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Considerations on the Multi-Index Transportation Problem for the Case of Symmetrical Constraints

Practice requires a thorough study of the general facets of phenomena.

The multi-index transportation problem is a generalization of the two-index transportation problem. It is a wellknown fact that the two-index transportation problem arises when a certain product has to be shipped at a minimum cost from some manufacturing sites to specific consumption sites, without taking into account such factors as the existence of several transportation types, of several types of products, of several delivery dates, a.s.o.

Therefore, when a single type of a product requires transportation from n_1 manufacturing sites to n_2 consumption sites, and we denote by $C_{\alpha_1 \alpha_2}$ ($\alpha_1 = 1, 2, ..., n_1$; $\alpha_2 = 1, 2, ..., n_2$) the transportation cost per unit of that product from manufacturing site α_1 to consumption site α_1, by $x_{\alpha_1 \alpha_2}$ the number of units shipped from site α_1 to site α_2 and by a_{α_1} and b_{α_2} the stocks available at site α_1 and the demand at site α_2, respectively, then the mathematical model of the double-index transportation problem is as follows:

Minimize the function

$$f = \sum_{\alpha_1,\ \alpha_2}^{n_1,\ n_2} C_{\alpha_1 \alpha_2}\ x_{\alpha_1 \alpha_2}$$

subject to

$$\sum_{\alpha_2 = 1}^{n_2} x_{\alpha_1 \alpha_2} = a_{\alpha_1} \qquad \alpha_1 = 1,\ 2, \ldots, n_1$$

$$\sum_{\alpha_1 = 1}^{n_1} x_{\alpha_1 \alpha_2} = b_{\alpha_2} \qquad \alpha_2 = 1, 2, \ldots, n_2$$

$$x_{\alpha_1 \alpha_2} \geqslant 0.$$

If we also take into account, for example, that several kinds of products are available, we are lead to a transportation problem with three indexes. Hence, assuming that n_3 kinds of products are available and denoting by $C_{\alpha_1 \alpha_2 \alpha_3}$ the transportation cost per unit type α_3 ($\alpha_3 = 1, 2, ..., n_3$) product, from manufacturing site α_1 to consumption site α_2, $x_{\alpha_1 \alpha_2 \alpha_3}$ being the number of units of type α_3 product shipped from site α_1 to site α_2, then the mathematical model of the three-index transportation problem will be:

Minimize the linear functional:

$$f = \sum_{\alpha_1 \alpha_2 \alpha_3} C_{\alpha_1 \alpha_2 \alpha_3} \, x_{\alpha_1 \alpha_2 \alpha_3}$$

subject to

$$\sum x_{\alpha_1 \alpha_2 \alpha_3} = a^{(k)}_{\alpha_{k-1} \alpha_{k+1}} \qquad k = 1, 2, 3$$

$$x_{\alpha_1 \alpha_2 \alpha_3} \geqslant 0.$$

In the most general case, when the transportation problem includes r factors. we are dealing with a s-index transportation problem, whose mathematical model is as follows:

Minimize the linear functional

$$f = \sum_{\alpha_1 \ldots \alpha_s} c_{\alpha_1 \alpha_2 \ldots \alpha_s} \, x_{\alpha_1 \alpha_2 \ldots \alpha_s}$$

where $C_{\alpha_1 \alpha_2 \ldots \alpha_s}$ stands for the transportation cost per unit of product, from manufacturing site α_1 to consumption site α_2, the product being of type α_3, the shipment being effected by $\alpha_4 \ldots$, transportation means, and the delivery date being α_s. The significance of the $x_{\alpha_1 \alpha_2 \ldots \alpha_s}$ variables is quite obvious.

As to the constraints imposed on the variables comprised by the multi-index transportation problem, the following preparatory considerations are necessary.

1. It can be noted that the $n_1 n_2 \ldots n_s$ number of $X_{\alpha_1 \alpha_2 \ldots \alpha_s}$ terms can be viewed as being the elements of a matrix in an s-dimensional space, the location of each element of this matrix being settled within this space by indexes $\alpha_1, \alpha_2, ..., \alpha_s$.

If in this space we consider the set of $X\alpha_1 \alpha_2, ..., \alpha_s$ elements, having m fixed elements $\alpha_{t_1}. \alpha_{t_2}, ..., \alpha_{t_m}$ with $(1 \leqslant m \leqslant s - 1)$ and $t_1 < t_2 < \ldots < t_m$, then these elements make up a new matrix in a space with $s - m$ dimensions, and we shall term it a multiple section matrix of order m, determined by indexes $(\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_m})$ or a $\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_m}$ - directed multiple section matrix.

If $m = s - 1$, we obtain a section determined by indexes $(\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_{s-1}})$ which is a one-sized matrix and which we shall term a α_{t_s} - directed line matrix (Fig. 1).

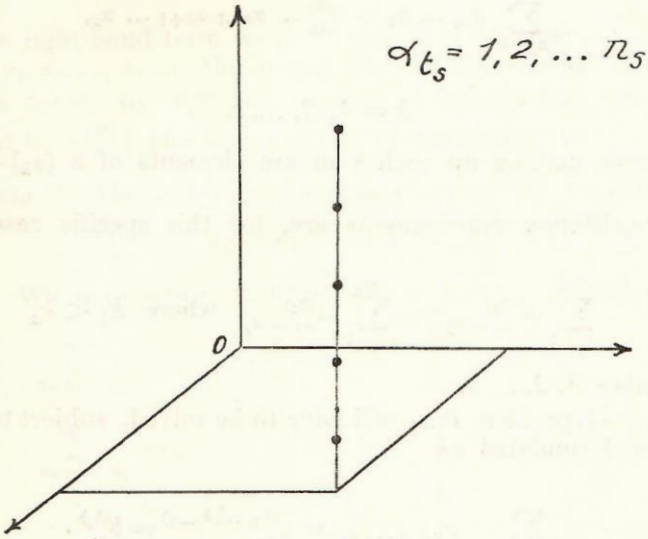


Figure 1

2. The most generalized case of this problem occurs when the constraints are not symmetrical with respect to the indexes, that is when the order in which the S factors are accomplished plays an essential part in this problem.

If, however, this facet is not of interest, then the problem is simplified, as the constraints imposed on variables become symmetrical with respect to the indexes.

It is the purpose of this paper to explore this last case in which the constraints are given by relationships such as:

$$\sum_{\alpha_{k_1}, ..., \alpha_{k_m}} x_{\alpha_1 \alpha_2 ... \alpha_s} = a^{(k_1, k_2, ..., k_m)}_{\alpha_1 ... \alpha_s} \tag{1}$$

where the numbers $k_1, k_2, ..., k_m$ assume all possible integer values, such that

$$1 \leqslant k_1 < ... < k_m \leqslant s \text{ and } 1 \leqslant m \leqslant s - 1.$$

The terms making up each sum are elements of a section matrix of the s—m order, in an m-dimensional space.

It is assumed that the consistency conditions regarding the constraints system (1), are met.

We shall denote by P_m the problem in which the constraints can be formulated as in (1) and which is solved for each value of m.

If $m = 1$, we have to solve problem P_1 subject to:

$$\sum_{\alpha_k=1}^{nk} x_{\alpha_1 \ldots \alpha_s} = a^{(k)}_{\alpha_1 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_s}, \tag{2}$$

$$k = 1, 2, \ldots, s.$$

The terms making up each sum are elements of a (α_k)-directed line matrix.

The consistency requirements are, for this specific case,

$$\sum_{\alpha_{k_1}=1}^{n_{k_1}} a^{(k_2)}_{\alpha_1 \ldots \alpha_s} = \sum_{\alpha_{k_2}=1}^{n_{k_2}} a^{(k_1)}_{\alpha_1 \ldots \alpha_s} \quad \text{where} \quad k_1 < k_2 \tag{3}$$

for any number $1, 2, \ldots, s$.

If $m = s-1$, problem P_{s-1} will have to be solved, subject to constraints which can be formulated as

$$\sum_{\alpha_{k_1}, \ldots, \alpha_{k_{s-1}}} x_{\alpha_1, \alpha_2, \ldots \alpha_s} = a^{(k_1, \ldots, k_{s-1})}_{\alpha_{k_s}} = b^{(k_s)}_{\alpha_{k_s}}. \tag{4}$$

$$k_s = 1, 2, \ldots n_s.$$

The consistency conditions are written as follows:

$$\sum_{\alpha_{k_s}=1}^{n_{k_s}} b^{(k_s)}_{\alpha_{k_s}} = N \quad \text{for any } k_s = 1, 2, \ldots, s \tag{5}$$

where

$$N = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_s} x_{\alpha_1, \alpha_2 \ldots \alpha_s}.$$

In order to develop the algorithm designed for the solution of the multi-index transportation problem, the following theorems referring to the case P_{s-1} are necessary.

Theorem 1. Problem P_{s-1} has always a solution.

Proof. Let us show that

$$x_{\alpha_1 \alpha_2 \ldots \alpha_s} = b^{(1)}_{\alpha_1} b^{(2)}_{\alpha_2} \ldots b^{(s(}_{\alpha_s} \cdot \frac{1}{N^{s-1}},$$

is a solution to the problem. Indeed, by effecting the substitutions in relationships (4), we obtain

$$\sum_{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_{s-1}}} x_{\alpha_1 \alpha_2 \ldots \alpha_s} = b^{(k_s)}_{\alpha_{k_s}} \left(\sum_{\alpha_{k_1}} b^{(k_1)}_{\alpha_{k_1}} \right) \cdots \left(\sum_{\alpha_{k_{s-1}}} b^{(k_{s-1})}_{\alpha_{k_{s-1}}} \right) \cdot \frac{1}{N^{s-1}} = l^{(k_s)}_{\alpha_{k_s}} \qquad (6)$$

where in the right-hand term we have split in a multiple sum, with respect to indexes $\alpha_1, \alpha_2, \ldots, \alpha_{k_s-1}$, the simple sum of the left-hand term.

Let us denote by $T_s^{(m)}$ the number of independent equations of the problem and by $[b^{(k)}_{\alpha_k}]$ the second terms of conditions (4).

Theorem 2. The following inequality occurs for problem P_{s-1}:

$$T_s^{(s-1)} \leqslant n_1 + n_2 + \ldots + n_s - (s-1)$$

Proof. We shall write conditions (4) in a more extended form:

$$\left.\begin{aligned}
&\sum_{\alpha_2, \alpha_3 \ldots \alpha_s} x_{1\,\alpha_2\,\alpha_3 \ldots \alpha_s} = b^{(1)}_{\alpha_1} \\
&\sum_{\alpha_2 \alpha_3 \ldots \alpha_s} x_{2 \alpha_2\,\alpha_3 \ldots \alpha_s} = b^{(1)}_{\alpha_1} \\
&\vdots \\
&\sum_{\alpha_2\,\alpha_3 \ldots \alpha_s} x_{n_1\,\alpha_2\,\alpha_3 \ldots \alpha_s} = b^{(n_1)}_{\alpha_1}
\end{aligned}\right\} \; n_1 \text{ equations}$$

$$\cdots\cdots\cdots

It can easily be noticed that we have $n_1 + n_2 + \ldots + n_s$ equations in all. By eliminating N from the s consistency conditions

$$\sum_{k=1}^{n_1} b_{\alpha_1}^{(k)} = \sum_{k=1}^{n_2} b_{\alpha_2}^{(k)} = \ldots = \sum_{k=1}^{n_s} b_{\alpha_s}^{(k)} = N \tag{7}$$

we find $s - 1$ restrictions relations among the $n_1 + n_2 + \ldots + n_s$ constraints.

Hence, the number of independent equations belonging to this problem is $n_1 + n_2 + \ldots + n_s - (s - 1)$. Q.E.D.

From Theorems 1 and 2 it follows, in particular, that the two-index transportation problem has always a solution and that the number of independent constraints imposed on this problem $T_2^{(1)}$ does not go beyond the number $n_1 + n_2 - 1$.

With respect to problem P_1, we may state the following theorem.

Theorem: Inequality

$$T_s^{(1)} \leqslant n_1 n_2 \ldots n_s - (n_1 - 1)(n_2 - 1) \ldots (n_s - 1) \tag{8}$$

holds for problem P_1.

We can prove the above assertion by means of the inductive method. If $s = 2$, based on the previous result, the theorem is true, since

$$T_2^{(1)} \leqslant n_1 + n_2 - 1 = n_1 n_2 - (n_1 - 1)(n_2 - 1).$$

We assume this theorem to hold for $s - 1$, that is

$$T_{s-1}^{(1)} \leqslant n_1 n_2 \ldots n_{s-1} - (n_1 - 1)(n_2 - 1) \ldots (n_{s-1} - 1). \tag{8'}$$

Let us show that it is also true for s. In order to do this, conditions (2) will be written in an extended form:

$$\begin{aligned}
&\sum_{\alpha_1} x_{\alpha_1 \alpha_2 \ldots \alpha_s} = a_{\alpha_1 \alpha_2 \ldots \alpha_s}^{(1)} \qquad (9_1) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\sum x_{\alpha_1 \alpha_2 \ldots \alpha_s} = a_{\alpha_1 \alpha_2 \ldots \alpha_{s-2} \alpha_s}^{(s-1)} \qquad (9_{s-1}) \\
&\sum x_{\alpha_1 \alpha_2 \ldots \alpha_s} = a_{\alpha_1 \alpha_2 \ldots \alpha_{s-1}}^{(s)} \qquad (9_s)
\end{aligned} \tag{9}$$

Since system (2) is consistent, it follows that conditions (3) are satisfied.

We shall now consider those equations belonging to (3) where k_1 is any number among the numbers 1, 2, ..., $s-1$ and $k_2 = s$.

$$\sum_{\alpha_k} \sum_{\alpha_s} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} \alpha_s} = \sum_{\alpha_2} \sum_{\alpha_k} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} \alpha_s}$$

consequently

$$\sum_{\alpha_k} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} n_s} = \sum_{\alpha_k} \sum_{\alpha_s} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} \alpha_s} - \sum_{\alpha_s = 1} \sum_{\alpha_k} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} \alpha_s}.$$

Thus, out of equations (9_1) and (9_{s-1}), those in which index α_s is equal to n_s can be removed. The remaining equations can be written as

$$\sum_{\alpha_k} x_{\alpha_1 \alpha_2 \ldots \alpha_{s-1} \alpha_s} = a^{(k)}_{\alpha_1 \alpha_2 \ldots \alpha_s}. \tag{10}$$

$$k = 1, 2, \ldots, s-1; \quad \alpha_k = 1, 2, \ldots, n_k; \quad \alpha_s = 1, 2, \ldots, n_{s-1}$$

with

$$\sum_{\alpha_s} x_{\alpha_1 \alpha_2 \ldots \alpha_s} = a^{(s)}_{\alpha_1 \alpha_2 \ldots \alpha_{s-1}}. \tag{11}$$

For each fixed α_s, system (10) represents a problem P_1 with $s-1$ indexes.

We have now $s-1$ problems and for each one relation (8′) holds by assumption. Consequently, the number of independent equations in system (10) does not exceed

$$(n_s - 1)[n_1 n_2 \ldots n_{s-1} - (n_1 - 1)(n_2 - 1) \ldots (n_{s-1} - 1)].$$

System (10) contains $n_1, n_2, \ldots, n_s$ equations. The overall number of independent equations in (10) and (11) does not thus exceed the number

$$(n_s - 1)[n_1 n_2 \ldots n_{s-1} - (n_1 - 1)(n_2 - 1) \ldots (n_{s-1} - 1)] + n_1 n_2 \ldots n_{s-1} =$$

$$= n_1 n_2 \ldots n_s - (n_1 - 1)(n_2 - 1) \ldots (n_s - 1)$$

that is, relation (8).

If in system (2), $T_s^{(1)} = n_1 n_2 \ldots n_s - (n_1 - 1)(n_2 - 1) \ldots (n_s - 1)$ we shall term the system a non-degenerate system.

The theorem regarding the number of independent constraints can be generalized for problem P_m since the following inequality occurs:

$$T_s^{(m)} \leqslant \sum_{i=0}^{s-m} (-1)^i C^{m-1}_{m+i-1} (n_1 n_2 \ldots n_{s-m-i} + \ldots + n_{m+i+1} n_{m+i+2} \ldots n_s). \tag{12}$$

For example, if $n_1 = 4, \quad n_2 = 5, \quad n_3 = 3, \quad n_4 = 2$, then

$$T_4^{(1)} = 4 \cdot 5 \cdot 3 \cdot 2 - 3 \cdot 4 \cdot 2 \cdot 1 = 96$$

$$T_4^{(2)} = n_1 n_2 + n_1 n_3 + n_1 n_4 + n_2 n_3 + n_2 n_4 + n_3 n_4 -$$
$$- 2(n_1 + n_2 + n_3 + n_4) + 3 = 46$$

$$T_4^{(3)} = n_1 + n_2 + n_3 + n_4 - 3 = 11.$$

Unlike the double-index transportation problem, where the consistency conditions are necessary and sufficient for a solution which does not imply reverse transportation to exist, in the multi-index problem $P_m (s \geqslant 3, m \leqslant s - 1)$ only the consistency conditions are necessary.

A solution of problem P_m which does not contain reverse transportations will be termed a possible solution. In problem P_m, the smaller m is, the more restrictive the conditions placed on the unknowns are.

The most restrictive conditions prescribed for the unknowns are to be found in problem P_1; we shall therefore explore it in more detail.

Consider

$$m_{\alpha_1 \alpha_2 \ldots \alpha_s} = \min_{1 \leqslant k \leqslant s} [a^{(k)}_{\alpha_1 \alpha_2 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_s}]. \tag{13}$$

By using the image given by a space matrix, $m_{\alpha_1 \alpha_2 \ldots \alpha_s}$ is the minimum corresponding to box $x_{\alpha_1 \alpha_2 \ldots \alpha_s}$, namely it is the minimum among all $a^{(k)}$ corresponding to all channels concurring in the respective cell.

Under these circumstances, the following theorem can be stated.

Theorem: Problem P_1 has a possible solution only when all inequalities

$$a^{(k)}_{\alpha_1 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_s} \leqslant \sum_{\alpha_k = 1}^{n_k} m_{\alpha_1 \alpha_2 \ldots \alpha_s} \qquad k = 1, 2, \ldots, s \tag{13'}$$

are satisfied.

Since all $x_{\alpha_1 \alpha_2 \ldots \alpha_s} \geqslant 0$, we shall also have

$$x_{\alpha_1 \alpha_2 \ldots \alpha_s} \leqslant a^{(k)}_{\alpha_1 \alpha_2 \ldots \alpha_s} \quad \text{for all } k = 1, 2, \ldots, s.$$

Hence,

$$x_{\alpha_1 \alpha_2 \ldots \alpha_s} \leqslant m_{\alpha_1 \alpha_2 \ldots \alpha_s} \tag{14}$$

and it follows that

$$\sum_{\alpha_k} x_{\alpha_1 \alpha_2 \ldots \alpha_s} \leqslant \sum_{\alpha_k} m_{\alpha_1 \alpha_2 \ldots \alpha_s}$$

that is

$$a^{(k)}_{\alpha_1 \alpha_2 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_s} \leqslant \sum m_{\alpha_1 \alpha_2 \ldots \alpha_s}$$

and we have thus shown the theorem to be true.

Note. If in (14) we have equalities everywhere, then there is a unique solution

$$x_{\alpha_1 \alpha_2 \ldots \alpha_s} = m_{\alpha_1 \alpha_2 \ldots \alpha_s}$$

$$\alpha_k = 1, 2, \ldots n_k; \; k = 1, 2, \ldots s$$

since a change made in the assignment, i.e. a modification of values $X_{\alpha_1 \alpha_2, \ldots, \alpha_s}$, would assume at least one $x_{\alpha_1, \alpha_2, \ldots, \alpha_s}$ to become smaller than the minimum value of $m_{\alpha_1, \alpha_2, \ldots, \alpha_s}$, which is impossible.

AN EXAMPLE OF A PROBLEM WHICH HAS NO POSSIBLE SOLUTION

Consider

$$a^{(1)}_{\bar{\alpha}_2 \alpha_3 \ldots \bar{\alpha}_s} > 2 \min_{\leqslant k \leqslant s} [a^{(k)}_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_{k-1} \bar{\alpha}_{k+1} \ldots \bar{\alpha}_s}], \quad \alpha_1 = 1, 2, \ldots n_1$$

(The dash placed over the index signals that the index has been fixed.)

Assume that:

$$a^{(2)}_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_s} \leqslant \frac{a^{(1)}_{\alpha_2 \ldots \bar{\alpha}_s}}{n_1} = \varepsilon$$

for $\alpha_1 = 1, 2, \ldots, E\left(\frac{n_1}{2}\right)$ where E is the entire part

$$a^{(3)}_{\alpha_1 \alpha_2 \alpha_4 \ldots \bar{\alpha}_s} \leqslant \varepsilon$$

for

$$\alpha_1 = E\left(\frac{n_1}{2}\right) + 1, \ldots, n_1 - 1$$

$$a^{(3)}_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_s} < \varepsilon$$

then all

$$m_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_s} = \min [a^{(1)}_{\alpha_2 \ldots \bar{\alpha}_s}, a^{(2)}_{\alpha_1 \alpha_3 \ldots \bar{\alpha}_s}, \ldots, a^{(s)}_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_{s-1}}] \leqslant \varepsilon$$

$$m_{n_1 \alpha_2 \ldots \bar{\alpha}_s} < \varepsilon, \quad \alpha_1 = 1, 2, \ldots, n_1 - 1$$

and we derive

$$\sum_{\alpha_1 = 1}^{n_1} m_{\alpha_1 \alpha_2 \ldots \bar{\alpha}_s} < \varepsilon_{n_1} = a^{(1)}_{\alpha_2 \alpha_3 \ldots \bar{\alpha}_s}$$

i.e. condition (13′) is not met.

In the above example,

$$a^{(2)}_{\alpha_1\alpha_2 \ldots \alpha_s} \quad \text{and} \quad a^{(3)}_{\alpha_1\alpha_2 \ldots \alpha_s}$$

are taken such that the consistency condition is satisfied.

It follows that the consistency condition is not sufficient for a possible solution to problem P_1, with $s \geqslant 3$, to exist.

A NUMERICAL EXAMPLE

Consider $s = 3$, $n_1 = n_2 = n_3 = 2$ and the constraints written on the surfaces of the space matrix illustrated in Fig. 2.

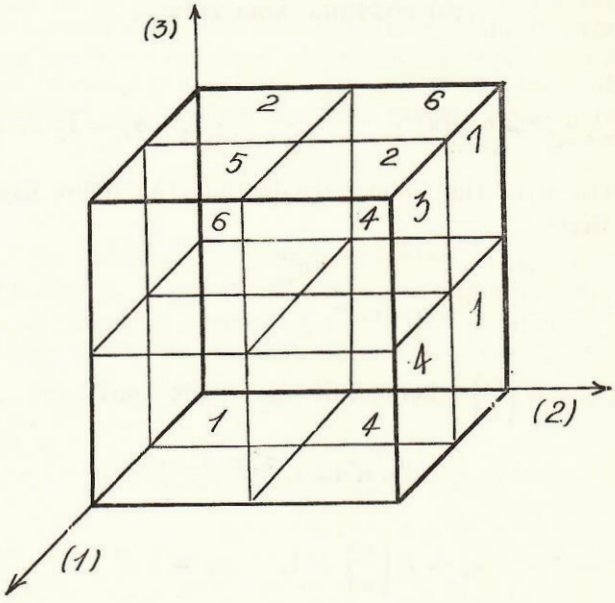


Figure 2

Actually, the consistency conditions are satisfied:

$$a^{(1)}_{11} + a^{(1)}_{12} = a^{(3)}_{11} = a^{(3)}_{21} = 7$$

$$a^{(1)}_{21} + a^{(1)}_{22} = a^{(3)}_{22} + a^{(3)}_{12} = 8$$

$$a^{(1)}_{11} + a^{(1)}_{21} = a^{(2)}_{11} + a^{(2)}_{21} = 5$$

$$a^{(1)}_{12} + a^{(1)}_{22} = a^{(2)}_{12} + a^{(2)}_{22} = 10$$

$$a^{(2)}_{11} + a^{(2)}_{12} = a^{(3)}_{11} + a^{(3)}_{12} = 8$$

$$a^{(2)}_{21} + a^{(2)}_{22} = a^{(3)}_{21} + a^{(3)}_{22} = 7$$

The problem has, however, no possible solution, since:

$$a_{12}^{(1)} = 6 > m_{112} + m_{212} = 2 + 3 = 5$$

that is, condition (13′) is not satisfied.

THE FRONTAL MATRIX METHOD FOR FINDING A POSSIBLE SOLUTION

We shall take

$$n_\alpha = min\,(n_1, n_2, .., n_s)$$

assuming $\alpha = 1$.

We shall term the matrix made up of $a^{(1)}_{\alpha_2 \dots \alpha_s}$, a frontal matrix and shall denote it

$$F_1 = \{a^{(1)}_{\alpha_2 \alpha_3 \dots \alpha_s}\} = \{f^{(1)}_{\alpha_2 \alpha_3 \dots \alpha_s}\}.$$

Consider now

$$m^{(1)}_{\alpha_1 \alpha_2 \dots \alpha_s} = min\,[a^{(k)}_{\alpha_1 \dots \alpha_{k-1}\, \alpha_{k+1} \dots \alpha_s}]$$

which makes up an s-dimensional matrix, whose (α_1)-directed sections will be denoted by

$$M_1^{(1)}, \dots, M_{n_1}^{(1)}$$

We apply to the values searched for, $x_{1 \alpha_2 \alpha_3 \dots \alpha_s}$, the requirements to have as an upper limit $m_{1 \alpha_2 \dots \alpha_s}$, while the lower limit will be

$$max\left[f^{(1)}_{\alpha_2 \dots \alpha_s} - \sum_{\alpha_1 = \alpha}^{n_1} m_{\alpha_1 \alpha_2 \dots \alpha_s},\ 0 \right].$$

These requirements are applied in order to insure that conditions (13′) are satisfied.

Consequently, $x_{1 \alpha_2 \dots \alpha_s}$ must meet conditions

$$\sum_{\alpha_k = 1}^{n_k} x_{1 \alpha_2 \dots \alpha_s} = a^{(k)}_{\alpha_2 \dots \alpha_{k-1}\, \alpha_{k+1} \dots \alpha_s} \qquad k = 2,\ 3,\ \dots.\ s$$

and

$$Max\left[f^{(1)}_{\alpha_2 \dots \alpha_s} - \sum_{\alpha_1 = 2}^{n_1} m^{(1)}_{\alpha_1 \dots \alpha_s},\ 0 \right] \leqslant x_{1 \alpha_2 \dots \alpha_s} \leqslant m^{(1)}_{1 \alpha_2 \dots \alpha_s}.$$

After all $x_{1 \alpha_2 \dots \alpha_s}$ have been selected in this way, we go over to

$$F_2 = \{f^{(2)}_{\alpha_2 \dots \alpha_s}\} = \{f^{(1)}_{\alpha_2 \dots \alpha_s} - X_{1 \alpha_2 \dots \alpha_s}\}$$

We find

$$M_2^{(2)} \dots M_{n_1}^{(2)}$$

where all

$$m^{(2)}_{\alpha_1 \alpha_2 \dots \alpha_s} = min\,[f^{(2)}_{\alpha_1 \alpha_2 \dots \alpha_s},\ m^{(1)}_{\alpha_1 \alpha_2 \dots \alpha_s}]$$

and so on, until we determine

$$F_{n_1} = \{x_{n_1 \alpha_2 \dots \alpha_s}\}$$

If index n_α determines the orientation of the frontal matrices, then the overall number of comparing and summing up operations designed with a view to finding out a possible solution is of an order equal to

$$2_{n_1, \dots, n_{\alpha-1}} (n_\alpha + 1)^2 n_{\alpha+1}, \dots, n_s$$

From these considerations it can be clearly seen why we must choose n_α to be a minimum, or, in other words, why we must take

$$n_\alpha = \min_{1 \leqslant k \leqslant s} n_k$$

In the following paragraphs we shall give an example of a problem in which, although conditions (13') are satisfied, the problem still has no possible solution.

By taking into account the constraints pointed out on the surfaces of the space matrix illustrated in Fig. 3, it can be seen that conditions (13') are satisfied.

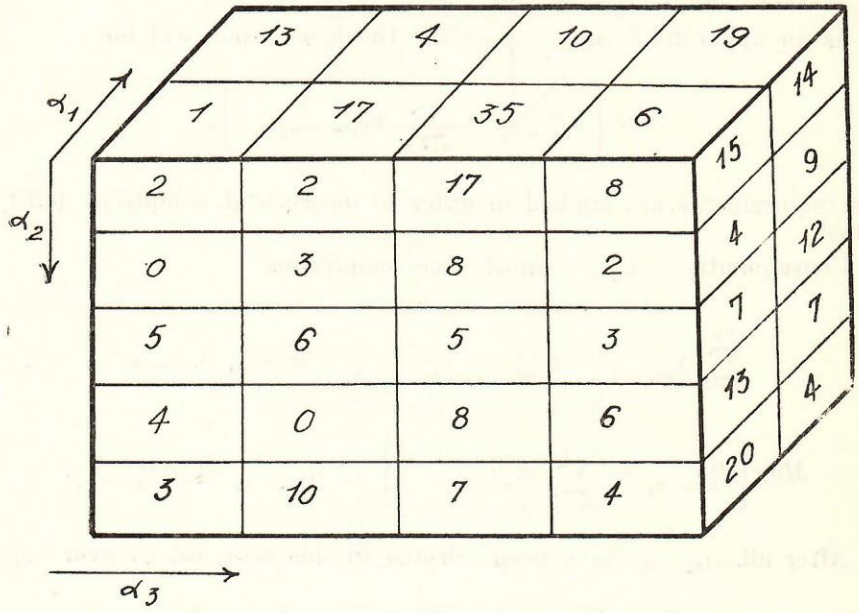


Figure 3

Thus, if we take

$$m_{113} = min\ (15,\ 17,\ 35) = 15$$

$$m_{123} = min\ (8,\ 4,\ 34) = 4$$

$$m_{133} = min\ (5,\ 35,\ 7) = 5$$

$$m_{143} = min\ (8,\ 35,\ 13) = 8$$

$$m_{153} = min\ (7,\ 35,\ 20) = 7$$

it follows that $\sum_{\alpha_2=1}^{5} min_{13}^{(2)} = 15 + 4 + 5 + 8 + 7 = 39;\quad a_{13}^{(2)} = 35.$

Since $a_{13}^{(2)} \leqslant 39$, it follows that conditions (13′) are met.

After making similar checks in all possible cases, we shall see that conditions (13′) are met everywhere.

By proceeding according to the algorithm described above, we find that the frontal matrix F_1 and the section matrices $M_1^{(1)}$ and $M_2^{(1)}$ are

F_1

2	2	17	8
0	3	8	2
5	6	5	3
4	0	8	8
3	10	7	4

$M_1^{(1)}$

1	2	15	6
0	3	4	2
1	6	5	3
1	0	8	6
1	10	7	4

$M_2^{(1)}$

2	2	10	8
0	3	8	2
5	4	5	3
4	0	7	7
3	4	4	4

We further draw up table X_1 and enter in each box of this table the upper and lower limits for the possible values the variable could assume.

X_1

1 0 1	14 0 2	0 7 15	0 0 16	15
0 0 0	3 0 3	1 0 4	0 0 2	4
0 0 1	0 2 6	7 0 5	0 0 3	7
0 0 1	0 0 0	13 1 8	0 1 6	13
0 0 1	0 6 10	14 3 7	6 0 4	20
1	17	35	6	

We can easily realize that we cannot find a value system for the variables, to meet the constraints existing in each box. Thus, for instance, if we look for a possible solution via the North-West corner method, we shall see that in the second box in the upper row the value 14 which has been found does not meet the constraints associated to that box. Hence, although conditions (13′) are satisfied, we cannot, however, find a possible solution.

In case the problem admits possible solutions and such a solution has actually been found, then after selecting a basis (for the non-degenerate problem the number of basic variables equals $T_s^{(1)}$) we ascertain to all $C^*_{\alpha_1 \alpha_2 \dots \alpha_s}$ corresponding to the variables present in that basis, the values

$$\mu^{(k)}_{\alpha_1 \alpha_2 \dots \alpha_{k-1} \alpha_k \dots \alpha_s},$$ such that we obtain

$$\sum_{k=1}^{s} \mu^{(k)}_{\alpha_1 \alpha_2 \dots \alpha_s} = C^* \alpha_1 \alpha_2 \dots \alpha_s$$

Then, by means of variables $\mu^{(k)}_{\alpha_1 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_s}$, we ascertain the values

$$\bar{C}_{\alpha_1 \alpha_2 \dots \alpha_3} = \sum_{k=1}^{s} \mu^{(k)}_{\alpha_1 \alpha_2 \dots \alpha_{k-1} \alpha_{k+1} \dots \alpha_s}$$

corresponding to the boxes which are not included in the basis.

If all $\bar{C}_{\alpha_1 \alpha_2 \dots \alpha_s} \leqslant C_{\alpha_1 \alpha_2 \dots \alpha_s}$, then the possible solutions thus found is optimal. If the possible solution contains non-zero non-basic components (i.e. which are not included in the basis), then it is necessary and sufficient for the optimal solution to be optimal that the following conditions are fulfilled.

a) $\bar{C}_{\alpha_1\alpha_2 \dots \alpha_s} \leqslant C_{\alpha_1\alpha_2 \dots \alpha_s}$, for the boxes in which the values assumed by the variables are equal to the lower limits obtained when applying the algorithm described for the finding of the possible solution;

b) $\bar{C}_{\alpha_1\alpha_2 \dots \alpha_s} \geqslant C_{\alpha_1\alpha_2 \dots \alpha_s}$, for the boxes in which the values assumed by the variables are equal to the upper limits given by the algorithm for the finding of the possible solution.

If these conditions are not met for one box only, then the possible solution found is not optimal, hence it can be improved.

In order to accomplish this improvement, we proceed to a new basis, by means of the known procedure consisting of introducing θ in that box which does not meet the optimum conditions. We shall call this kind of box, a θ—box.

After proceeding to a new basis, the value of the functional decreases:

$$| \bar{C}_{\alpha_1\alpha_2 \dots \alpha_s} - C_{\alpha_1\alpha_2 \dots \alpha_s} | \theta$$

This shows that it is appropriate to choose the θ—box such that by proceeding to the new basis this value is a maximum.

Suppose we have selected a θ-box.

In order to go over to a new basis it is required to build a closed s-dimensional contour-line, just like in the case of the double-index problem.

We shall term a variable of the basis which is not included in the given θ-contour line, an independent variable.

Theorem. If there exists an α_i directed line containing a single variable of the basis, and the θ-box is not placed on this line, then this variable of the basis is independent.

Proof. This variable of the basis must meet s conditions which, when proceeding to the new basis, must still remain in force. If the variable was included in the θ-contour line, then conditions $a^{(\alpha_i)}_{\alpha_1\alpha_2 \dots \alpha_s}$ would not be met any more on the α_i—directed line.

Theorem. If in some unspecified line which does not contain a θ—box, r variables out of this basis are included, and r—1 such variables are independent, then the rth variable is also independent. The proof is made through induction and is based on the fact that if the rth variable of the basis was be included in the θ-contour line, this would lead to one constraint being met.

We shall consider all independent variables in the basis. All remaining variables in the basis are included in the θ-contour line. In order to establish the most adequate manner for changing the variables in the basis, we shall introduce the auxiliary unknowns $\beta_{\alpha_1\alpha_2 \dots \alpha_s}$ corresponding to the boxes of the θ-contour line. Part of the $\beta_{\alpha_1\alpha^2 \dots \alpha_s}$ will be positive, the other part will be negative. Unlike the double-index problem, where all $\beta_{\alpha_1\alpha_2}$ are equal either to θ or to $-\theta$, in the multi-index problem $P_1(s \geqslant 3)$, $\beta_{\alpha_1\alpha_2 \dots \alpha_s}$ are the multiples of θ.

The fact that θ may not be an integer leads to the possibility that also the solution is not a whole number.

All the above considerations concerning problem P_1 can be generalized in order to find a solution to problem P_m. The building through iteration of the successive bases by means of the procedure described here continues until the optimal solution is obtained.

Example

Consider Problem P_1 for the specific case where $s = 3$. All $a^{(1)}_{\alpha_2\alpha_3}$, $a^{(2)}_{\alpha_1\alpha_2}$, $a^{(3)}_{\alpha_1\alpha_2}$ can be found on the surfaces of the space matrix shown in Fig. 4.

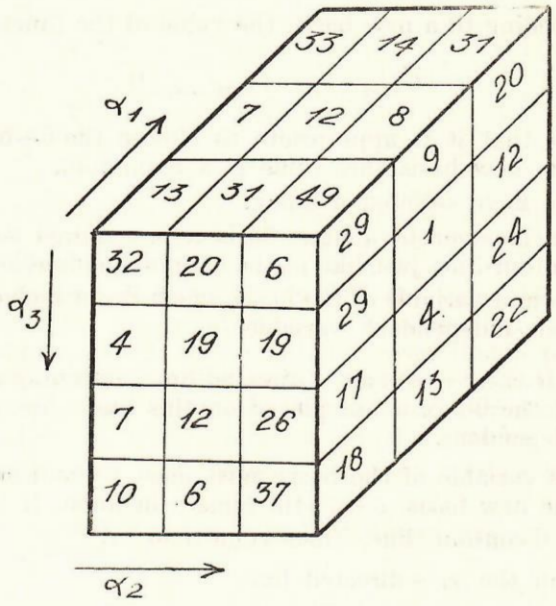


Figure 4

The indexes $\alpha_1 = 1, 2, 3$; $\alpha_2 = 1, 2, 3$ and $\alpha_3 = 1, 2, 3, 4$ are increasing in the sense shown by the arrows. By applying the frontal matrix algorithm in view of finding out a possible solution, we obtain

$\longrightarrow \alpha_2$ $M_1^{(1)}$

$\downarrow \alpha_3$

13	20	6
4	19	19
7	12	17
10	6	18

$\alpha_1 = 1$

$M_2^{(1)}$

7	9	6
1	1	1
4	4	4
7	6	8

$\alpha_1 = 2$

$M_3^{(1)}$

20	14	6
4	12	12
7	12	24
10	6	22

$\alpha_1 = 3$

Conditions (13′) are satisfied.

Since $n_1 = min(n_1, n_2, n_3)$, we shall select as a frontal matrix the matrix $F_1 = \{a^{(1)}_{\alpha_2\alpha_3}\} = \{f^{(1)}_{\alpha_2\alpha_3}\}$

13 5 13	16 0 20	0 6	29
15 0 4	6 19	14 6 19	29
17 0 7	0 12	0 17	17
18 0 10	0 6	0 18	18
13	31	49	

In matrix $X_1 = \{x_{1\alpha_2\alpha_3}\}$, within each box placed in the lower right-hand corner the lower limits are entered, i.e. $m_{1\alpha_2\alpha_3}$, while in the lower left-hand corner the upper limits are entered, i.e.

$$max\left[f^{(1)}_{\alpha_2\alpha_3} - \sum_{\alpha_1=2}^{n_1} m^{(1)}_{\alpha_1\alpha_2\alpha_3},\ 0\right]$$

By computing $f^{(2)}_{\alpha_2\alpha_3}$, $m^{(2)}_{2\alpha_2\alpha_3}$ and $m^{(2)}_{3\alpha_2\alpha_3}$ we obtain

F_2

19	4	6
4	4	5
7	12	9
10	6	19

$M^{(2)}_2$

7	4	6
1	1	1
4	4	4
7	6	8

$M^{(2)}_3$

19	4	6
4	4	5
7	12	9
10	6	19

Conditions (13′) are once again satisfied.

We obtain $X_2 = \{x_{2\alpha_2\alpha_3}\}$ and $X_3 = F_3$ through a similar procedure as that used for obtaining X_1.

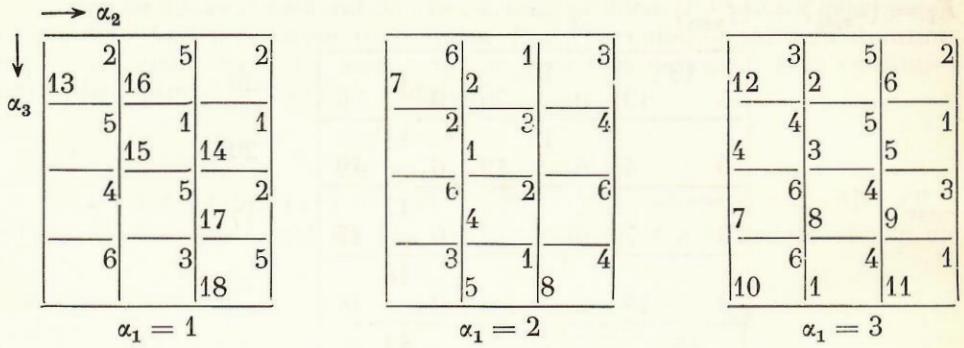
X_2

7 0 7	2 0 4	0 6	9
1 0 1	0 1	0 1	1
4 0 4	0 4	0 4	4
5 0 7	0 5	8 0 8	13
7	12	8	

$X_3 = F_3$

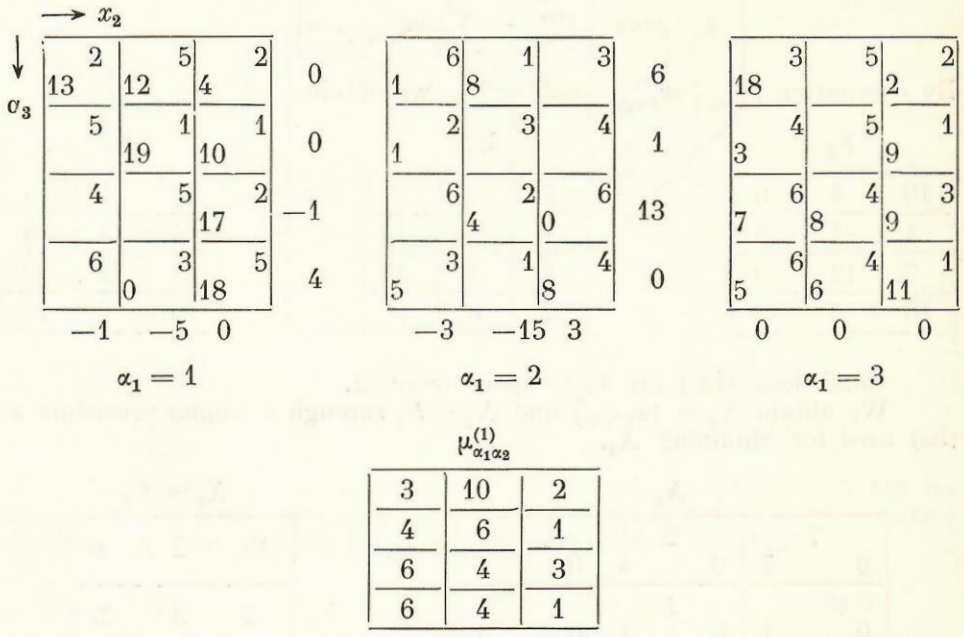
12	2	6
4	3	5
7	8	9
10	1	11

It follows that the possible solution is



In the upper right-hand corner the values $C_{\alpha_1\alpha_2\alpha_3}$ are entered. The possible solution found for the transportation cost is $L^{(1)} = 613$.

After four iterations, we obtain a new possible solution



$$\mu^{(1)}_{\alpha_1\alpha_2}$$

3	10	2
4	6	1
6	4	3
6	4	1

Let us find the values $\mu^{(1)}_{\alpha_2\alpha_3}$, $\mu^{(2)}_{\alpha_1\alpha_3}$, $\mu^{(3)}_{\alpha_1\alpha_2}$ which meet conditions

$$\mu^{(1)}_{\alpha_2\alpha_3} + \mu^{(2)}_{\alpha_1\alpha_3} + \mu^{(3)}_{\alpha_1\alpha_2} = C^*_{\alpha_1\alpha_2\alpha_3}$$

where $C^*_{\alpha_1\alpha_2\alpha_3}$ are the transportation costs corresponding to the boxes of the basis.

The values $\mu^{(1)}_{\alpha_2\alpha_3}$ will be entered in a separate matrix, while the values $\mu^{(2)}_{\alpha_1\alpha_2}$ will be recorded on the right side of the corresponding matrices and the values $\mu^{(3)}_{\alpha_1\alpha_2}$ beneath the corresponding matrices.

We shall introduce θ into the box with the co-ordinates (2, 3, 3). The following boxes are independent: (1, 1, 1), (1, 3, 3), (2, 1, 2), (1, 3, 2), (3, 1, 2). The remaining boxes are included in the θ-contour line. We shall now ascertain $\beta_{\alpha_1\alpha_2\alpha_3}$.

We obtain:

$$\beta_{121} = -\theta \quad \beta_{211} = -\theta \quad \beta_{223} = -\theta \quad \beta_{333} = -\theta$$

$$\beta_{131} = +\theta \quad \beta_{221} = +\theta \quad \beta_{311} = +\theta \quad \beta_{314} = -\theta$$

$$\beta_{124} = +\theta \quad \beta_{214} = +\theta \quad \beta_{331} = -\theta \quad \beta_{324} = -\theta$$

$$\beta_{134} = -\theta \quad \beta_{234} = -\theta \quad \beta_{323} = +\theta \quad \beta_{334} = 2\theta$$

As $\beta_{334} = 2\,\theta$, we have $\theta = 1$.

As a result of four more iterations we obtain the following possible solution:

$\longrightarrow \alpha_2$

$\alpha_3 \downarrow$

13	10	6	2	4
5	15	14	1	0
4	5	17	2	−1
6	6	12	3	4
	−5	−4	0	

$\alpha_1 = 1$

7	9	1	6	−2
2	1	3	4	0
6	2	2	6	0
3	2	1	4	0
	−3	−2	3	

$\alpha_1 = 2$

19	1	3	2	0
4	3	5	1	0
7	10	7	3	0
3	6	4	19	0
	4	5	5	

$\alpha_1 = 3$

3	5	−2
4	5	1
6	4	3
6	3	1

This solution meets all optimal conditions, hence it cannot be improved any more. Thus, following eight iterations, starting with the first possible solution when the transportation cost was 613, we reach the optimal solution for which the transportation cost is $L^{(9)} = 525$.

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QUELQUES CONSIDERATIONS CONCERNANT LE PROBLEME DES TRANSPORTS A PLUSIEURS INDICES DANS LE CAS DES RESTRICTIONS SYMETRIQUES

Les auteurs traitent du problème des transports à plusieurs indices, c'est-à-dire une généralisation du problème doublement indicial.

Cette généralisation présente de l'intérêt vu que, dans la pratique, on rencontre de tels problèmes lorsque, sauf les facteurs pris en considération pour le problème doublement indicial, apparaissent aussi d'autres facteurs: le genre de transport, le terme de livraison, le type de marchandise transportée, etc.

On envisage le cas des restrictions imposées aux variables ayant un caractère symétrique. Pour déterminer une solution possible, on expose la méthode de la matrice frontale en présentant, soutenu par des arguments théoriques, l'algorithme utilisé pour obtenir et améliorer la solution.

СООБРАЖЕНИЯ ПО ВОПРОСУ ПЕРЕВОЗОК С НЕСКОЛЬКИМИ ПОКАЗАТЕЛЯМИ В СЛУЧАЕ СИММЕТРИЧНЫХ ОГРАНИЧЕНИЙ

Авторы рассматривают вопрос перевозок с несколькими показателями, являющийся обобщением известного вопроса с двумя показателями.

Производимое обобщение представляет интерес, так как на практике встречаются подобные вопросы, когда, помимо факторов, принимаемых в расчет в случае двух показателей, появляются и другие факторы, как, например, вид транспорта, срок поставки, вид перевозимых грузов и т. д.

Рассматривается случай, когда применяемые ограничения переменных имеют симметричный характер. Для определения возможного решения излагается метод фронтальной матрицы и приводится с теоретическими обоснованиями соответствующий алгоритм изыскания и улучшения решения.