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A Dynamic Programming Method Using Markov Chains, Applied to the Maximization of the Average Yield of Wheat

To try means to state, to confirm or to refute something; to try means the ability of turning uncertainty into certainty.

The processes and phenomena whose optimal development we wish to ascertain in this paper are liable to become an appropriate area for the utilization of dynamic programming. In such fields with a prevailing stochastic side the models to be applied are more sophisticated.

In a simple form [1], the function objective to be optimized can be expressed as:

$$R(x_1, x_2, ..., x_n) = g_1(x_1) + g_2(x_2) + ... + g_n(x_n)$$

where:

$$x_i \geqslant 0$$

$$\sum_{i=1}^{n} x_i = X$$

while vector X stands for the amounts of available resources which are used for n independent activities.

With a view to achieving a sequential optimization, materialized in a recursive expression, the following series of functions is considered:

$$\{f_n(X)\}$$

where:

$$f_n(X) = \max_{\{x_i\}} R(x_1, x_2, \ldots, x_n) = \max_{0 \leqslant x_n \leqslant X} [g_n(X_n) + f_{n-1}(X - x_n)]$$

on condition that:

$$\sum_{i=1}^{n} x_i = X.$$

Function $f_n(X)$ is the optimal expected result of assigning the amount X of resources to n activities. In two specific cases function $f_n(X)$ assumes the values:

$$f_n(0) = 0 \quad \text{if} \quad g_i(X) = 0$$

and

$$f_1(X) = X.$$

The recursive relationship existing between $f_n(X)$ and $f_{n-1}(X)$ is to be explained as follows: if we take x_n, $0 \leqslant x_n \leqslant X$, to represent the assignment for the nth activity, then, no matter what the value of x_n is, the remaining amount of resources $(X - x_n)$ will be utilized in such a manner that a maximum result is obtained from the rest of $(n-1)$ activities. In other words, by optimally selecting x_n, function $R(x_1, \ldots, x_n)$ is maximized.

Since the model is referring to a stochastic process where the most frequent dependence links are of the Markov type, we shall outline only those elements which are strictly necessary for the purposes aimed at in this paper.

A Markov chain [3], [4] is defined through the following elements:

1° — initial probabilities:

$$P(x_i = j) = p_j \quad \text{and} \quad j \in I$$

conditioned by $p_j \geqslant 0$

$$\sum_{j=1}^{m} p_j = 1$$

where I is the finite set of states $I = (1, 2, \ldots, m)$ which the random variable assumes.

2° — the transition probabilities (or the stochastic matrix) making up a quadratic matrix:

$$\Pi = \| p_{ij} \| \qquad ij \in I$$

$$p_{ij} \leqslant 0$$

$$\sum_{i=1}^{m} p_{ij} = 1 \qquad j = 1, 2, \ldots, m.$$

The equations which caracterize a Markov process are:

$$1^\circ \qquad p_j(n+1) = \sum_{j=1}^{m} p_j(n)\, p_{ij} \qquad i = 1, \ldots, m$$

or, written in a matricial form,

$$p(n+1) = p(n),$$

which indicates that the transition probabilities (i.e. the probability that at time $n+1$, a process exists in state S_j, after being in state S_i at time n) do not depend upon the time development of the process, hence they make up a steady state chain.

$$2^\circ \qquad p_j(n+1) = \sum_{j=1}^{m} p_j(n)\, p_{ij} \qquad i = 1, 2, \ldots, m$$

or

$$p(n+1) = p(n)\,\Pi(n).$$

It can be seen that probabilities p_{ij} depend upon time n, hence the chain becomes non-homogeneous or unsteady.

Supposing each transition from state S_i to state S_j to be associated to a decision r, the transition probabilities $p_{ij}^{(r)}$ depende upon i, j, r and possibly upon n. A value $R_{ij}^{(r)}$ is associated to each change from S_i to S_j and to each decision r.

We shall use the concept of "decision vector in phase n", for the set of decisions selected in this phase; of "strategy" for a series of decision vectors from time zero to time N; of "sub-strategy" for an uninterrupted series of decision vectors from $N-n$ up to N.

We shall now refer to some investigations made at the experimental station for the homologation of species at Dealul Ocnei — Sibiu. Tables I and II show the data which reflect the influence of chemical fertilizers

Table I

Fertilizing alternatives	First section — wheat after peas							
	Berzostaia I species				Harrach species			
	Average yield per hectare	Percentage increase	Absolute increase, kg.	Increased yield per 1 kg. fertilizer	Average yield per hectare	Percentage increase	Absolute increase, kg.	Increased yield per 1 kg. fertilizer
Unfertilized	2,100	—	—	—	1,860	—	—	—
$N_{16}P_{26}$	2,900	42	800	19	2,600	39	740	17.6
$N_{48}P_{54}$	3,460	64	1,360	13.3	3,000	60	1,140	11.1

Table II

Fertilizing alternative	Second section — wheat after potatoes							
	Berzostaia 1 species				Harrach species			
	Average yield per hectare	Percentage increase	Absolute increase kg.	Increased yield per 1 kg. fertilizer	Average yield per hectare	Percentage increase	Absolute increase kg.	Increased yield per I kg. fertilizer
Unfertilized	2,418	—	—	—	2,519	—	—	—
$N_{16}P_{26}$	2,506	3.6	88	2.0	2,653	5.3	134	3.1
$N_{48}P_{54}$	2,482	2.6	64	0.62	2,553	1.3	34	0.33

($N_{16}P_{26}$ and $N_{48}P_{54}$) on the yield of wheat (Berzostaia 1 and Harrach species) under the circumstances that the preceding crop was potatoes or peas.

The process with dependence relationships between the various states represents the crop alternation, while states S_i are specified as follows: S_1 = peas; S_2 = potatoes; S_3 = "Berzostaia 1" wheat and S_4 = "Harrach" wheat.

The decisions which are made during the period from $N—n$ to N and which determine the optimal result are: lack of fertilization, the application of nitrogeneous and phosphoric fertilizers in the ratio N_{16}: P_{26}, and in the ratio N_{48}:P_{54}. (The ratio N_{16}:P_{26} of active materials corresponds to an amount of 50 kg. nitrogen and 150 kg. phosphoric anhydride — superphosphate — per hectare, while the ratio N_{48}: P_{54} corresponds to an amount of 150 kg. nitrogen and 300 kg. phosphoric anhydride per hectare).

If we consider the problem as a whole, with all its potential dependence relationships, we must analyze all possible mutual effects, by taking into account the soil fertilization, as well as all possible variants of alternating the crops. Given the limited amount of results obtained at the Dealul Ocnei — Sibiu experimental station we shall restrict ourselves in this paper to studying only the effect of the two factors — chemical fertilization and the kind of the preceding crop, respectively — on the wheat species "Berzostaia 1" and "Harrach". Considering the kind and amount of currently available data, it is not possible, in our opinion, to also analyze the other potential relationships, hence we have not attempted such a complete analysis.

In order to extend the data obtained as a result of dynamic programming, a territorial planning is necessary, that is a quantitative estimation of the average yield per hectare for a whole agricultural zone or area and not for a single farm only.

We must first ascertain the transition probabilities of the system. These transitions are analyzed as follows: we consider all farms included in a selection sample, chosen with this end in view, which had potato and bean crops during the basic year, followed, however, by wheat crops of the "Berzostaia 1" and "Harrach" species.

Consequently, after an adequate processing of data resulting from the inquiry, we assume to have obtained the following matrices concerning the

above mentioned crop alternation, at the same time specifying the fertilization method used in each case:

$$\Pi = P^r_{ij} = \begin{bmatrix} 0 & 0 & P^r_{13} & P^r_{14} \\ 0 & 0 & P^r_{23} & P^r_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} i = 1, \ldots, 4 \\ j = 1, \ldots, 4 \\ r = 1, \text{ unfertilized} \\ r = 2, \text{ fertilized with } N_{16}P_2 \\ r = 3, \text{ fertilized with } N_{48}P_5 \end{array}$$

$$\text{i.e. } p^1_{ij} = \begin{bmatrix} 0 & 0 & 0.40 & 0.60 \\ 0 & 0 & 0.38 & 0.63 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p^2_{ij} = \begin{bmatrix} 0 & 0 & 0.65 & 0.35 \\ 0 & 0 & 0.44 & 0.56 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p^3_{ij} = \begin{bmatrix} 0 & 0 & 0.55 & 0,45 \\ 0 & 0 & 0.31 & 0.69 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To these matrices we can associate the matrices of results, made up according to Tables I and II. These are:

$$R^r_{ij} = \begin{bmatrix} 0 & 0 & R^r_{13} & R^r_{14} \\ 0 & 0 & R^r_{23} & R^r_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} 1 \leqslant i \leqslant 4 \\ 1 \leqslant j \leqslant 4 \\ 1 \leqslant r \leqslant 3 \end{array}$$

that is,

$$R^1_{ij} = \begin{bmatrix} 0 & 0 & 2,100 & 1,860 \\ 0 & 0 & 2,418 & 2,519 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^2_{ij} = \begin{bmatrix} 0 & 0 & 2900 & 2600 \\ 0 & 0 & 2506 & 2653 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R^3_{ij} = \begin{bmatrix} 0 & 0 & 3460 & 3000 \\ 0 & 0 & 2482 & 2553 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The change prescribed by the prevailing conditions can be easily noticed, i.e. the non-zero partition of matrix R^r_{ij}, where the chain assumes the values.

In order to accomplish the calculations, we shall draw up Table III of transitions from S_i to S_j.

Table III

Year n	Decisions D_i^r	Year $n+1$	Transition probabilities p_{ij}^r	Average yield per ha. R_{ij}^r	Remarks
S_1	D_1^1	S_3	$0.40 = P_{13}^1$	$2{,}100 = R_{13}^1$	unfertilized
S_1	D_1^1	S_4	$0.60 = P_{14}^1$	$1{,}860 = R_{14}^1$	
S_1	D_1^2	S_3	$0.65 = P_{13}^2$	$2{,}900 = R_{14}^2$	$N_{16} - P_{26}$
S_1	D_1^2	S_4	$0.35 = P_{14}^2$	$2{,}600 = R_{13}^2$	
S_1	D_1^3	S_3	$0.55 = P_{13}^3$	$3{,}460 = R_{13}^3$	$N_{48} - P_{54}$
S_1	D_1^3	S_4	$0.45 = P_{14}^3$	$3{,}000 = R_{14}^3$	
S_2	D_2^1	S_3	$0.38 = P_{23}^1$	$2{,}418 = R_{23}^1$	unfertilized
S_2	D_2^1	S_4	$0.62 = P_{24}^1$	$2{,}519 = R_{24}^1$	
S_2	D_2^2	S_3	$0.44 = P_{23}^2$	$2{,}506 = R_{23}^2$	$N_{16} - P_{26}$
S_2	D_2^2	S_4	$0.56 = P_{24}^2$	$2{,}653 = R_{24}^2$	
S_2	D_2^3	S_3	$0.31 = P_{23}^3$	$2{,}482 = R_{23}^3$	$N_{48} - P_{54}$
S_2	D_2^3	S_4	$0.69 = P_{24}^3$	$2{,}553 = R_{24}^3$	

We shall now compute the highest average yield per hectare for a five-year period, assuming that the fertilizers exert a proportionate effect on the yield. This assumption does not hold over a longer period, as it is presumed that the fertilizing effect tends towards a saturation point. This implies an analytical understanding of this curve, in order to make the correction of results possible.

Hence, consider $N = 5$ years. We shall denote the mathematical expectation of the average yield per hectare by q_i^r. If at time n the chain was in state $S_i (i = 1, 2)$ and the decisions were $r (r = 1, 2, 3)$, then it follows that the mathematical expectation of the overall yield Q_i per hectare during n years, i.e. starting with year $N—n$ until year N (assuming the system to be in state S_i in the year $N—n$) will be:

$$Q_i(N - n, N) = \max_r \left[g_i^r + \sum_j p_{ij} Q_j(N - n + 1, N) \right]$$

with the distinct case

$$Q_i(N, N) = 0,$$

The optimal average wheat yield over five years is computed by means of the expression:

$$\bar{q}_i(N) = \frac{Q_i(O, N)}{N}.$$

In order to determine g_i^r, we shall use Table IV, which includes the corresponding estimation for each decision and for each transition from one state to another.

Table IV

Decision D_i^r	Mathematical expectation of the average yield per hectare: $g_i^r = \sum_{j=3}^{4} p_{ij}^r R_{ij}^r$
D_1^1	$g_1^1 = \sum_j p_{1j}^1 R_{1j}^1 = 1{,}956$
D_1^2	$g_1^2 = \sum_j p_{1j}^2 R_{1j}^2 = 2{,}795$
D_1^3	$g_1^3 = \sum_j p_{1j}^3 R_{1j}^3 = 3{,}253$
D_2^1	$g_2^1 = \sum_j p_{2j}^1 R_{2j}^1 = 2{,}280.6$
D_2^2	$g_2^2 = \sum_j p_{2j}^2 R_{2j}^2 = 2{,}588.8$
D_2^3	$g_2^3 = \sum_j p_{2j}^3 R_{2j}^3 = 2{,}430.9$

If $r = 1$ and the system is in state S_3, we shall have:

$$Q_3(N-1, N) = \max_r [(g_1^r + \sum_j p_{ij}^r Q_j(N, N)] = \max_r (g_j^r + 0) =$$

$$= \max_r (1{,}956;\ 2{,}795;\ 3{,}253) = 3{,}253$$

With the system in state S_4, we have:

$$Q_4(N-1, N) = \max_r [(g_2^r + \sum_j p_{ij} Q_j(N, N)] = \max_r (g_2 + 0) = 2{,}588$$

The optimal decision for Q_1 is D_1^3, while for Q_2 it is D_2^2. Similarly, for $n = 2$, we shall have:

$$Q_3(N-2, N) = 6{,}205.7$$

$$Q_4(N-2, N) = 6{,}047.12$$

with the pair of optimal decisions D_1^3 and D_2^2 a.s.o.

The data thus obtained are put together in Table V.

An optimal permanent strategy D_1^3 can be outlined for S_2. Conversely, for state S_3, we have an alternation of optimal substrategies D_2^2 and D_2^3. In order to optimize the wheat yield over a five-year period, we must take into account this result, knowing that an optimal strategy is made up of optimal sub-strategies.

If at time $n = 5$ we are in state S_3, the optimal annual average wheat yield per hectare will be:

$$\bar{q}_3 = \frac{Q_3(O, N)}{N} = 3{,}196.1 \text{ kg./ha.}$$

and for state S_4 we shall obtain:

$$\bar{q}_4 = \frac{Q_4(O, N)}{N} = 3{,}008.5 \text{ kg./ha.}$$

Table V

Date n	State S_i at time $n+1$	Optimal increase $Q_i(N-n, N)$	Optimal decision
1	S_3	$Q_3(N-1, N) = 3{,}253$	D_1^3
	S_4	$Q_4(N-1, N) = 2{,}588$	D_2^3
2	S_3	$Q_3(N-2, N) = 6{,}205.7$	D_1^3
	S_4	$Q_4(N-2, N) = 6{,}047.12$	D_2^3
3	S_3	$Q_3(N-3, N) = 9{,}387.3$	D_1^3
	S_4	$Q_4(N-3, N) = 9{,}349.2$	D_2^3
4	S_3	$Q_3(N-4, N) = 1{,}262.31$	D_1^3
	S_4	$Q_4(N-4, N) = 12{,}854.8$	D_2^3
5	S_3	$Q_3(0, N) = 15{,}980.5$	D_1^3
	S_4	$Q_4(0, N) = 15{,}342.7$	D_2^3

These results are useful in planning the average yield per hectare over a five-year period for an agricultural area, knowing that it will be comprised within the limits 3.008.5 kg/ha, and 3,196.1 kg/ha, when for each subperiod, the optimal decision has been selected as related to the states sequence

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UN MODELE DE PROGRAMMATION DYNAMIQUE A CHAINES MARKOV EN VUE DE LA MAXIMISATION DE LA PRODUCTION MOYENNE DE BLE

L'alternance des cultures, étudiée dans un contexte dynamique, implique des effets économiques de grande importance pour une planification zonale de la production agricole. L'auteur essaie de surprendre et de quantifier ces effets à l'aide de la théorie des processus stochastiques, utilisant la méthodologie élaborée pour la programmation dynamique.

Les données expérimentales employées, bien qu'elles se résument à un cercle relativement simplifié, peuvent former un support suffisant pour suggérer une voie attrayante d'envisager le problème de la planification de la production moyenne agricole.

МОДЕЛЬ ДИНАМИЧЕСКОГО ПРОГРАММИРОВАНИЯ С ПРИМЕНЕНИЕМ ЦЕПЕЙ МАРКОВА ДЛЯ МАКСИМИЗАЦИИ СРЕДНЕЙ УРОЖАЙНОСТИ ПШЕНИЦЫ

Чередование культур, рассматриваемое в динамическом разрезе, дает экономические эффекты большого значения для зонального планирования сельскохозяйственного производства. В статье автор делает попытку выявить и количественно выразить эти эффекты при помощи теории стохастических процессов, применяя разработанную методику динамического программирования.

Используемые экспериментальные данные, хотя и ограничены сравнительно упрощенным кругом, могут создать достаточную основу для того, чтобы наметить перспективные пути рассматривания вопросов планирования средней урожайности в сельском хозяйстве.