I.M. STANCU-MINASIAN
Department of Economic Cybernetics
Academy of Economic Studies

Analysing Several Points of View in Mathematical Programming with Several Objective-Functions

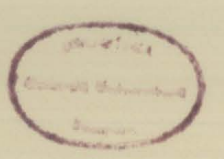
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In this paper, the author is trying to establish certain relationships between the various points of view on solving the problem of mathematical programming with several objective-functions.

Mathematical programming with several objective-functions represents a recent chapter of mathematical programming and has appeared out of the necessity of solving the many problems occuring in science, industry, economy, etc. which, through their complexity require the simultaneous consideration of several goals. The problem consists in optimizing several objective-functions (some of them having to be maximized and others minimized) provided that the variables should verify a system of linear or non-linear constraints.

As the vectorial space with its elements having as components the values of the objective-functions is not totally ordered, as a rule, it is impossible to find a point in the field of admisible solutions which should optimize the set of objective-functions. Most often, the optimal solution after a function is not optimal for the other functions as well, sometimes being even very disadvantageous. For this reason, in the case of several objective-functions, the notion of optimal solution is replaced by the notion of solution which achieves "the best compromise".

Even in the case of a single objective-function, having as coefficients random variables with a certain distribution function two deterministic objective-functions are naturally appearing which should be taken into account: the maximizing of the mean value is pursued on the one hand and the minimizing of the objective-function dispersion on the other hand. It is not sufficient to find a solution which maximizes the mean value because if it leads to a high dispersion, a practically inacceptable solution is resulting.



1. PRESENTING THE PROBLEM

Consider the problem of linear programming with several objective-functions.

$$Ax \leqslant b \tag{1}$$

$$x \geqslant 0 \tag{2}$$

$$(\text{optimum})\ F = Cx \tag{3}$$

where

$A = ((a_{ij}))\ \ i = 1, ...,m,\ j = 1, ..., n$	is a matrix $m \times n$,
$b = (b_1, ..., b_m)' \in R^m$	is a column vector,
$x = (x_1, ..., x_n)' \in R^n$	is an unknown column vector,
$F = (F_1, ..., F_r)'$	is a column vector with r components representing the objective-functions.
$c = ((c_{ij}))\ \ i = 1, ..., r,$	$j = 1, ..., n$, is a matrix $r \times n$.

We note with

$$D = \{ x = (x_1, ..., x_n)' \mid Ax \leqslant b, \quad x \geqslant 0 \}$$

the set of admissible solutions for the enunciated problem.

For fixing the ideas and without the loss of the generality we shall suppose in the following that all functions are of maximum. If this condition is not fulfilled, by transforming $F(x) = -\ max\ [-F(x)]$ we can make all functions to be of maximum.

The problem of programming with several objective-functions consists in finding that vector $X^* = (x_1^*, ..., x_n^*)' \in D$ which should be "as good as possible" from the point of view of the set of objective functions $F_h (h = 1,...,r)$.

The definition of X^* is ambiguous (the expression "as good as possible" is unquantifiable and insufficiently clear) and for this reason, each of its mathematical formulation generates a point of view each and thus, a method each solving the multiple criteria problem.

In the case of several objective functions, the optimal solution for a function is not optimal for the others too — that is why we introduce one of the notions known as a solution "achieving the best compromise, an "undominated" solution, an efficient solution, an efficacious solution, an optimal Pareto solution etc.

The various attempts to define vector X^* may be grouped as follows:

1. X^* is the vector optimizing a synthesis-function of the r efficiency functions

$$h(F) = h\ [F_1, F_2, ..., F_r].$$

Function h may be defined in various ways [5]:

a) $h[F_1, ..., F_r] = \text{optimum}\{F_i(x)\}$ [10], [14];

$$i = 1, ..., r.$$

b) for $r = 2$ it is possible that $h[F_1, F_2] = \frac{F_1(x)}{F_2(x)}$ which leads us to a usual problem of fractionary programming;

c) $h[F_1, ..., F_r] = \sum_{i=1}^{r} \alpha_i [F(x)]^{\beta_i}$; $\alpha_i, \beta_i \geqslant 0$;

d) $h[F_1, ..., F_r] = -\sum_{i=1}^{r} \alpha_i \exp[-F_i(x)]$; $\alpha_i \geqslant 0$;

2. X^* is the vector minimizing an optimality criterion of the form [11].

$$\Phi(X^*) = \min_{x \in D} h(\psi_1(x - X_1), ..., \psi_r(x - X_r)) \tag{4}$$

where: $X_j = (x_{1j}, ..., x_{nj})$, is the optimal solution of the objective function F_j and ψ_r is a function of distance between vector $x \in D$ and the optimal solution X_k of function F_k.

The concrete selection of functions h and ψ_k allows to obtain particular cases of expression (4) for example:

a) $h(x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} (x_j - x_{jk})^2$; $\alpha_k \geqslant 0$

b) $h(x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} |x_j - x_{jk}|$;

c) $h(x) = \max_{k=1, ..., r} \psi_k(x - X_k)$;

d) $h(x) = \sum_{k=1}^{r} \psi_k(x - X_k)$;

e) $h(x) = \prod_{k=1} \psi_k(x - X_k)$

3. X^* is the vector belonging to a set of efficient points [4] which is defined as follows:

A point $x^\circ \in D$ is said to be efficient if and only if does not exist another point $x \in D$ so that $F_h(x) \geqslant F_h(x^\circ)$, $h = 1, ..., r$ and for at least one h_0 we should have $F_{h_0}(x) > F_{h_0}(x^\circ)$, supposing that all functions are of maximum.

In other words, x° is efficient if there is no point x which should improve at least one function, while the others remain unchanged.

The notion of efficiently optimal solution which we find with various authors under various names, as was shown before, is playing an important part in economy, in the game theory, in the statistical decisions theory and,

generally, in any problems of decisions with several incomparable criteria. As we shall see later the determining of the efficient points includes most of the optimizing methods in mathematical programming.

4. X^* is an optimal solution obtained by ordering criteria [8] and which is attained as follows:

We solve r problems of mathematical programming, each time restricting the field D by turning into constraints the optimal solutions obtained by solving a certain problem with a single function. More exactly, we determine successively the sets:

$$D_0^* = D$$

$$D_1^* = \{ x \mid F_1(x) = \underset{y \in D_0^*}{\text{optimum}} F_1(y);\ x \in D_0^* \},$$

$$D_2^* = \{ x \mid F_2(x) = \underset{y \in D_1^*}{\text{optimum}} F_2(y);\ x \in D_1^* \},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_k^* = \{ x \mid F_k(x) = \underset{y \in D_{k-1}^*}{\text{optimum}} F_k(y)\ x \in D_{k-1}^* \},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_r^* = \{ x \mid F_r(x) = \underset{y \in D_{r-1}^*}{\text{optimum}} F_r(y);\ x \in D_{r-1}^* \}.$$

The solving of the problem with several objective functions represents the determining of one or several points of set D_r^* [8]. Obviously this set D_r^* is closely linked with the order of functions. Generally, different sets are corresponding to two different orders.

5. X^* is a point of the field of admissible solutions which is obtained by a seeking method according to certain criteria. Such methods are the POP method (Method of Gradual Orientation) [1] and the STEM method (Step method) [2].

6. X^* belongs to a set of properly efficient points which is defined as follows:

A point x^* is a properly efficient solution, if it is efficient, and if there is a scalar $M > 0$ such that $f_i(x) > f_i(x^*)$ implies

$$\frac{f_i(x) - f_i(x^*)}{f_j(x^*) - f_j(x)} \leqslant M$$

for some j weith $f_j(x) < f_j(x^*)$.

In the linear case the two sets coincide.

2. RELATIONSHIPS BETWEEN VARIOUS METHODS OF SOLVING THE PROBLEMS OF PROGRAMMING WITH SEVERAL OBJECTIVE FUNCTIONS

We note with $D^{**} \subset D$ the set of efficient solutions and with D_r^* the set of optimal solutions obtained by ordering the criteria. The following theorem shows the connection between the two sets.

Theorem 1. Any optimal solution obtained by ordering the criteria is an efficient point of these criteria, namely:

$$D_r^* \subseteq D^{**}.$$

Proof:

Let us suppose, against all reason, that $D_r^* \not\subseteq D^{**}$ that is, there is a point $x' \in D_r^*$ so that $x' \not\subseteq D^{**}$. Point x' not being efficient it results that there exists another point $x'' \subseteq D$ with the characteristic that $F_k(x'') \geqslant F_k(x')$, $k = 1, ..., r$ and for at least an index l (one of indices $1, ..., r$), the inequality is strict

$$F_l(x'') > F_l(x'). \tag{5}$$

Consider $k = 1$. As $x' \in D_r^*$ and $D_r^* \subset D_1^*$, it results that $x' \in D_1^*$ thus x' is a maximum point of function F_1.

As we have $F_1(x'') \geqslant F_1(x')$ it results that this ratio can be verified only with the equal sign, that is $F_1(x'') = F_1(x')$. For $k = 2$, resulting again will be $F_2(x') = F_2(x'')$ thus $x'' \in D_2^*$. Continuing the reasoning in a similar way we obtain:

$F_k(x'') = F_k(x')$ for any $k = 1, ..., r$ which contradicts ratio (5).

With this, the theorem is proved.

J. Saska [11] suggests for solving the problem of linear programming with several objective functions a method consisting in finding vector $X^* \in D$ which minimizes the functions.

$$h(x) = \max_{1 \leqslant h \leqslant r} \frac{|X_h - F_h(x)|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} \tag{6}$$

where

$$X_h = \underset{x \in D}{(\text{optimum})}\ F_h(x).$$

In this way, vector $X^* = (x_1^*, ..., x_n^*)' \in D$ has the characteristic that it is the least distant modulus from the hyperplanes determined by the r objective-functions, that is

$$\max_{1 \leqslant h \leqslant r} \frac{|X_h - F_h(X^*)|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} = \min_{x \in D} \left[\max_{h=1,...,r} \frac{|X_h - F_h(x)|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} \right] \equiv \delta. \tag{7}$$

Function $h(x)$ is convex (thus the existence of the minimum value is ensured) but non-linear, a fact which makes difficult the effective numerical determination of the optimal solution.

In the above-mentioned work, J. Saska restricts the nonlinear problem to an equivalent, but linear, one of the form

$$\begin{gathered} Ax \leqslant b \\ x \geqslant 0 \\ V^{(k)}x + x_{n+1} \geqslant \widetilde{X}_k \\ V^{(k)}x - x_{n+1} \leqslant \widetilde{X}_k \\ min\ \psi(x) = x_{n+1} \end{gathered} \tag{8}$$

where:

$$V^{(k)} = (V_{ki}) = \left(\frac{c_{ki}}{\sqrt{\sum_{j=1}^{n} c_{kj}^2}} \right),\ k = 1, ..., r;\ i = 2, ..., n.$$

$$\widetilde{X}_k = \frac{X_k}{\sqrt{\sum_{j=1}^{n} c_{kj}^2}},\ k = 1, ..., r$$

and x_{n+1} is a complementary variable so that

$$\frac{|\ X_k - F_k(x)\ |}{\sqrt{\sum_{j=1}^{n} c_{kj}^2}} \leqslant x_{n+1};\ k = 1, ..., r. \tag{9}$$

The connection between the two problems is given by the following theorem:

Theorem 2

The optimal solution $(\hat{x},\ \hat{x}_{n+1})$ of the programming problem (8) is equal to the optimal solution $(x^*,\ \delta)$ of the problem of nonlinear programming having the objective-function (6).

Proof:

$$\text{We have: } \min_x \psi(x) = \hat{x}_{n+1} = \max_{1 \leqslant k \leqslant r} \frac{|\ X_k - F_k(\hat{x})\ |}{\sqrt{\sum_{j=k}^{n} c_{kj}^2}} =$$

$$= \max_{1 \leqslant k \leqslant r} |\ \widetilde{X}_k - V^k\, \hat{x}\ | \geqslant \min_x \{ \max_{1 \leqslant k \leqslant r} |\widetilde{X}_k - V^k\ | \} =$$

$$= \min_x \left\{ \max_{1 \leqslant k \leqslant r} \frac{|\ X_k - F_k(x)\ |}{\sqrt{\sum_{j=1}^{n} c_{kj}^2}} \right\} = \max_{1 \leqslant k \leqslant r} \frac{|\ X_k - F_k(x^*)\ |}{\sqrt{\sum_{j=1}^{n} c_{kj}^2}} \equiv \delta \tag{10}$$

Ratio (10) shows that

$$\hat{x}_{n+1} \geqslant \delta \tag{11}$$

As we can write that $\min x_{n+1} = \hat{x}_{n+1}$ also resulting is the inequality:

$$\hat{x}_{n+1} \leqslant \delta. \tag{12}$$

Resulting from (10) and (12) is

$$\hat{x}_{n+1} = \max_{1 \leqslant k \leqslant r} | \widetilde{X} - V^{(k)} \hat{x} | = \max_{1 \leqslant k \leqslant r} | \widetilde{X}_k - V^{(k)} x^* | = \delta.$$

J. Saska gives another method for solving the multiple criteria problem considering instead of function 6, function

$$h_1(x) = \max_{1 \leqslant k \leqslant r} \frac{| X_k - F_k(x) |}{X_k}$$

which must be minimized.

The same idea is to be found in the papers [9] and [16].

Let us consider that

$$\psi_i(x - X_i) = F_i(X_i) - F_i(x)$$

and that $h(\psi_1(x - X_1), ..., \psi_r(x - X_r)) = \sum_{i=1}^{r} [F_i(X_i) - F_i(x)]^2 = \Phi(x)$.

S. C. Huang [6] considers the problem of determining that vector x^* which minimizes function $\Phi(x)$ and defines it as the vector solving the problem with several objective-functions. Thus x^* minimizes the sum of quadrates of deviations from the absolute optimal values. It is proved that x^* is an efficient point.

Theorem 3. Any vector x^* minimizing on the field D function

$$\Phi(x) = \sum_{i=1}^{r} [F_i(X_i) - F_i(x)]^2 \tag{13}$$

is an efficient point.

Proof.

Let us admit that x^* minimizes function $\Phi(x)$ that is, for any other point x we have

$$\sum_{i=1}^{r} [F_i(X_i) - F_i(x^*)]^2 \leqslant \sum_{i=1}^{r} [F_i(X_i) - F_i(x)]^2 \tag{14}$$

but x^* is not efficient, thus there is a point $\bar{x}$ with the characteristic:

$$F_i(x^*) \leqslant F_i(\bar{x}), \quad i = 1, ..., k$$

$$F_i(x^*) < F_i(\bar{x}), \quad i = k + 1, ..., r. \tag{14'}$$

On the other hand:

$$F_i(X_i) - F_i(x^*) \geqslant 0, \qquad i = 1, ..., r \tag{15}$$

$$F_i(X_i) - F_i(\bar{x}) \geqslant 0, \qquad i = 1, ..., r. \tag{16}$$

From $F_i(X_i)$ we subtract ratios (14) and taking into account (15)—(16) we have:

$$F_i(X_i) - F_i(x^*) \geqslant F_i(X_i) - F_i(\bar{x}) \geqslant 0, \; i = 1, ..., k,$$

$$F_i(X_i) - F_i(x^*) \geqslant F_i(x_i) - F_i(\bar{x}) \geqslant 0, \; i = 1, ..., k. \tag{17}$$

If we square and total according to $i = 1, ..., r$ on both sides of (17) we obtain

$$\sum_{i=1}^{r} [F_i(X_i) - F_i(x^*)]^2 > \sum_{i=1}^{r} [F_i(X_i) - F_i(\bar{x})]^2$$

which contradicts ratio (14). Thus x^* is efficient.

Consider now

$$\psi_i(x - X_i) = F_i(X_i) - F_i(x)$$

and

$$h(\psi_1(x - X_1), ..., \psi_r(x - X_1)) = \max_i [F_i(X_i) - F_i(x)] = \psi(x).$$

It can be agreed that vector x^* which minimizes function $\psi(x)$ should be the solution of the problem with several objective functions.

Point x^* minimizes the maximum deviation from the optimal values of the objective functions.

Theorem 4. Any solution x^* which minimizes function

$$\psi(x) = \max_i [F_i(X_i) - F_i(x)]$$

on the field D is an efficient solution.

Proof. Indeed, if x^* is not efficient there will exist x' so that $F_i(x') \geqslant F_i(x^*)$ and at least one of the inequalities is strict.

Hence

$$F_i(X_i) - F_i(x') \leqslant F_i(X_i) - F_i(x^*)$$

that is

$$\max_i [F_i(X_i) - F_i(x')] < \max_i [F_i(X_i) - F_i(x^*)]$$

which contradicts the hypothesis that x^* minimizes function $\psi(x)$.

3. GOAL-PROGRAMMING

In tackling so far the problem with several objective functions we have admitted that r problems of mathematical programming are solved first and then, in keeping with the optimal values found, a synthesis function is built.

We shall present another group of methods known in literature under the name of goal-programming.

We consider a vector $\bar{F} = (\bar{F}_1, \bar{F}_2 ..., \bar{F}_r)$ whose components, represent the levels to be attained by the objective functions. For a $x \in D$ there will be more or less deviations from these values and the problem is to minimize a function measuring the distance between vector $\bar{F}$ and the vector whose components represent the possible values of the objective functions.

We shall consider the vectorial space n — dimensional R^n endowed with the norm $\| \cdot \|$ classically defined.

Considering two points $q = (q_1, ..., q_n)$ and $r = (r_1, ..., r_n)$ as $d(q, r) = \| q - r \|$ we shall further use instead of the distance between q and r the norm of the difference of vectors q and r, as a measure of approach between $\bar{F}$ and F.

The best known norm is norm L_p or the Hölder norm

$$\| x \|_P = \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p}, \quad p \geqslant 1.$$

In particular cases the following norms are obtained:

$$p = 1$$

$$\| x \|_1 = \sum_{i=1}^{n} | x_1 | ;$$

$$p = 2$$

$$\| x \|_2 = \left(\sum_{i=1}^{n} | x_i |^2 \right)^{1/2}$$

$$p = \infty$$

$$\| x \|_\infty = \max_i \{ | x_i | \}.$$

Norm $\| \cdot \|_2$ represents the Euclidean distance.

If we take the norms $F - \bar{F}$ the following models are obtained which solve the goal programming problem:

Model 1

$$\min_x \{ \| F - \bar{F} \|_p = \left[\sum_{i=1}^{n} |F_i - \bar{F}_i|^p \right]^{1/p} | \; Ax = b, \; x \geqslant 0 \}.$$

Model 2

$$\min_x \{ \| F - \bar{F} \|_1 = \sum_{i=1}^{r} | F_i - \bar{F}_i | \quad | \; Ax = b, \; x \geqslant 0 \}.$$

Model 3

$$\min_x \{ \| F - \bar{F} \|_2 = \left[\sum_{i=1}^{n} | F_i - \bar{F}_i |^2 \right]^{1/2} | \; Ax = b, \; x \geqslant 0 \}.$$

Model 4

$$min\,\{\|F - \bar{F}\|_\infty = max\,|F_i - \bar{F}_i| \;\; | \; Ax = b, x \geqslant 0\}.$$

Taking into account Hőlder's inequalities norm $\|\cdot\|_p$ is a convex function. We also note other forms of model 3.

Model 3'

$$\min_x \{\|F - \bar{F}\|_2 = \left[\sum_{i=1}^{r} (F_i - \bar{F}_i)^2\right] \;\; | \; Ax = b, \; x \geqslant 0\}.$$

Model 3''

$$\min_x \{\|F - \bar{F}\|_2 = \left[\sum_{i=1}^{r} \alpha_i (F_i - \bar{F}_i)^2\right] \;\; | \; Ax = b, x \geqslant 0\}$$

where α_i are the importance coefficients of functions F_i.

The minimizing of norm $\|F - \bar{F}\|_p$, for $p \geqslant 2$ leads to nonlinear programming problems whereas the minimizing of norms $\|F - \bar{F}\|_1$ and $\|F - \bar{F}\|_\infty$ can be done by the simplex method of linear programming.

Indeed, let us consider model 2 in which the norm of the absolute value is minimized. This model was studied by A. Charnes and W.W. Cooper [4] and Y. Ijiri [7].

We note with $d_k^+(x)$ and $d_k^-(x)$ for each function, the deviations with plus or minus, from values $\bar{F}_k$. The problem is to minimize the total of these deviations.

Model 2 becomes:

$$min \sum_{k=1}^{r} (d_k^+(x) + d_k^-(x))$$

$$F_k(x) + d_k^- - d_k^+ = \bar{F}_k \quad k = 1, ..., r \tag{18}$$

$$d_k^+(x), \; d_k^-(x) \geqslant 0$$

$$x \in D$$

As for a function F_k the deviation from $\bar{F}_k$ in a point takes place only in one direction, it will result that if $d_k^+ > 0$ in this case $d_k^- = 0$ and reciprocally if $d_k^- > 0$ in this case $d_k^+ = 0$.

Note with e a column vector having r elements, all equal to 1, to I^+ the unit matrix of the order r and to d^+ and d^- the component vectors d_i^+ and d_i^-. Model (18) will be written in the form:

$$min\,[ed^+ + ed^-]$$

$$F(x) - I\,d^+ + Id^- = \bar{F}$$

$$Ax = b$$

$$x, \; d^+, \; d^- \geqslant 0.$$

Theorem 5. Any optimal solution $\tilde{x}$ for problem (18) is an efficient solution.

Proof.

Let us suppose that $\tilde{x}$ is not efficient, thus there is a point $x \in D$ so that:

$$F_k(x) \geqslant F_k(\tilde{x}), \quad k = 1, ..., r \tag{19}$$

and for at least a k we have

$$F_k(x) > F_k(\tilde{x}).$$

We note with $\tilde{d}_k^+$ and $\tilde{d}_k^-$ the deviations of the values of objective functions from the values $\bar{F}_k$ in point x.

On the other hand, both x and $\tilde{x}$ verifies the constraints of problem (18), that is

$$F_k(x) + d_k^- - d_k^+ = \bar{F}_k$$

and

$$F_k(\tilde{x}) + \tilde{d}_k^- - \tilde{d}_k^+ = \bar{F}_k$$

or

$$d_k^+ - d_k^- = \bar{F}_k - F_k(x)$$

and

$$\tilde{d}_k^+ - \tilde{d}_k^- = \bar{F}_k - F_k(\tilde{x}).$$

Hence, resulting also from ratios (19) are:

$$\tilde{d}_k^+ - \tilde{d}_k^- \geqslant d_k^+ - d_k^-, \text{ for all } k = 1, ..., r$$

and

$$\tilde{d}_k^+ - \tilde{d}_k^- > d_k^+ - d_k^- \text{ for at least a } k.$$

By summation we obtain:

$$\sum_{k=1}^{r} (\tilde{d}_k^+ - \tilde{d}_k^-) > \sum_{k=1}^{r} (d_k^+ - d_k^-)$$

which contradicts the fact that $\tilde{x}$ is an optimal solution for problem (18).

Model 4 was studied by Zuhovitkii S.I. and L.I. Avdeeva [15] and was replaced by the following linear model.

$$min \ \lambda$$

$$-\lambda \leqslant \bar{F}_i - F_i \leqslant \lambda$$

$$Ax = b$$

$$x, \lambda \geqslant 0.$$

As regards model 3 it can be solved by the method of generalized inverses [7]. A solution is

$$x = R^+ \bar{F} + R^\circ \beta$$

where R^+ having the order $n \times r$ is the generalized inverse of R of the degree q, R° is a matrix $n \times (n-q)$ with the characteristics that $RR^\circ = 0$ and β is an arbitrary vector with $n - r$ components.

R° and β are determined from the other conditions on x, for example from $x \geqslant 0$, that is from $R^+\bar{F} + R^\circ \beta \geqslant 0$ we determine R°; and from $Ax = b$ that is $AR^+\bar{F} + AR^\circ\beta = b$ we determine β.

We have presented only some aspects of mathematical programming with several objective functions. Though by taking into consideration several objective functions the decision problem becomes very much complicated, yet this field is extremely rich and interesting.

Stochastic programming with several objective functions represents a topical direction of research.

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ANALYSE DE CERTAINS POINTS DE VUE DANS LA PROGRAMMATION MATHEMATIQUE À PLUSIEURS FONCTIONS OBJECTIFS

L'auteur essaie d'établir certaines relations entre les différents points de vue concernant la solution du problème de programmation mathématique à plusieurs fonctions objectifs

АНАЛИЗ НЕКОТОРЫХ ТОЧЕК ЗРЕНИЯ ПРИ МАТЕМАТИЧЕСКОМ ПРОГРАММИРОВАНИИ С НЕСКОЛЬКИМИ ЦЕЛЕВЫМИ ФУНКЦИЯМИ

В настоящей статье автор пытается установить некоторые отношения между различными точками зрения в связи с решением задачи математического программирования с несколькими целевыми функциями.