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DYNAMIC PROGRAMMING AND OPTIMAL TRAJECTORIES FOR QUADRATIC VARIATIONAL PROCESSES

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The problem is considered, of minimizing the quadratic functional $I(w) = \int_a^T [w^2 + g(t)w]\, dt$ with the restriction $w(a) = c$.

The solution is based on the method of dynamic programming and on the method of invariant sinking. The main steps of the solution are the following: First, to minimize the value of the performance $\varphi(c, a) = \min_w I(w)$, the fundamental equation of dynamic programming is written:

$$\varphi(c, a) = \min_v [(v^2 + g(a)\ c^2)\ \Delta + \varphi\ (c + v\,\Delta,\ a + \Delta) + 0(\Delta)$$ which,

at the limit, becomes

$$-\varphi a(c, a) = \min_v [v^2 + g(a)\, c^2 + v\varphi_c(c, a)]$$

with the initial condition $\varphi(C, T) = 0$.

The method of the separation of variables requires the solution of this equation to be represented in the form:
$u(t; c, t) = m(t, a)\, c$, where the function $m(t, a)$ satisfies the differential equation:

$$m(t, a) = \rho(a)\ m(t, a), \qquad m(t, t) = 1.$$

The last stage consists in checking the fact that, in the hypothesis there is a limited differentiable solution of the equation, the function $u(t; c, a) = m(t, a)\, c$ satisfies Euler's differential equation

$$U = g(t)u$$

with the limit conditions $u(a; c, a) = c$ and $u(T; c, a) = 0$, $0 = a = T$

I. S.