \leqslant 0$ at least for an $i = 1, 2, ..., m$; we have the strict inequality < 0; the system is inconsistent;

c) $\sum_{i=1}^{m} (t_i - \varphi_i(y_i^0) < 0$; the system is inconsistent;

d) $\sum_{i=1}^{m} (t_i - \varphi_i(y_i^0) \geqslant 0$; the system may be consistent; by applying the method of comparative differences, situation a) or b) will be reached.

NOTE. The number k of components $y_{ij} \neq 0$ in a basic solution satisfies the following double inequality: $m + n \leqslant k \leqslant 2m + n$. Among these components $y_{ij} \neq 0$, a number of $m + n$ components meet conditions

$$t_i - \sum_{j=1}^{n} t_{ij} y_{ij} > 0, \quad i = 1, 2, ..., m$$

Case II. $c_{ij} \neq t_{ij}$. The problem is stated as follows:

$$min\, g(y(= \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} y_{ij} \tag{11}$$

under the following conditions:

$$\text{(a)} \quad \sum_{i=1}^{n+1} y_{ij} = t_i, \quad i = 1, 2, ..., m$$

$$\text{(b)} \quad \sum_{i=1}^{m} y_{ij} = a_j, \quad j = 1, 2, ..., n$$

$$\text{(c)} \quad \sum_{i=1}^{m} y_{i,n+1} = \sum_{i=1}^{m} t_i - \sum_{j=1}^{n} a_j \tag{12}$$

$$\text{(d)} \quad y_{ij} \geqslant 0, \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n$$

$$\text{(e)} \quad \begin{cases} y_{i,n+1} - \sum_{j=1}^{n} (t_{ij} - 1)\, y_{ij} \geqslant 0 \quad \text{or} \\ t_i - \sum_{j=1}^{n} t_{ij} y_{ij} \geqslant 0 \end{cases} \qquad i = 1, 2, ..., m$$

A basic solution of this problem has k components $y_{ij} \neq 0$, where $n + m \leqslant k \leqslant n + 2m$.

Components $y_{i,\,n+1}$ for which we have

$$y_{i,\,n+1} - \sum_{j=1}^{n} (t_{ij} - 1)\ y_{ij} = 0$$

will be called non-essential components, while those for which we have

$$y_{i,\,n+1} - \sum_{j=1}^{n} (t_{ij} - 1)\ y_{ij} > 0$$

will be called essential components of the solution. The number of components included in a solution, except the non-essential ones, is $n + m$, at most. If this number of components is precisely $n + m$, the solution will be called non-degenerate, while otherwise we have a degenerate solution.

The dual problem corresponding to this problem is stated as follows: Determine

$$max\ w(u,\ v) = \sum_{i=1}^{m} t_i u_i + \sum_{j=1}^{n} a_j v_j \tag{13}$$

under the following conditions

$$\begin{aligned} &u_i + v_j - (t_{ij} - 1)\, z_i \leqslant c_{ij}, \quad i = 1,\ 2,\ ...,\ m \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad j = 1,\ 2,\ ...,\ n \\ &u_i + v_j - z_i \leqslant 0, \qquad\qquad\quad i = 1,\ 2,\ ...,\ m \\ &z_i \geqslant 0. \end{aligned} \tag{14}$$

Let Y^0 be a solution of problem (11), (12). Based on the dual theorem, the optimum criteria for solution Y^0 are:

$$\begin{aligned} &\text{(a)}\ u_i + v_j - (t_{ij} - 1)\, z_i = c_{ij},\ \text{if}\ y_{ij} \in Y^0, \\ &\text{(b)}\ u_i + v_j - (t_{ij} - 1)\, z_i \leqslant C_{ij},\ \text{if}\ y_{ij} \notin Y^0, \\ &\text{(c)}\ u_i = v_{n+1} - z = 0,\ z_i > 0. \end{aligned} \tag{15}$$

In particular, if $v_{n+1} = 0$, these optimum criteria [2] will be written as:

$$\begin{aligned} &t_{ij} u_i + v_j = c_{ij},\ \text{if}\ y_{ij} \in Y^0,\quad j \neq n + 1, \\ &t_{ij} u_i + v_j \leqslant c_{ij},\ \text{if}\ y_{ij} \notin Y^0, \\ &z_i = -u_i,\quad i = 1,\ 2,\ ...,\ m. \end{aligned} \tag{15'}$$

If $y_{k,\,n+1}$ is a non-essential component of solution Y^0, then by writing its corresponding condition (17 c), we obtain

$$u_k + v_{n+1} - z_k = 0,\quad z_k \geqslant 0,$$

and consequently, we must have

$$u_k + v_{n+1} \geqslant 0. \tag{16}$$

Thus, for a solution to be optimal, condition (16) must be satisfied by all non-essential components $y_{i,\,n+1}$. In this way we arrive at the following system of conditions:

$u_i + v_{n+1} - z_i = 0$, for essential components;

$u_i + v_{n+1} \geqslant 0$, for non-essential components;

$$u_i + v_j - (t_{ij} - 1)\, z_i = c_{ij},\ y_{ij} \in Y^0,\quad j \neq n+1 \tag{17}$$

$$u_i + v_j - (t_{ij} - 1)\, z_i \leqslant c_{ij},\ y_{ij} \notin Y^0$$

consisting of $n + m$ equations, with $n + 2m + 1$ unknowns.

In the number of non-essential components included in the solution is less than m, the system (17), assuming $z_i = 0$, leads to the following system of conditions:

(a) $u_i + v_{n+1_l} = 0$, for essential components;

(b) $u_i + v_{n+1} \geqslant 0$, for non-essential components;

(c) $u_i = v_j = c_{ij},\ y_{ij} \in Y^0$; (18)

(d) $u_i + v_j \leqslant c_{ij},\ y_{ij} \notin Y^0$;

the system made up of (18 a, c) is consistent.

If the number of non-essential components (unknowns) is equal to m, then in solution Y^0 we have a closed cycle. Let this cycle be: (l, k), (l, h), (j, h), (j, k) and consider $z_1 = z_2 = \ldots = z_{l-1} = 0$, $z_l \neq 0$, $z_{l+1} = z_{l+2} = \ldots = z_m = 0$. In this case the optimum conditions become:

$$u_i + v_{n+1} \geqslant 0,\quad \text{for } i \neq l;$$

$$u_i + v_j = c_{ij},\quad \text{for } i \neq l,\ y_{ij} \in Y^0$$

$$t_{lj} u_l + v_j = c_{lj}$$

$$u_i + v_j \leqslant c_{ij},\quad y_{ij} \notin Y^0.$$

In order to improve the solution we shall proceed as follows:

Let Y^0 be a basic solution of problem (11), (12). If condition (18) or (19) is not met for Y°, then there is a

$$u_i + v_j \geqslant c_{ij},\ y_{ij} \notin Y^0.$$

Let be $y_{hk} \notin Y^0$, for which we have:

$$u_h + v_k > c_{hk}.$$

Then we can write a cycle for element (h, k), namely: (h, k), (h, j), (i, j), (i, k). We denote by Δy_{hk}, Δy_{hj}, Δy_{ij}, Δy_{ik} the amounts by which the component

of solution Y° are modified in this cycle. The following relationships will prevail among these amounts:

(a) $\Delta y_{hk} = \Delta y_{ik}$

(b) $\Delta y_{hj} = \Delta y_{ij}$

(c) $t_{hk}\Delta y_{hk} - t_{hj}\Delta y_{hj} \leqslant t_h - \varphi_h(y_h)$ (20)

(d) $t_{ij}\Delta y_{ij} - t_{ik}\Delta y_{ik} \leqslant t_i - \varphi_i(y_i)$

(e) $\Delta y_{hk} \leqslant y_{ik}, \quad \Delta y_{ij} \leqslant y_{hj}$

	k		j		
i	$i,\ k^-$	$\cdots$	$i,\ j^+$		$t_i - \varphi_i(y_i)$
	$\vdots$		$\vdots$		
h	$h,\ k^+$	$\cdots$	$h,\ j^-$		$t_h - \varphi_h(y_h)$

If at least one of the quantities $t_i - \varphi_i(y_i)$, $t_h - \varphi_h(y_h)$ is not equal to zero, by taking into account relationships (20 a) and (20 b), inequalities (20 c) and (20 d) become:

$$\begin{aligned} t_{hk}\Delta y_{hk} - t_{hj}\Delta y_{hj} &\leqslant t_h - \varphi_h(y_h) \\ -t_{ik}\Delta y_{hk} + t_{ij}\Delta y_{hj} &\leqslant t_i - \varphi_i(y_i) \end{aligned} \tag{21}$$

and the following double inequality results for Δy_{hk}:

$$\frac{t_{ij}\Delta y_{hj} - t_i + \varphi_i(y_i)}{t_{ik}} \leqslant \Delta y_{hk} \leqslant \frac{t_{hj}\Delta y_{hj} + t_h - \varphi_h(y_h)}{t_{hk}} \tag{22}$$

We must thus have inequality

$$\frac{t_{ij}\Delta y_{hj} - t_i + \varphi_i(y_i)}{t_{ik}} \leqslant \frac{t_{hj}\Delta y_{hj} + t_h - \varphi_h(y_h)}{t_{hk}} \tag{22'}$$

or, written in another form

$$(t_{ij}t_{hk} - t_{hj}t_{ik})\,\Delta y_{hj} \leqslant (t_i - \varphi_i(y_i))\,t_{hk} + (t_h - \varphi_h(y_h))\,t_{ik}. \tag{23}$$

If $t_{ij}t_{hk} - t_{hj}t_{ik} \leqslant 0$, then inequality (23) always occur.

If $t_{ij}t_{hk} - t_{hj}t_{ik} > 0$, then for inequality (23) to occur Δy_{hj} must meet the following condition:

$$\Delta y_{hj} \leqslant \frac{t_i - \varphi_i(y_i))\, t_{hk} + (t_h - \varphi_h(y_h))\, t_{ik}}{t_{ij}t_{hk} - t_{hj}t_{ik}}. \tag{24}$$

By using an analogous procedure and starting again from relationships (21), we also find for Δy_{hj} the following double unequality:

$$\frac{-t_h + t_{hk}\Delta y_{hk} + \varphi_h(y_h)}{t_{hj}} \leqslant \Delta y_{hj} \leqslant \frac{t_i + t_{ik}\Delta y_{hk} - \varphi_i(y_i)}{t_{ij}} \tag{25}$$

it follows from (25) that we must have:

$$\frac{-t_h + t_{hk}\Delta y_{hk} + \varphi_h(y_h)}{t_{hj}} \leqslant \frac{t_i + t_{ik}\Delta y_{hk} - \varphi_i(y_i)}{t_{ij}} \tag{26}$$

which can be expressed in another form, as

$$(t_{ij}t_{hk} - t_{hj}t_{ik})\, \Delta y_{hk} \leqslant (t_i - \varphi_i(y_i))\, t_{hj} + (t_h - \varphi_h(y_h))\, t_{ij}. \tag{26'}$$

Assuming that:

$$t_{ij}t_{hk} - t_{hj}t_{ik} > 0.$$

Δy_{hk} will be selected such that we shall have:

$$\Delta y_{hk} \leqslant \frac{(t_i - \varphi_i(y_i))\, t_{hj} + (t_h - \varphi_h(y_h))\, t_{ij}}{t_{ij}t_{hk} - t_{hj}t_{ik}} \tag{27}$$

Thus, $\Delta\, y_{hk}$ will be selected such that:

$$\Delta y_{hk} = min \left\{ y_{ik}, \frac{(t_i - \varphi_i(y_i))\, t_{hj} + (t_h - \varphi_h(y_h))\, t_{ij}}{t_{ij}t_{hk} - t_{hj}t_{ik}} \right\} \tag{28}$$

If this minimum is y_{ik}, then we take $\Delta y_{hk} = y_{ik}$; otherwise we consider Δy_{hk} as being equal to the other expression within the braces and, as a result, $t_h - \varphi_h(y_h)$ becomes equal to zero.

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L'OPTIMISATION DE LA DISTRIBUTION DES TÂCHES DE PRODUCTION ENTRE LES EXÉCUTANTS

Les auteurs essaient de résoudre la fonction suivante qui survient dans les divers problèmes d'organisation:

A obtenir le minimum de la fonction

$$f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

dans les conditions:

$$\sum_{j=1}^{n} t_{ij} x_{ij} \leqslant t_i, \quad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = a_j. \quad j = 1, 2, ..., n$$

$$x_{ij} \geqslant 0; \quad i = 1, ..., m \qquad j = 1, ..., n$$

Bien qu'il existe des méthodes de résolution pour ce genre de problèmes — les méthodes de la programmation linéaire — les auteurs présentent un procédé nouveau qui peut être appliqué au problème considéré.

ОПТИМИЗАЦИЯ РАСПРЕДЕЛЕНИЯ ПРОИЗВОДСТВЕННЫХ ЗАДАЧ МЕЖДУ ИСПОЛНИТЕЛЯМИ

Авторы занимаются решением следующей зависимости, возникающей в различных задачах по организации:

Найти минимум зависимости:

$$f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

в условиях:

$$\sum_{j=1}^{n} t_{ij} x_{ij} \leqslant t_i, \quad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = a_j, \quad j = 1, 2, ..., n$$

$$x_{ij} \geqslant 0 \quad i = 1, ..., m, \quad j = 1, ..., n$$

Несмотря на то, что существуют методы решения этого типа задач — методы линейного программирования, авторы пытаются представить новый способ решения, применяемый к рассматриваемой задаче.