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The Utilization of the Graph Theory in Solving the Assignment Problem

The assignment problems, also known under the name of choice problems or allocation problems, set up a special type of correspondence among the elements of two sets, the so-called "one-to-one" correspondence.

The assignment problems can be solved by using a linear model, that is through the algorithms peculiar to linear programming and to transportation problems.

There are several possible approaches to the assignment problem through the use of the graph theory, however, without a comprehensive systematization. It is the purpose of this paper to develop a synthetic presentation of the assignment problem and, at the same time, to apply it by using the graph theory.

The graphs to be considered here are bipartite graphs, i.e. graphs of the type $\mathbf{G} = (X \cup Y, \Gamma) = (X, Y, \Gamma)$ (Fig. 1), with the property that:

(1) $X \cap Y = \emptyset$

(2) $(\forall) x_i \in X$ we have $\Gamma x_i \subset Y$ or $\Gamma x_i = \emptyset$

(3) $(\forall) y_j \in Y$ we have $\Gamma y_j = \emptyset$

An important concept in studying the assignment problems is related to the matching of a bipartite graph. Let $\mathbf{G} = (X, Y, \Gamma)$ be a bipartite graph and let $A \subset X$, $B \subset Y$ be two sub-sets, such that $|A| = |B|$. The matching of A on Y or the matching of A on B is a one-to-one Δ application of A on B, such that:

$$(\forall)\, x_i \in A\colon \Delta\{x_i\} \subset \Gamma\{x_i\}.$$

If we denote by $\mathcal{A}$ the family of all matchings of a graph G; a maximal matching C_0 is the one having a maximal number of arcs, i.e. $|C_0| = \max_{C \in \mathcal{A}} C$ The number $|C_0| = Q(G)$ is called the *checquering* index.

In practical activities, the assignment problem is encountered under various names and aspects. Even some theoretical problems can be formulated as assignment problems. Among the possible assignment problems we shall give the following examples:

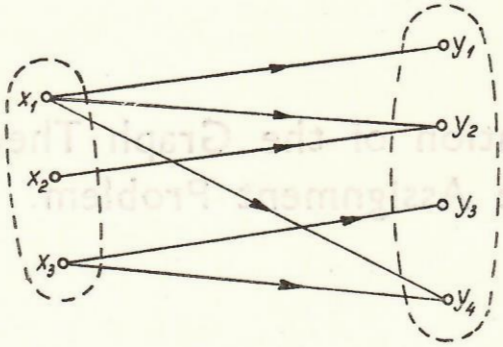


Figure 1.

1) In a workshop, m workers have to perform n tasks, each worker being able to perform only certain specific jobs. It is required to make an assignment of each worker to only one of the jobs for which he has been qualified, so that as many jobs as possible can be achieved. Similar problems are: the assignment of m pieces of equipment to n tasks, the assignment of messages to communication channels, the assignment of berths on ships, etc.

2) In a locality there is the same number of young men and young women wishing to get married. Each young man is acquainted with a number m of young women and each young woman is acquainted with m young men. Is it possible for each young man to find his partner within that same locality, knowing that marriages between close relatives up to the fourth degree of relation are not allowed?

3) Again we consider example (1) where, this time, each worker is able to perform any job. As the workers have various capabilities, two workers need not the same amount of one job to each worker, so as to minimize the total time. Similar problems are: the program assignments to computers, the loading of workshops, the assignment of graduates to various jobs so that their preferences can be satisfied as much as possible.

4) The Crew Problem. We want to schedule the flights of some crews between two towns, A and B, so as to minimize the maximum amount of time spent by the crews away from their homes. The crews having their residences in A and B are known, as well as the number of daily flights of the airplanes between both towns. This problem might also be formulated as a minimization problem of the total amount of time spent by the crews away from their homes. In both cases, additional constraints may appear, namely [17]:

— the uninterrupted flight time of each crew shall not exceed 8 hours or, also including their ground activities, it will not exceed 14 hours;

— the rest time after a flight shall be at least equal to the double of the flight time, plus a quarter of an hour; a.s.o.

5) A company manufactures m screws and n nuts which can be matched. Let a_i be the weight of any screw and b_j the weight of a nut. It is required to match the greatest possible number of screws and nuts, provided that

$$u_1 \leqslant a_i - b_j \leqslant v_1$$
$$u_2 \leqslant a_i + b_j \leqslant v_2$$

i.e. the sum and difference of the weights of these matching parts be within certain specific limits, namely u_1, u_2, v_1, v_2.

From the above examples it follows that:

(a) In some assignment problems one seeks an answer to the question if it is feasible to make pairs out of the elements belonging to two disjoint sets, so that any element of a set shall be a member of only a single pair.

b) In other problems, it is required to determine the specific matching for which the sum of the values associated to the arcs shall be a maximum or a minimum.

c) Another category states the problem of finding a matching for which the highest value associated to the arcs shall be a minimum.

d) The purpose of still another group of problems is to find a very special type of matching (example 5).

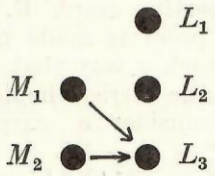
It is noteworthy that in order to obtain practical results, the assignment problems are not to be treated in general, but by considering each individual case.

Before pointing out how each of the above-mentioned problems is solved, we shall introduce some concepts belonging to the graph theory and we shall demonstrate some theorems, the purpose of this section being to answer the following questions:

A. — Is it possible to assign each worker to perform one and only one job for which he is competent?

B. — Which is the maximal number of arcs associated to a matching?

A. — It is not always possible to assign each worker to one job and only to that one for which he is qualified. A necesary condition is that the number of available workers does not exceed the number of jobs. This condition is not, however, sufficient; for instance, if we have two workers, M_1 = a concreter and M_2 = a brick layer, and three jobs, L_1 = assembling and joinery, L_2 = parquet laying and L_3 = concrete casting, the associated graph will be as follows:



Although the number of workers is less than the number of jobs, it is, however, not possible to assign each worker to a job; the assignment of M_1

to job L_3 precludes the possibility of assigning a job to M_2. The following theorem gives the necessary and sufficient condition for the existence of a solution:

The König-Hall Theorem, [1], [19]

Let $\mathbf{G} = (X, Y, \Gamma)$ be a bipartite graph, $|X| = m$, $|Y| = n$ and $A \subset X$ a set of vertices. It is possible to match X on Y if and only if whichever the set A may be, we have $|\Gamma A| \geqslant |A|$.

Practically, a transportation network is associated to the bipartite graph $\mathbf{G}$, as follows: a vertex y_0 (network input) is added, which is connected to the vertices of Y through the arcs (y_0, y_j) with capacity 1, and a vertex x_{m+1} (network output) is also added, which is connected to the vertices of X through arcs (x_i, x_{m+1}) with capacity 1. The vertices of Y are connected to the vertices of X through arcs (y_j, x_i) of infinite capacity, only if in the original bipartite graph the arcs (x_i, y_j) are present.

In order to find the maximal matching of a graph, the associated transportation network is set up and the maximal flow is determined by using the Ford-Fulkerson algorithm, the graphical method, the tabular method, a.s.o.

B. — A number of works deal with the problem of estimating the number of arcs associated to a matching [1], [2], [19]. We shall not attempt to describe all estimations made up to now; instead, we shall confine ourselves to the presentation of the König-Hall theorem, which is fundamental in the theory of bipartite graphs. With this end in view it is necessary to recall some pertinent concepts belonging to the graph theory.

The *deficiency* of a bipartite graph is defined as the number

$$\delta_0 = \max_{A \subset X} (|A| - |\Gamma A|).$$

The *carrier* of a bipartite graph is any subset $S \subset X \cup Y$ with the property that the arc of the graph has at least one end in S. If we denote by $\mathfrak{L}$ the family of all carriers of a graph $\mathbf{G}$. one carrier S_0 is called minimal carrier if $|S_0| = \min_{S \in \mathfrak{L}} |S|$. The number $|S_0| = D(\mathbf{G})$ is termed *scattering index.*

The König Theorem, [19]

In a bipartite graph $\mathbf{G} = (X, Y, \Gamma)$ ($|X| \leqslant |Y|$), the lowest number of vertices of a carrier is equal to the highest number of arcs of a matching, i.e.,

$$\min_{S \in \mathfrak{L}} |S| = \max_{C \in \mathfrak{C}} |C|$$

Proof

Starting from the bipartite graph $\mathbf{G}$, the associated transportation network is considered. The proof is made in two stages:

a) We can proceed in such a way that a cut whose capacity is equal to the number of vertices of the carrier should correspond to the respective carrier. To do this, we consider a carrier $S = X^* \cup Y^*$ and denote it by $\overline{X}^* = X - X^*$ and $\overline{Y}^* = Y - Y^*$. As S is a carrier, it follows that the subgraph $\mathbf{G}^* = [(X \cup Y) - X^* \cup Y^*, \Gamma]$ contains no arc, which means that in the associated transportation network there is no arc from any vertex of $\overline{Y}^*$ towards any vertex of $\overline{X}^*$.

If $E = \{X_{m+1}\} \cup \overline{X}^* \cup Y^*$ denotes a set of vertices which does not include the input, but which includes the output, then $U_{\overline{E}}$ is a cut of the network and is made up of the arcs incident towards the inner part of E and whose original end does not belong to E, i.e.,

$$U_{\overline{E}} = U_{\overline{X}_{m+1}} \cup U_{\overline{X}^*} \cup U_{\overline{Y}^*}$$

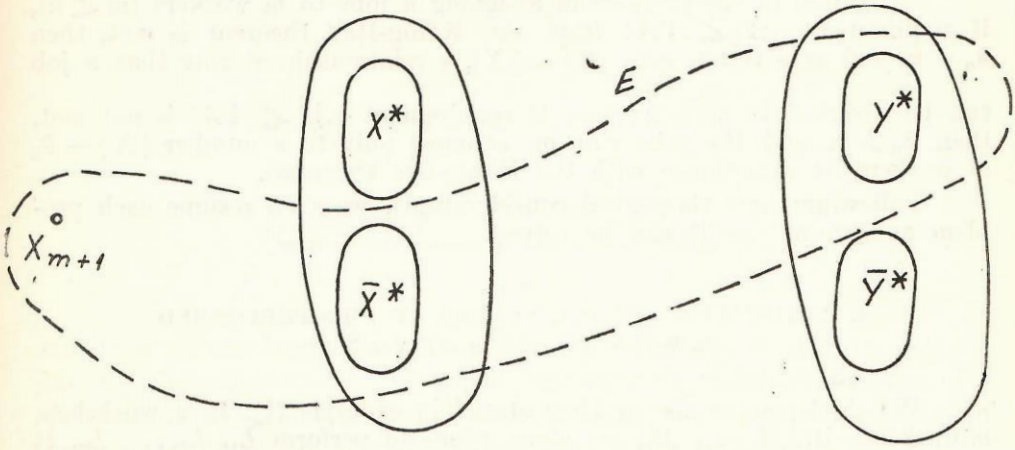


Figure 2.

Set $U_{\overline{X}_{m+1}}$ is made up of the arcs coming from X^* (Fig. 2), whose total capacity is $|X^*|$. Set $U_{\overline{X}^*}$ is void, as there are no arcs from $\overline{Y}^*$ towards $\overline{X}^*$. Finally, set $U_{\overline{Y}^*}$ is made up of the arcs coming from Y_0, whose total capacity is $|Y^*|$. It follows that cut $U_{\overline{E}}$, corresponding to carrier S. has the capacity $c(U_{\overline{E}}) + c(U_{\overline{X}_{m+1}}) + c(U_{\overline{X}^*}) + c(U_{\overline{Y}^*}) = |X^*| + |Y^*|$ equal to the number of arcs in the carrier.

(b) It is possible to set up a correspondence between any cut with a finite capacity and a carrier, whose number of vertices equals the capacity of the cut.

Let $U_{\overline{E}}$ be a cut in the network, the set E being constituted of the vertices X_α of X and X_β of Y. As the cut has a finite capacity, it follows that there is no arc from the vertices of $\overline{Y}_\beta = Y - Y_\beta$ towards the vertices of X. As a result, subset $(X - X_\alpha) \cup Y_\beta$ is a carrier and, as has been previously shown, the capacity of the cut equals the number of vertices of the carrier.

A minimal cut $U_{\overline{E}^*}$ is considered to which a minimal carrier S^*, $|S^*| = \min\limits_{S \in \mathcal{L}} |S| = c(U_{\overline{E}^*})$ corresponds. According to the Ford-Fulkerson theorem, the minimum capacity of a cut is equal to the maximal flow. On the other hand, the maximal flow equals the maximal number of arcs belonging to a matching. Consequently

$$\min_{S \in \mathcal{L}} |S| = \max_{C \in \mathcal{A}} |C|$$

and the theorem has been proved.

The König-Ore Theorem

In a bipartite graph $G = (X, Y, \Gamma)$ ($|X| \leqslant |Y|$).

$$\max_{C \in \mathcal{A}} |C| = |X| - \delta_0$$

Returning to the problem of assigning n jobs to m workers ($m \leqslant n$), if requirement $|A| \leqslant |\Gamma A|$ from the König-Hall theorem is met, then $\delta_0 = 0$, and as a result, $\max_{C \in \mathcal{A}} |C| = |X|$, a relationship stating that a job can be assigned to each worker. If requirement $|A| \leqslant |\Gamma A|$ is not met, then $\delta_0 > 0$, and the jobs can be assigned only to a number $|X| - \delta_0$ of workers (in accordance with the König-Ore theorem).

Following these theoretical considerations, we shall resume each problem and show how it can be solved.

I. MAXIMIZATION OF THE NUMBER OF JOBS PERFORMED (m WORKERS AND n JOBS, $n \geqslant m$)

We shall resume the problem stated in example (1). In a workshop, m workers, $M_1, M_2, \ldots, M_m$ are given n jobs to perform $L_1, L_2, \ldots, L_n$. It is required to assign each worker to perform only one job among those for which he is qualified, provided the highest possible number of jobs is accomplished.

The solution of this problem is the same as the solving of a linear programming problem. Consider the numbers a_{ij} defined as:

$$a_{ij} = \begin{cases} 1, \text{ if the worker } M_i \text{ is able to perform job } L_j; \\ 0, \text{ in the reverse case.} \end{cases}$$

Consider also the numbers

$$x_{ij} = \begin{cases} 1, \text{ if worker } M_i \text{ performs job } L_j; \\ 0, \text{ in the reverse case.} \end{cases}$$

The problem is reduced to find numbers x_{ij} which satisfy conditions:

$$\sum_{j=1}^{n} x_{ij} \leqslant 1 \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \leqslant 1 \qquad j = 1, 2, \ldots, n$$

and maximize the expression

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_{ij}.$$

We shall not enter into more details concerning this particular way of handling the problem, and shall choose to solve it through the graph theory instead. Hence, we associate to this problem a bipartite graph $\mathbf{G} = (X, Y, \Gamma)$ (Fig. 3), denoting $X = \{M_1, \ldots, M_m\}$, $Y = \{L_1, \ldots, L_n\}$,

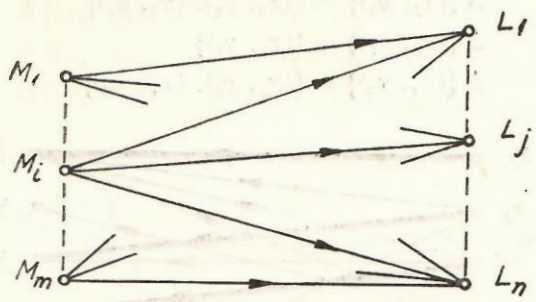


Figure 3

and draw an arc from M_i to L_j only when worker M_i is able to perform job L_j. Consequently, the problem amounts to the determination of a maximal matching. Considering that the maximal matching problems solved through the flow in a transportation network are sufficiently well-known, we shall give here another solving method, [19].

Given a bipartite graph $\mathbf{G} = (X, Y, \Gamma) = (X, Y, U)$ and a matching C, the arcs of the matching are termed "thick" arcs, while the arcs $U-C$ are termed "thin" arcs. A vertex serving as a starting point for at least one "thick" arc is considered a saturated vertex. The graph of the matching arcs is the graph $\mathbf{G}^* = (C, \Delta)$, the application Δ leading the "thick" arcs with the terminal end adjacent to the initial end of ν, to correspond to a "thick" arc $\nu \in C$. Two sets C^+ (C^-) are further defined as being the sets of "thick" arcs with the terminal (initial) end adjacent to an unsaturated vertex.

An algorithm for finding the maximal matching is as follows:

Step 1. A matching C_1 is determined through estimation.

Step 2. The graph of arcs belonging to the matching (C_n, Δ) is set up for a matching C_n.

Step 3. Sets C_n^+ and C_n^- are established.

Step 4. A path leading from C_n^+ to C_n^- is sought. If such a path exists, one proceeds to step 5. In the reverse case, one goes over to step 6.

Step 5. An alternate chain is established, serving to find a matching C_{n+1}, which would have a larger number of arcs. The algorithm is iterated for C_{n+1}, starting with step 2.

Step 6. This last matching is selected as a maximal matching.

Example. Consider the graph shown in figure 4.

A chain obtained through estimation is $C_1 = \{(x_2, y_6), (x_3, y_1), (x_4, y_5)\}$. This matching is *complete*. The graph of arcs belonging to the matching (C_1, Δ) is shown in figure 5, where

$$\Delta\{(x_2, y_6)\} = \{(x_2, y_6), (x_3, y_1)\}$$
$$\Delta\{(x_3, y_1)\} = \{(x_3, y_1)\}$$
$$\Delta\{(x_4, y_5)\} = \{(x_3, y_1), (x_4, y_5)\}$$

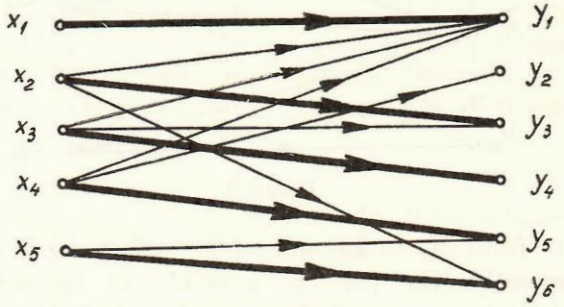


Figure 4

On the same figure are also pictured sets C_1^- and C_1^+ which in this case are equal. It is easy to note that there is a path leading from C_1^+ to C_1^-, for instance the arc $[(x_2, y_6), (x_3, y_1)]$, a path of length equal to 1. As a result, the selected matching is not maximal. We consider the chain $x_1\overline{y_1}\overline{x_3}y_4$ which allows to select matching C_2 rather than matching C_1.

$$C_2 = \{(x_1, y_1), (x_2, y_6), (x_3, y_4), (x_4, y_5)\}$$

The new matching contains one additional arc as compared to C_1.

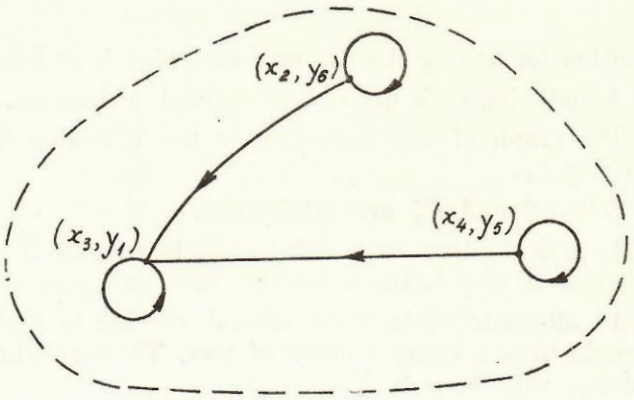


Figure 5

By following the same route as for matching C_1, our purpose is to check if C_2 is maximal. The graph of the arcs associated to matching (C_2,Δ) is illustrated in figure 6, where

$$\Delta\{(x_1, y_1)\} = \{(x_1, y_1\}$$
$$\Delta\{(x_2, y_6)\} = \{(x_1, y_1), (x_2, y_6)\}$$
$$\Delta\{(x_3, y_4)\} = \{(x_1, y_1), (x_3, y_4)\}$$
$$\Delta\{(x_4, y_5)\} = \{(x_1, y_1), (x_4, y_5)\}$$

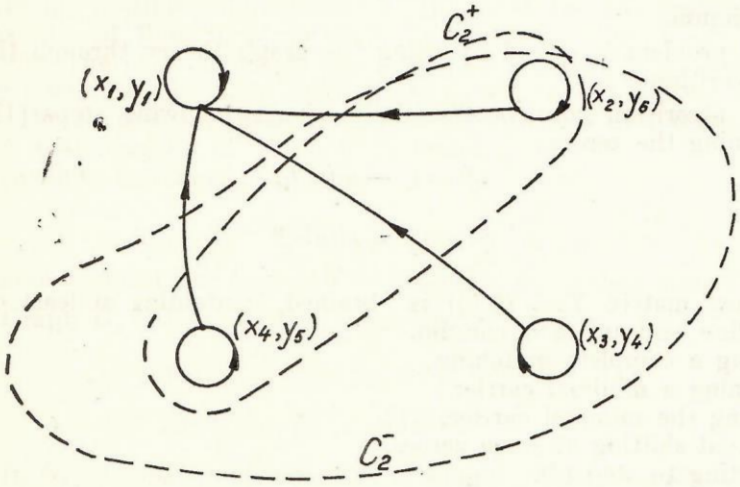


Figure 6

In this case, sets C_2^+ and C_2^- are:

$$C_2^+ = \{(x_2, y_6), (x_4, y_5)\}$$
$$C_2^- = \{(x_2, y_6), (x_3, y_4), (x_4, y_5)\}$$

and this time there exists a path leading from C_2^+ to C_2^-, namely the zero-length path, because vertices (x_2, y_6) and (x_4, y_5) are common. The chain $\overline{x_5 y_6 x_2 y_3}$ allows to proceed to matching C_3 obtained from C_2, through the removal of arc (x_2, y_6) and through introducing arcs (x_5, y_6) and (x_2, y_3), i.e. $C_3 = \{(x_1, y_1), (x_2, y_3), (x_3, y_4), (x_4, y_5), (x_5, y_6)\}$. (Fig. 4.). The matching now obtained is maximal, a property which can also be seen directly, without needing to build up the graph of arcs associated to this matching.

II. MINIMIZATION OF A TOTAL

Assign to n workers a number m of jobs so that all workers have a job to do and only one ($n = m$) and the total time be a minimum.

Let t_{ij} be the time needed by worker x_i to perform job y_i. The problem amounts to determining the numbers x_{ij} subject to conditions

$$\sum_{i=1}^{n} x_{ij} \leqslant 1, \qquad j = 1, 2, ..., m$$

$$\sum_{j=1}^{m} x_{ij} \leqslant 1, \qquad i = 1, 2, ..., n$$

such that expression

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} t_{ij}$$

be a minimum.

The problem is solved by using the graph theory through the Hungarian algorithm.

The algorithm consists essentially of the following steps [17], [19]:
(1) Obtaining the zeroes

$$t_{ij}^{(1)} = t_{ij} - \min_{i} t_{ij}$$

$$t_{ij}^{(2)} = t_{ij}^{(1)} - \min_{j} t_{ij}^{(1)}$$

This is how matrix $T_2 = ((t_{ij}^{(2)}))$ is obtained, containing at least one zero in each line and in each column.
(2) Seeking a complete matching.
(3) Obtaining a minimal carrier.
(4) Marking the minimal carrier.
(5) Eventual shifting of some zeroes.
(6) Reverting to step (3).

If $m < n$, $n - m$ lines equal to zero are added to matrix $T = ((t_{ij}))$ and the procedure is continued as in the case when $n = m$.

If $m > n$, $m - n$ columns equal to zero are added.

In a number of practical problems, the circumstances are such as to oblige us to maximize a total. The problem amounts to a minimum one, after covering the following steps:

1) $t_0 = \max_{i,j=1,...,n} t_{ij}$

2) $t'_{ij} = t_0 - t_{ij}$

3) $\max \sum_{i,j} t_{ij} x_{ij} = n t_0 - \min \sum_{i,j} t'_{ij} x_{ij}$

III. MINIMIZATION OF MAXIMAL TIME

In a workshop, n workers are to perform n jobs. It is required to assign the tasks to all the workers, so that the longest among the times spent by the n workers on the respective task be a minimum. We assume that all workers are capable to perform any job. Let X be the set of workers and Y the set of jobs. We denote by t_{ij} (i, j=1... n) the duration of job y_j performed by worker x_i,. A bipartite graph $\mathbf{G} = (X, Y, \Gamma)$ can illustrate the relevant data.

Under the above-stated conditions, the König-Hall theorem shows that any worker could be entrusted with the performance of a job. There are $n!$ matchings (assignments). Among these matchings, it is required to find one, such that

$$max\ \{t_{ij};\ (x_i, y_j) \in C\}$$

be a minimum.

For a very small n, the solution can be obtained by considering all $n!$ alternatives. As soon as n increases, however, it is actually unfeasible to explore all possible alternatives. In this case too the problem can be solved as one of flow in a network. We shall describe an algorithm by *Ducamp* (5). The idea on which this algorithm is based is to consider any matching (assignment) and to successively pass to matchings for which the value of the longest arc decreases. The algorithm is as follows:

Step 1. A matching C_1 of X on Y is found.

Step 2. Given a matching C_n,

$$D_n = max\ \{t_{ij};\ (x_i, y_j) \in C_n\}$$

is calculated and the arc (x_{i_n}, y_{i_n}) with length D_n is determined.

Step 3. Graph $\mathbf{G}_n = (X \cup Y,\ \Gamma^*)$ is set up as follows:

a) $y_j \in \Gamma^* x_i$, if $(x_i, y_j) \notin C_n$ and $t_{ij} < D_n$

b) $x_i \in \Gamma^* y_j$, if $(x_i, y_j) \in C_n$

Step 4. In $\mathbf{G}_n$ an elementary circuit γ_n is sought, which includes the critical (reversed) arc (y_{j_n}, x_{i_n}), If there is no such circuit, one proceeds to step 5, while if this circuit exists, one takes the matching $C_{n+1} = C_n + \gamma_n$ instead of C_n and one proceeds to step 2.

Step 5. C_n is taken as the optimal matching.

The following remarks can be made regarding this problem:

(a) The algorithm remains valid even if the workers are capable of performing only some specific tasks. We shall take $t_{ij} = \infty$ if worker x_i is unable to perform job y_j. With a view that any job be performed, even in this case, the condition specified in the König-Hall theorem will have to be met.

(b) The problem considered here appears as a *mini-max* problem. Actually, maxi-min problems could also occur, such as, for instance, within the production of an assembly line, where as a rule, each equipment can perform any task, however, with various yields. As the yield of the entire assembly line is determined by the yield resulting in its weakest point, it follows that we are to ascertain that assignment of tasks to the equipment which maximizes the operation with the least output. The previously described algorithm can also be applied to this case, naturally, modified as needed.

In order to illustrate the algorithm, we consider five pieces of machinery and five operations to be performed, the processing times being shown

in the following table, while the associated bipartite graph is shown in figure 7.

	Y_1	Y_2	Y_3	Y_4	Y_5
X_1	9	5	7	9	11
X_2	8	∞	∞	10	9
X_3	8	7	3	7	∞
X_4	13	6	∞	13	17
X_5	∞	6	7	4	12

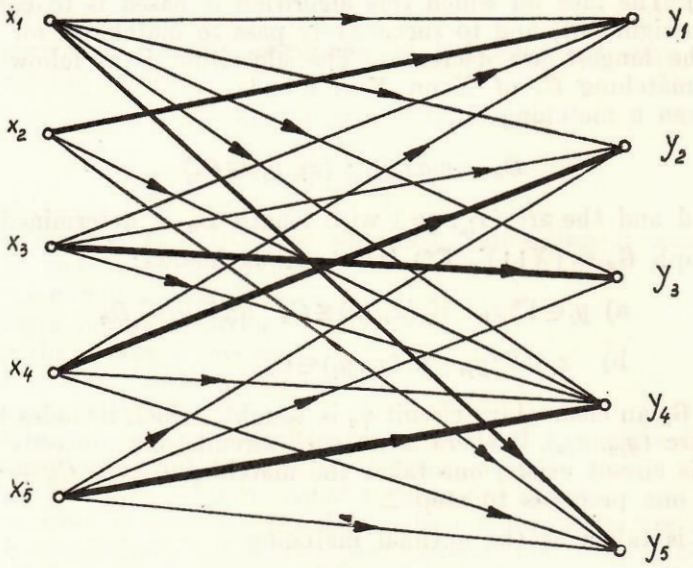


Figure 7

A random selected matching C_1 corresponds to arc (x_1, y_5) (Fig. 7) with a length:

$$D_1 = max \{12, 8, 3, 6, 4\} = 12$$

Graph $\mathbf{G}_1 = (X \cup Y, \Gamma^*)$ associated to matching C_1 (Fig. 8) is obtained from the original bipartite graph, by reverting the arrows at the matching's arc, while out of the remaining arcs only those with a value less than D_1 are retained. We have considered the circuit $\gamma_1 = \{(x_2, y_5), (y_5, x_1), (x_1, y_1), (y_1, x_2)\}$ which will permit us to pass to matching $C_2 = C_1 + \gamma_1 = \{(x_1, y_1), (x_2, y_5), (x_3, y_3), (x_4, y_2), (x_5, y_4)\}$.

Notation $C_1 + \gamma_1$ signifies the union of arcs belonging to C_1 and to γ_1 less the arcs circulated in both senses. The highest value of an arc for matching C_2 is 9. We build up again graph $\mathbf{G}_2 = (X, \cup Y, \Gamma^*)$ (Fig. 9). It can easily be seen that there is no circuit to include the reverted critical arcs, which means that matching C_2 is the optimal matching, in the previously defined terms.

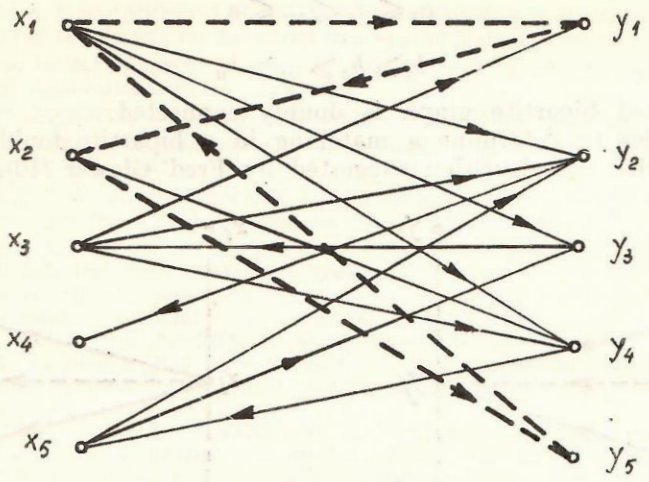


Figure 8

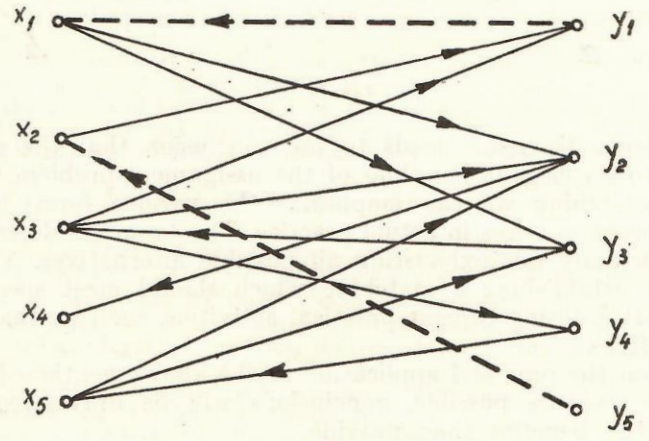


Figure 9

IV. DETERMINATION OF A MATCHING WHICH VERIFIES CERTAIN CONDITIONS

Let $G = (X, Y, \Gamma) = (X, Y, U)$ be a bipartite graph, $|X| = m$, $|Y| = n$. Such a graph has *the property of connexion with respect to X* if, whichever the index $j = 1,\ldots, n$ in $(x_h, y_j) \in U$ and $(x_k, y_j) \in U$, it follows that $(x_i, y_j) \in U$, whichever the index i $(h \leqslant i \leqslant k)$ (Fig. 10 a).

A bipartite graph has *the property of connexion with respect to Y* if, whichever the index $i = 1,\ldots,n$ in $(x_i, y_h) \in U$ and $(x_i, y_k) \in U$, it follows that $(x_i, y_j) \in U$, whichever the index j $(h \leqslant j \leqslant k)$ is. (Fig. 10 b) A *bipartite doubly connected graph* is a graph having the connexion property with respect to X and Y. Resuming example (5) and numbering the parts so that

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a$$

and

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n$$

the associated bipartite graph is doubly connected.

In order to determine a matching in a bipartite doubly connected graph, we cite the algorithm suggested by Fred Glower (10).

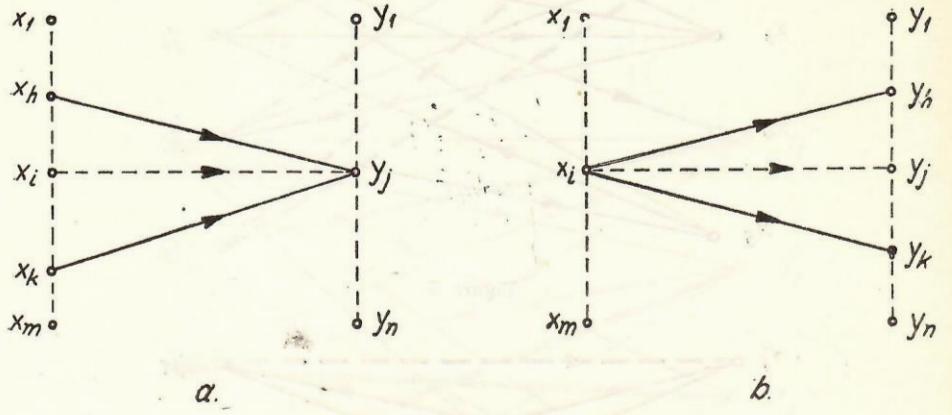


Figure 10

The above discussion leads to the conclusion that the graph theory lends itself to an elegant handling of the assignment problem. Through the concept of matching we can emphasize the various forms in which the assignment problem arises in actual practice. The examples described in this paper are certainly not exhausting all possible alternatives. Various other problems for establishing a matching which should meet specific requirements can arise during current practical activities, such as that mentioned in example (5).

Based on the practical application of the above-mentioned algorithms, to as many cases as possible, conclusions will be drawn regarding their utility and the benefits they provide.

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L'ETUDE DU PROBLEME DE L'AFFECTATION EN UTILISANT LA THEORIE DES GRAPHES

L'ouvrage constitue une synthèse du problème de l'affectation, rencontré dans la pratique sous des aspects et noms divers. Les graphes pris en considération sont bipartis. Tout le matériel présenté se fonde sur la notion de couplage d'un graphe simple. On donne une série d'exemples de problèmes d'affectation: la répartition de m ouvriers à n travaux, le problème des équipages, l'allocation de messages à certains cannaux de communication, etc. On distingue quatre types de tels problèmes.

Avant de présenter la modalité de résoudre chacun de ces quatre types, on donne quelques notions de la théorie des graphes et on cherche à répondre aux suivants problèmes:

A. Peut-on affecter chaque ouvrier à un travail et uniquement à un seul, pour lequel il est qualifié?

B. Quel est le nombre maximum des arcs d'un couplage?

C'est à cet effet qu'on donne le théorème de König-Hall et celui de König, essentiels dans la théorie des graphes simples.

Dans la seconde partie de l'ouvrage on donne les algorithmes de résolution de chaque problème énoncé.

Chaque algorithme est illustré par un exemple numérique.

ИЗУЧЕНИЕ ВОПРОСА АФФЕКТИРОВАНИЯ С ПРИМЕНЕНИЕМ ТЕОРИИ ГРАФОВ

Статья представляет собой синтез вопроса аффектирования, встречаемого в практике с различными наименованиями. Принимаются во внимание простые (двусторонние) графы. Весь приведенный материал основывается на понятии сочетания простого графа. Приводится ряд примеров задач аффектирования: распределение m рабочих на n работ, задача экипажей, распределение сообщений по каналам связи и т. д. Выявляются 4 типа таких задач.

Прежде чем указать, как решается каждый из четырех типов задач, приводятся некоторые понятия теории графов и делается попытка дать ответ на вопросы:

А. Можно ли распределить каждого рабочего на одну работу и только на ту, в которой он квалифицирован?

Б. Каково наибольшее число дуг сочетания?

С этой целью приводятся теорема Кёнига-Халла и основная теорема Кёнига в теории простых (двусторонних) графов.

Во второй части статьи указываются алгоритмы решения каждой изложенной задачи. Каждый алгоритм поясняется числовым примером.