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An Algorithm for Solving some Problems Connected with the Optimum Use of the Investment Fund in an Industrial Branch

In this paper the author suggests a mathematical model whose solution allows to determine the size up to which existing centres should be developed, as well as the number location and capacity of the new industrial objectives.

The most efficient instrument for increasing the economic potential of the country and for a continuous improvement in the productivity of social labour is the endowment of national economy with ever better and more productive fixed production funds. In our country, as it is shown in the Document of the IXth Congress, during the period of the next 5-year plan, the volume of investments from state funds will rise up to the impressive sum of more than 280 billion lei, a sum equal to the value of investments for the last ten years.

The choice, in each case, of the most efficient solution from the economic standpoint, is a decisive condition for the good management of the investment fund.

Among the many problems raised by the rational use of the investment funds, there is also the one regarding the location of the new industrial objectives, linked to the mode of development and modernization of the present ones.

The problem is often set before the engineers and economists in the planning institutions of the distribution of the investment fund among enterprises producing the same product, with respect to their reconstruction and modernization, as well as in view of the construction of new enterprises.

The aim is always well defined and it refers either to bring about an increase in the physical volume of production, or to reduce the cost price, or to improve some technical-economical indicators, etc.

The use of mathematical methods is imperative in view of finding the *most rational solutions for each case in part. One of the most useful methods* for solving some problems linked with the rational use of the investment fund is the method of mathematical programming.

In this paper, the author approaches one of the problems which arise in the case where one is faced with the problem of using a given investment fund to develop and modernize a sector in which a homogenous product is produced.

The first part of the paper gives a mathematical model of the problem to be solved.

The model obtained is one of linear programming. On account of the great number of unknowns, the solution of this model by means of the common methods for solving models of linear programming would imply a large volume of computing.

In the second part of the paper a computing algorithm is suggested, special for the model considered, and in the third part a numerical example is given.

1. SETTING THE PROBLEM

As already mentioned, this paper gives a mathematical model of an investment problem under conditions of the existence of a certain investment fund for a branch, with the help of which new objectives are to be constructed, besides developing and modernizing the existing ones.

The problem arising from an economic standpoint is that of distributing in a optimum manner the investment fund between the new constructions and the development and modernizing operations.

The following are assumed to be known :

— The existing production centres and maximum production capacity up to which they can be expanded ;

— The points where the new industrial objectives may be located, the existing projects for each point and the maximum capacity designed ;

— The specific investment (the share of investment for every unit of product) for each production centre (both the new ones and those subject to expansion) ; it is assumed hereafter that investment costs are directly proportional with the installed production capacity ;

— The efficiency coefficient of investments in the branch considered ;

— The investment fund allotted to the branch considered ;

— The position of the users and the quantity of their requirements. Since any industrial objective is built for use over a longer period, one must, in the computation, take into account not only the quantities required by the users within a certain year, but also the necessary average for the period considered, determined on basis of the knowledge of the upward trend of these requirements ;

— Production costs for every unit of the product at every production centre (including the new ones) ;

— Transportation costs of a unit of the product from every production centre to every user.

Based on these data the optimum solution should be determined regarding the size to which the existing centres should be expanded, as well as the location, adopted project and capacity of the new objectives.

The following notations will be introduced in view of writing the mathematical model :

m_1 — number of points where the new objectives may be located. They will be noted with the index i ($i = 1, 2, ..., m_1$)

m_2 — number of existing production centres. They will be noted with the index i $(i = m_1 + 1,\ m_1 + 2,...,\ m_1 + m_2)$ $m = m_1 + m_2$

p_i — number of existing design variants for the location point i $(i = 1, 2,...,\ m_1)$

r — number of new objectives to be built.

n — number of users. They will be noted with the index j $(j = 1, 2,...,n)$

$a_i(V_k)$ — maximum production capacity of the new production centre i in case design V is adopted $(i = 1, 2,...,\ m\ ;\ k = 1, 2,...,\ p\ )$

a_i — existing production capacity at centre i $(i = m_1 + 1,\ m_1 + 2,...,\ m_1 + m_2)$

a_i' — production capacity by which the centre i can be expanded $(i = m_1 + 1,\ m_1 + 2,...,\ m_1 + m_2)$

b_j — quantity required by user j $(j = 1, 2,...,\ n)$

$d_i(V_k)$ — cost price* (computed) at new production centre i in case design variant V_k is adopted $(i = 1, 2,...,\ m_1;\ k = 1, 2,...,\ p\ )$

d_i — cost price* at existing production centre i $(i = m_1 + 1,\ m_1 + 2,...,\ m_1 + m_2)$

d_i' — cost price* corresponding to new production capacity at existing centre i $(i = m_1 + 1,\ m_1 + 2,...,\ m_1 + m_2)$

t_{ij} — transportation cost for unit of product from centre i to user j j $(i = 1, 2,...,\ mi = 1, 2,...,\ m_1 + m_2;\ j = 1, 2,...,\ n)$

$e_i(V_k)$ — specific investment at production centre i in the variant k $(i = 1, 2,...,\ m_1;\ k = 1, 2,...,\ p_i)$

e_i — specific investment necessary to expand the production centre i $(i = m_1 + 1,\ m_1 + 2,\ ...,\ m_1 + m_2)$

E — efficiency coefficient of investments in the branch considered

I — volume of investment funds allotted to the sector considered.

The unknowns of the problem are:

X_i — volume of production at centre $i(i = 1, 2,...,\ m)$

x_{ij} — quantity to be delivered by production centre i to the user $j(i = 1, 2,...,\ m;\ j = 1, 2,...,\ n)$

X_i' — supplementary production volume at centre i $(i = m_1 + 1\ ..\ m)$ consequently to production capacity expansion

x_{ij}' — quantity to be delivered from supplementary production by production centre i to user j $(1 = m_1 + 1\ ...\ m;\ j = 1,2,...n)$

We introduce the following bivalent variables:

$t_i = \begin{cases} 1 \text{ if at point } i \text{ a new objective will be built} \\ 0 \text{ if at point } i \text{ no new objective will be built } (i = 1,2,...,\ m_1) \end{cases}$

$S_i(V_k) = \begin{cases} 1 \text{ if at point } i \text{ a new objective will be built on basis of design } V_k \\ 0 \text{ if at point } i \text{ no new objective will be built on basis of design } V_k \end{cases}$

$$\left\{ \begin{array}{l} i = 1,2,...,\ m_1 \\ k = 1,2,...,\ p_i \end{array} \right\}$$

*) only direct costs

We also note:

$$C_{ij}(V_k) = d_i(V_k) + t_{ij} + E_{ei}(V_k) \qquad \begin{pmatrix} i = 1,2,..., m_1 \\ j = 1,2,..., n \end{pmatrix}$$

$$C_{ij} = d_i + t_{ij}$$

$$C_{ij} = d_i + t_{ij} + E_{ei} \qquad \begin{pmatrix} i = m_1 + 1,..., m \\ j = 1,2,..., n \end{pmatrix}$$

With the above notations the mathematical model to be analysed further is the following:

$$\sum_{i=1}^{m_1} t_i = r \sum_{k=1}^{pi} S_i(V_k) = t_i \qquad (1.1)$$

$$\sum_{j=1}^{n} x_{ij} = X_i \leqslant t_i S_i(V_k)_{ai}(V_k) \qquad (i = 1,2,..., m_1) \qquad (1.2')$$

$$\sum_{j=1}^{n} x'_{ij} = X_i \leqslant a_i \qquad (i = m_1 + 1,..., m) \qquad (1.2'')$$

$$\sum_{j=i}^{n} x'_{ij} = X_i \leqslant a'_i \qquad (i = m_1 + 1, m_1 + 2,..., m) \qquad (1.2''')$$

$$\sum_{i=1}^{m} x_{ij} + \sum_{i=m_1+1}^{m} x'_{ij} = b_j \qquad (j = 1,2,..., n) \qquad (1.3')$$

$$\sum_{i=1}^{m_1} e_i(V_k) \cdot t_i S_i(V_k) \cdot X_i + \sum_{i=m_1+1}^{m} eX_i' \leqslant \mathrm{I} \qquad (1.4')$$

$$x_{ij} \geqslant 0; \quad x'_{ij} \geqslant 0 \begin{cases} i = 1,2,..., m \\ j = 1,2,... n \end{cases} \qquad (1.5')$$

$$\sum_{i=1}^{\cdots 1} \sum_{j=1}^{n} C_{ij}(V_k)\, t_i s_i(V_k)\, x_{ij} + \sum_{i=m_1+1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} + \sum_{i=m_1+1}^{m} \sum_{j=1}^{n} C'_{ij} x'_{ij} = Z \text{ (min)} \qquad (1.6')$$

To solve the model formed by the relations *(1.1)*, *(1.2')* *(1.2'')*, *(1.2''')*, *(1.3')*, *(1.4')*, *(1.5')* and *(1.6')*, since it contains the variable $S_j(V_k)$ and t_i which can only have integer values, one may apply, for instance, Gomory's algorithm (1). However, taking into account the considerable dimensions of the problems, the practical solution by means of Gomory's method is difficult even with the help of modern electronic machines.

For our scope, we shall hereafter consider that instead of one production centre $i(i = 1,2,..., m_1)$ for which there are p_i design variants, there are p_i production centres, and instead of on old production centre we shall consider two centres, one with the old production capacity, which will require no investment, and the second with a production capacity equal to the one up to which the centre may be expanded.

In this case the solution of our model is easily reduced to the solution of a number of C_p^r problems (where p denotes the number of all the variants $p = \sum_{i=1}^{m_1} p_i$), of the type:

$$\sum_{j=1}^{n} x_{ij} = X_i \leqslant a_i \qquad (i = 1,2,\ldots, m \qquad (1.2)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1, 2,\ldots, n) \qquad (1.3)$$

$$\sum_{i=1}^{m} e_i X_i \leqslant I \qquad (1.4)$$

$$x_{ij} \geqslant 0 \begin{Bmatrix} i = 1,2,\ldots, m \\ j = 1,2,\ldots, n \end{Bmatrix} \qquad (1.5)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} = Z \text{ (min)} \qquad (1.6)$$

In the case, for instance, where we shall have to locate 3 new plants for which there are 5 designs, we shall have to solve $C_5^3 = 10$ problems of the type *(1.2) — (1.6)*, and then choose the best among the solutions obtained.

Obviously, the problem *(1.2) — (1.6)* is interesting in the case where

$$\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j \qquad (1.7)$$

As it can be seen the model *(1.2) — (1.6)* cannot be solved by means of the usual methods for solving the transportation problem, because of the supplementary relation *(1.4)*.

It is clear that the model can be solved by means of one of the general methods for solving models of linear programming (e.g. the simplex method). This, however, may be difficult even for electronic computers because in such problems the number of x_{ij} unknowns is rather large.

The model may be solved by using Dantzig-Wolfe's decomposition principle (*2*). However, this has the disadvantage that a great number of iterations is required in the application of this principle. Finkelstein (*3*) has suggested an iterative method to determine an aproximate solution of the model *(1.2) — (1.6)*, the degree of aproximation up to which the computation is carried out being set from the start.

The procedure suggested by Finkelstein essentially comes down to the successive solution of several transportation problems where functions of the following form are considered as objective functions:

$$Z(\lambda) = \sum_{i=1}^{m} \sum_{j=1}^{n} [(1 - \lambda)\, C_{ij} + \lambda\, e_i]\, x_{ij} \qquad (1.8)$$

where $\lambda \in [0, 1]$

An algorithm is suggested hereafter, by means of which it will be possible to determine, in a finite number of steps, an optimum solution of the model *(1.2) — (1.6)*

2. COMPUTATION ALGORITHM

The algorithm we suggest starts from the optimum solution of the model considered from which, however, the relation *(1.4)* is excluded. In case the optimum of the model formed from relations *(1.2)*, *(1.3)*, *(1.5)* and *(1.6)* also verifies relation *(1.4)*, it constitutes an optimum solution of the problem considered.

This does not generally occur and in this case a series of modifications is carried out on the initial solution, so as to finally reach an optimum solution of the model *(1.2) — (1.6)*.

In order to describe the algorithm suggested we shall introduce the following definitions :

Definition 2.1

The mathematical model formed of relations *(1.2) — (1.6)* will be called model A and the model formed of relations *(1.2)*, *(1.3)*, *(1.5)* and *(1.6)* will be called model B.

Definition 2.2

A system of values $X=(x_{ij})_{mxn}$ which verifies the relations *(1.2) — (1.6)* is called an *optimum solution of model A*.

Definition 2.3

A system of values $X = (x_{ij})_{mxn}$ which verifies the relations *(1.2)*, *(1.3)*, *(1.5)*, and *(1.6)* is called an *optimum solution of model B*.

Definition 2.4

We call cell of the matrix $C=(c_{ij})_{mxn}$ any pair of indices (i, j) $(1 \leqslant i \leqslant m;$ $1 \leqslant j \leqslant n)$

Let X be an optimum solution of problem B. According to this solution we shall divide the set of indices i :

$$(M = 11,\ 2,...,\ i,...\ m) \tag{2.1}$$

in two sub-sets :

$$M^{\circ} = \left(i \left| \sum_{j=1}^{n} x_{ij} = a_i \right.\right) \tag{2.2}$$

$$M^{-} = \left(i \left| \sum_{j=1}^{n} x_{ij} < a_i \right.\right) \tag{2.3}$$

Obviously :

$$M^{\circ} \cap M^{-} = \Phi \tag{2.4}$$

and

$$M = M^{\circ} \cup M^{-} \tag{2.5}$$

Definition 2.5

We call connecting path between the lines i_0 and i_s of the matrix $C = (c_{ij})_{mxn}$ the following set of cells:

$$\{(i_0, j_0), (i_1, j_0), (i_1, j_1), (i_2, j_1),..., (i_{s-1}, j_{s-1}), (i_s, j_{s-1})\} \qquad (2.6)$$

where

$$i_0 \in M,\ i_s \in M^-\ i_k \in M \qquad (k = 1,2,...,\ s-1) \qquad (2.7)$$

and

$$x_{ik\ jk} > 0 \quad (k = 0,1,2,...,\ s-1) \qquad (2.8)$$

Definition 2.6

We call lenght of the connecting path *(2.6)* the following magnitude:

$$l = \sum_{k=0}^{s-1} [C_{i_{k+1}jk} - C_{i_kjk}] \qquad (2.9)$$

Definition 2.7

One calls minimum distance between the lines $i_0 =$ and $i_s =$ the following magnitude:

$$d_{i_0}i_s = \min\ l \qquad (2.10)$$

the minimum being chosen taking into account all the connecting paths between the two lines.

Let $X^0 = x^0_{ij})_{mxn}$ be an optimum solution of model B and $d_{i_0}i_s (i_0 \in M,\ i_s \in M^-)$ the minimum distance between the lines $i_0 =$ and i_s. We shall assume that the minimum *2.10* is achieved for the connecting path *(2.6)*

We shall note

$$u' = \min\ x^0_{ikjk} \quad k = 0,1,...,\ s-i \qquad (2.11)$$

$$u'' = ai_s - \sum_{j=1}^{n} x^0_{isj} \qquad (2.12)$$

$$u\ \min\ (u',\ u'') \qquad (2.13)$$

Since u > 0, we shall define as follows a new solution $X' = (x'_{ij})_{mxn}$ of problem B:

$$x'_{ij} = x^0_{ij} - u\ \ (i,\ j) \in \{(i_0, j_0), (i_s, j_1),..., (i_{s-1}, j_{s-1})\} \qquad (2.14)$$

$$x'_{ij} = x^0_{ij} + u\ \ (i,j) \in \{(i_1, j_0), (i_2, j_1),..., (i_1, j_{s-1})\} \qquad (2.15)$$

$$x'_{ij} = x^0_{ij} \text{ for the remaining pairs of indices} \qquad (2.16)$$

We shall give without proof*) the following lemma and theorem:

Lemma 2.1

The solution X' defined by relations *(2.14)* — *(2.16)* does not contain cycles.

Theorem 2.1 (fundamental)

The solution X' obtained from the solution X^0 by means of relations *(2.14)* — *(2.16)* is an optimum solution for the following B-type model.

$$\sum_{j=1}^{n} x_{ij} = X_i^0 - \partial_i^{i_0} u + \partial_i^{i_s} u \quad (i = 1,2,..., m) \tag{2.17}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1,2,..., n) \tag{2.18}$$

$$x_{ij} \geqslant 0 \qquad \begin{Bmatrix} (i = 1,2,..., m) \\ (j = 1,2,..., n) \end{Bmatrix} \tag{2.19}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} = Z \text{ (min)} \tag{2.20}$$

where ∂ is Kroneker's symbol.

As a consequence of the above theorem we have the following

Corollary 2.1

The following relation exists between the values of the objective function corresponding to the optimum solutions X^0 and X^1

$$Z(X^1) = Z(X^0) + u d_{i_0 i_s} \tag{2.21}$$

where $d_{i_0 i_s}$ is the minimum distance between the lines i_0 an i_s, and u is defined by relation *(2.13)*.

Further, based on theorem *(2.1)* we shall give the algorithm for solving model *A* *((2.1)* — *(1.6))*.

Let X^0 be an optimum solution of model *B*. In the case where

$$\sum_{i=1}^{m} e_i X_i^0 = I_0 \leqslant I. \tag{1.4'}$$

we have an optimum solution of model A and the problem is solved. Let us assume that this relation is not verified.

In this case we note by:

$$\Delta = \min \frac{d_{i'i''}}{ei' - ei''} = \frac{d_{i_0 i_s}}{e_{i_0} e_{i_s}} \tag{2.22}$$

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*) These proofs will appear in another paper where an algorithm is given for solving transportation-type problems (classical transportation problem, transportation problem with parameters, transportation problems with limited capacity, multi-dimensional transportation problems, non-linear transportation problems, and so on).

the minimum being chosen, all the i lines being taken into account on condition that :

$$\sum_{j=1}^{n} x^0_{i'',j} < a_{i''} \text{ and} \qquad (2.23)$$

$$e_{i'} - e_{i''} > 0 \qquad (2.24)$$

In other words only those pairs of indices i are chosen (those pairs of production centres) for which, according to optimum solution X^0 at production centre i'' there still is an available production capacity, and the specific investment is smaller than at production centre i'.

In case there are several pairs of indices for which the minimum of *(2.22)* is achieved, one choses among them those for which the maximum of the difference

$$e_{i'} - e_{i''}, \qquad (2.22')$$

is also achieved.

In this case we shall pass from the optimum olution X^0 of model B, to the solution X^1 by means of the formulae *(2.14)* — *(2.16)* where, however, nstead of u given by formula *(2.13)* we take

$$\bar{u} = \min\left(u',\ u'',\ \frac{I^0 - I}{e_{i_0} - e_{i_s}}\right) \qquad (2.13')$$

where u', u'' are those yielded by the formulae *(2.11)* — *(2.12)*, and

$$\sum_{i=1}^{m} e_i X_i^0 = I^0 \qquad (2.25)$$

Let be :

$$\sum_{i=1}^{m} e_i X_i^1 = I^1 \qquad (2.26)$$

Obviously

$$I^1 < I^0 \qquad (2.27)$$

Theorem 2.2

The solution X^1 determined above is optimum for the model formed from relations *(1.2)*, *(1.3)*, *(1.5)*, *(1.6)* and relation

$$\sum_{i=1}^{m} e_i X_i \leqslant I^1 \qquad (2.28)$$

I^1 being defined by *(2.26)*

Proof

Let X^1 be the solution determined above and $\bar{X}$ a solution whatever of the model formed from relations *(1.2)*, *(1.3)*, *(1.5)*, *(1.6)* and *(2.28)*.

Let be

$$\sum_{j=1}^{n} \bar{x}_{ij} = \bar{X}_i = X_i^0 + \varepsilon_i \leqslant a_i \qquad (i = 1,2,..,\ m) \tag{2.29}$$

where ε_i was used to note the deviations

$$\varepsilon = \bar{X}_i - X_i^0 \tag{2.30}$$

From

$$\sum_{i=1}^{m} \bar{X}_i = \sum_{i=1}^{m} X_i^0 = \sum_{j=1}^{n} b_j \tag{2.31}$$

there results that:

$$\sum_{i=1}^{m} \varepsilon_i = 0 \tag{2.32}$$

We shall again divide the set of indices

$$M = (1,2,...\ i,...\ m) \text{ in two sub-sets}$$

$$M' = \{i \mid \varepsilon_i < 0\} \tag{2.33}$$

$$M'' = \{i \mid \varepsilon_i \leqslant 0\} \tag{2.34}$$

t i s obvious that

$$M' \bigcup M'' = M\,; \quad M' \bigcap M'' = \emptyset \tag{2.35}$$

By applying corollary *2.1* we have

$$Z(\bar{X}) \geqslant Z(X^0) + \sum_{i' \in M''} \varepsilon_{i'} d_{i',i''} \tag{2.36}$$

where $d_{i',i''}$ is the minimum distance between the lines i' and i''.

Noting by

$$d_{i_0' i_0''} = \min_{\substack{i' \in M' \\ i'' \in M''}} d_{i',i''} \tag{2.37}$$

we have:

$$Z(\bar{X}) \geqslant Z(X^0) + d_{i_0' i_0''} \sum_{i \in M''} \varepsilon_i \tag{2.38}$$

On the other hand, since $\bar{X}$ satisfies relation (*2.28*) we have:

$$\sum_{i=1}^{m} e_i \bar{X}_i = \sum_{i=1}^{m} e_i \bar{X}_i^0 + \sum_{i=1}^{m} e_i \varepsilon_i \leqslant I^1 \tag{2.39}$$

hence

$$\sum_{i=1}^{m} e_i \varepsilon_i \leqslant I^1 - I^0 \tag{2.40}$$

or

$$\sum_{i\in M'} e_i\varepsilon_i + \sum_{i\in M''} e_i\varepsilon_i \leqslant I^1 - I^0 \qquad (2.40')$$

But from (*2.32*) there results that

$$\sum_{i\in M'} \varepsilon_i = -\sum_{i\in M''} \varepsilon_i \qquad (2.41)$$

Noting by

$$e_{i'} = \max_{i\in M'} e_i; \qquad e_{i''} = \min_{i\in M''} e_i \text{ we have} \qquad (2.42)$$

$$e_{i'} \sum_{i\in M'} \varepsilon_i + e_i \sum_{i\in M''} \varepsilon_i = (e_{i''} - e_{i'}) \sum_{i\in M''} \varepsilon_i \leqslant I^1 - I^0 \qquad (2.43)$$

But since $I^1 - I^0 < O$, and $\sum_{i\in M''} \varepsilon_i > 0$, there results that:

$$e_{i''} - e_{i'} < 0 \qquad (2.44)$$

and hence

$$\sum_{i\in M''} \varepsilon_i \geqslant \frac{I^0 - I'}{e_{i''} - e_{i'}} \qquad (2.45)$$

In this case relation (*2.38*) becomes

$$Z(\overline{X}) \geqslant Z(X^0) + d_{i'i''} \frac{I^0 - I^1}{e_{i''} - e_{i'}} \qquad (2.46)$$

or

$$Z(\overline{X}) > Z(X^0) + \frac{d_{i'i''}}{e_{i''} - e_{i'}} (I^0 - I^1) \qquad (2.46')$$

But, on the other hand, according to corollary *2.1* we have

$$Z(X') = Z(X^0) + U \cdot d_{i_0 i_s} \qquad (2.47)$$

and

$$U \leqslant \frac{I^0 - I^1}{e_{i_0} - e_{i_s}} \qquad (2.48)$$

Hence

$$Z(X^1) \leqslant Z(X^0) + \frac{d_{i_0 is}}{e_{i_0} - e_{i_s}} (I^0 - I') \qquad (2.47')$$

From relations (*2.22*), (*2.46'*), (*2.47*) and (*2.48*) there results

$$Z(\overline{X}') \leqslant Z(X^0) + \frac{d_{i'i''}}{e_{i''} - e_{i'}} (I^0 - I') \geqslant Z(X') \qquad (2.49)$$

Since $\overline{X}$ is a solution whatever of the model formed from relations (*1.2*), (*1.3*), (*1.5*), (*1.6*) and (*2.28*), there results that X^1 is an optimum solution of this model.

On basis of theorem 2.2 the procedure to be followed in view of determining an optimum solution of the problem (*2.2*) — (*2.6*) results, viz.:

If I^1 *I*, X^1 is an optimum solution of the problem; if not, a new optimum solution X^2 is determined by a similar procedure, so that

$$\sum_{i=1}^{m} e_i X_i^2 = I^2 < I' \text{ and so on} \qquad (2.50)$$

In a finite number of steps, a solution X^p will be determined, for which

$$\sum_{i=1}^{m} e_i X_i^p = I^p \leqslant I \qquad (2.51)$$

which, on basis of theorem (*2.2*) will be an optimum solution of model A

3. NUMERICAL EXAMPLE

Let us assume that in a branch in which a homogenous product is manufactured there are two plants which we shall call A and B. The maximum production capacities of these plants are at present 10,000 t., respectively 5,000 t. yearly and both plants work at maximum capacity (three shifts).

In the years 1966—1970 of the 5-year plan it is expected that consumption will reach up to 28,000 tons yearly. To meet this increase it is known that the plants A and B may be so expanded that the first will reach a capacity of 14,000 tons yearly, and the second 7,000 t. yearly. Likewise, a new plant, according to designs, may be located at points C or D, both having a potential capacity of maximum 15,000 t. yearly.

From the state funds 7,000,000 lei have been allotted for the development of this branch. The construction and assembly work for the new plant amounts to 1,400,000 lei in both variants.

We further assume that the expansion of plants A and B does not require construction work.

The specific investments* are the following:

— for the expansion of plant *A*	400 lei/ton
— for the expansion of plant B	350 lei/ton
— for the design of plant C	450/ton
— for the design of plant D	460/ton

The efficiency coefficient of investments in the branch considered is 0.20 (20%), in other words the period necessary to retrieve the investment is of $\frac{1}{0.20} = 5$ years.

*) Only that part of the investment costs which vary directly proportionnally with the capacity installed is included in the computation.

With regard to the users, they are seven in number. Their average yearly requirements in the period 1966—1970 is the following:

User 1	5000 t
„ 2	3000 t
„ 3	3500 t
„ 4	4000 t
„ 5	2000 t
„ 6	8000 t
„ 7	2500 t
Total	28000 t

As shown at point 1, since, in the computation, the expansion facilities of the existing plants are taken into account, instead of plants A and B which can each be expanded, we shall consider two units, the one having the old production capacity, and the second having a production capacity equal to the one by which the respective plant will be expanded. Obviously, the cost price for the two „units" which, in fact, represent the same plant, may differ. The table hereunder shows the specific investments for each plant, the cost price (direct costs), production capacities, requirements of users and transportation costs from each supplier to each user, the product Ee_i between specific investment and the efficiency coefficient of investments by branch, as well as the sum $Ee_i + d_i$.

To determine the optimum investment variant (which plants are to be expanded and up to what capacity and where and at what capacity the new plant will operate), we must solve two problems of the type *(1.2)* — *(1.6)*'

(lei/ton) *Table 1*

Plants.	Specific investment e_i	$E\,e_i$ $(E=0.2)$	Cost price (d_i)	$d_i + Ee_i$ $(3+4)$	Maximum production capacity (t)	Users and unit transportation costs (t_{ij})						
						1	2	3	4	5	6	7
1	2	3	4	5	6	7	8	9	10	11	12	13
A	0	0	200	200	10 000	40	35	25	50	10	15	30
A'	400	80	160	240	4 000	40	35	25	50	10	15	30
B	0	0	220	220	5 000	30	15	50	25	40	10	24
B'	350	70	180	250	2 000	30	15	50	25	40	10	24
C	450	90	145	235	15 000	20	20	30	15	10	25	40
D	460	92	138	230	15 000	40	10	15	30	5	5	20
Required quantity						5 000	3 000	3 500	4 000	2 000	8 000	2 500

one in which plants A, A', B, B' and C are included in the computation, and the second in which plants A, A', B, B' and D are taken into the computation.

The mathematical model of the first problem is

$$\sum_{j=1}^{7} x_{ij} = X_i \leqslant a_i \qquad (i = 1,\ 2,\ 3,\ 4,\ 5) \tag{3.1}$$

$$\sum_{i=1}^{5} x_{ij} = b_j \qquad (j = 1,\ 2,\ \ldots,7) \tag{3.2}$$

$$\sum_{i=1}^{5} e_i x_i \leqslant 7{,}000{,}000 - 1{,}400{,}000 = 5{,}600{,}000 \tag{3.3}$$

$$x_{ij} \geqslant 0 \begin{cases} i = 1,\ 2,\ \ldots,5 \\ j = 1,\ 2,\ \ldots,7 \end{cases} \tag{3.4}$$

$$\sum_{i=1}^{5}\sum_{j=1}^{7} C_{ij} x_{ij} = z \ (\min) \tag{3.5}$$

The magnitudes a_i, b_j as well as $C_{ij} = d_i + Ee_i + t_{ij}$, which represent the cost price, free at the user's to which is added the product Ee_i between the efficiency coefficient of the investment and the specific investment, are shown in Table 2.

Table 2.

(lei/ton)

Plants \ Users		1	2	3	4	5	6	7	Production capacity (a_i)
1	*A*	240	235	225	250	210	215	230	10 000
2	*A'*	280	275	265	290	250	255	270	4 000
3	*B*	250	235	270	245	260	230	244	5 000
4	*B'*	280	265	300	275	290	260	274	2 000
5	*C*	255	255	265	250	245	260	275	15 000
Required quantity (b_j)		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

As shown above, for the beginning, we shall solve the model excluding the restriction (*3.3*) related to the limitation of the investment fund. In this case the model is solved simply, being a common transportation model. The optimum solution is shown in Table 3.

Table 3

(variant I)

Users \ Plants		1	2	3	4	5	6	7	Production capacity (a_i)
1	*A*			1 500			8 000	500	10 000
2	*A'*								
3	*B*		3 000					2 000	5 000
4	*B'*								
5	*C*	5 000		2 000	4 000	2 000			13 000
Required quantity (b_j)		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

According to this variant the entire consumption increase (13,000 t) would be assured by the plant to be built in C. In this case the total production and transportation cost would be 5,490,500 lei and if we also add Ee_i it would be 6,660,500 lei.

The investment cost would be $13{,}000 \times 450 = 5{,}850{,}000$ lei. There results that this variant is not possible since the investment would exceed by 250,000 lei the available fund (4.4%).

We must, in this case, find the modifications leading to a decrease in the cost of investment.

To this end we must find the connection paths between the lines 5 (C) 2 (A') and 5(C) 4(B') *. We find that the minimum distance between the lines 5 and 2, respectively 5 and 4 are:

$$d_{52} = 0 \quad d_{54} = 4 \tag{3.6}$$

and one finds, respectively, for the connecting paths:

$$\{(5,3);\ (2.3)\} \tag{3.7}$$

$$\{(5.3);\ (1.3);\ (1.7):\ (4.7)\} \tag{3.8}$$

To compute the magnitude Δ (see 2.22), we have

$$\frac{d_{52}}{e_5 - e_2} = \frac{0}{450-400} = 0\,; \quad \frac{d_{54}}{e_5 - e_4} = \frac{4}{450-350} = 0.04 \tag{3.9}$$

Hence:

$$\Delta = \frac{d_{52}}{e_5 - e_2} = 0 \tag{3.10}$$

Making the modification corresponding to connecting path (3.7) we obtain the new variant.

Table 4

(Variant II)

Plants \ Users	1	2	3	4	5	6	7	Production capacity (a_i)
1 A			1 500			8 000	500	10 000
2 A′			2 000					2 000
3 B		3 000					2 000	5 000
4 B′								
5 C	5 000			4 000	2 000			11 000
Required quantity (t_j)	5 000	3 000	3 500	4 000	2 000	8 000	2 500	

In the new variant the investment costs have decreased to $11{,}000 \times \times 540 \times 2{,}000 \times 5{,}750{,}000$.

*) See (4) for the methodology to find the path of minimum distance between two lines.

This variant is not acceptable either, since the investment fund would be exceeded by 150,000 lei.

There results that a further series of modifications should be brought to it.

For the variant in Table 4 we find that the paths of minimum distance connecting the lines 5 (C) — 2 (A') and 5 (C) — 4 (B') are:

$$\{(5.5);\ (2.5)\} \tag{3.11}$$

$$\{(5.5);\ (1.5);\ (1.7);\ (4.7)\} \tag{3.12}$$

and the minimum respective distances are

$$d_{52} = 250 - 245 = 5 \tag{3.13}$$

$$d_{54} = 210 - 245 + 274 - 230 = 9 \tag{3.14}$$

To compute the magnitude Δ we have:

$$\frac{d_{52}}{e_5 - e_2} = \frac{5}{450 - 400} = \frac{5}{50} = 0{,}1\,; \quad \frac{d_{54}}{e_5 - e_4} = \frac{9}{450 - 350} = \frac{9}{100} = 0.09 \tag{3.15}$$

Hence:

$$\Delta = \frac{d_{54}}{e_5 - e_4} = 0.09 \tag{3.16}$$

By making the corresponding modification of the connecting path (*3.12*) we obtain the following variant.

Table 5

(Variant III)

Plants \ Users		1	2	3	4	5	6	7	Production capacities
1	A			1 500		500	8 000		10 000
2	A′			2 000					2 000
3	B		3 000					2 000	5 000
4	B′							500	500
5	C	5 000			4 000	1 500			10 500
Required quantity		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

In the new variant the investment costs have decreased to: 10,500 $\times$ $\times$ 450 + 2,000 $\times$ 400 + 350 $\times$ 500 = 5,700,000 lei, therefore the investment fund is exceeded by 100,000 lei. And therefore the new variant must also be modified.

For the variant in table 5 the paths of minimum distance are

$$\{(5.5);\ (2.5)\} \tag{3.17}$$

$$\{(5.5);\ (1.5);\ (1.6);\ 4.6)\} \tag{3.18}$$

and the respective minimum distances are

$$d_{52} = 250 - 245 = 5 \tag{3.19}$$

$$d_{54} = 210 - 245 + 260 - 215 = 10 \tag{3.20}$$

To compute the magnitude Δ we have:

$$\frac{d_{52}}{e_5 - e_2} = \frac{5}{450 - 400} = 0.1 \quad \frac{d_{54}}{e_5 - e_4} = \frac{10}{450 - 350} = 0.1 \tag{3.21}$$

Making the corresponding modification to the connecting path (*3.18*) we obtain the following optimum variant.

Table 6

(Optimum variant)

Plants \ Users		2	2	3	4	5	6	7	Production capacities
1	A			1 500		1 500	7 000		10 000
2	A′			2 000					2 000
3	B		3 000					2 000	5 000
4	B′						1 000	500	1 500
5	C	5 000			4 000	500			9 500
Required quantity		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

According to the optimum variant in Table 6 plant *A* should be expanded up to a capacity of 10,000 + 2,000 = 12,000 tons, plant *B*, up to a capacity of 5,000 + 1,500 = 6,500 t yearly, and plant *C* should have a capacity of 9,500 t yearly.

The investment cost would be:

$$9{,}500 \times 450 + 2{,}000 \times 400 + 1{,}500 \times 350 = 5{,}600{,}000 \text{ lei},$$

i.e. equal to the allotted fund.

In the optimum variant the investment cost was reduced in comparison to the initial variant, by

$$5{,}850{,}000 - 5{,}600{,}000 = 250{,}000 \text{ lei};$$

however, the total transportation and production cost ($+ Ee_i$) has increased from 6,660,500 to 6,675,000 lei, i.e. by 14,500 lei.

If we only consider the cost price and transportation costs, without adding the product Ee_i, the sum of 5,490,500 lei corresponds to the initial variant in Table 3, while the sum of 5,555,000 lei corresponds to the optimum variant in Table 6.

Now proceeding to solve the second model in which the plants A, A', B, B' and D will be taken into the computation, we should initially find the optimum solution of the transportation problem whose data are shown in Table 7.

Table 7

Plants \ Users		1	2	3	4	5	6	7	Production capacities
1	*A*	240	235	225	250	210	215	230	10 000
2	*A'*	280	275	265	290	250	255	270	4 000
3	*B*	250	235	270	245	260	230	244	5 000
4	*B'*	280	265	300	275	290	260	274	2 000
5	*D*	270	240	245	260	235	235	250	1 500
Requirep capacity		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

(Initial variant)

Table 8

Plants \ Users		1	2	3	4	5	6	7	Production capacity
1	*A*	4 000		3 500		2 000		500	10 000
2	*A'*								
3	*B*	1 000			4 000				5 000
4	*B'*								
5	*D*		3 000				8 000	2 000	13 000
Available quantity		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

The investment cost in the initial variant is 13,000 × 460 = 5,980.000 lei therefore it exceeds by 380,000 lei the allotted investment fund. By making, like in the preceding case, the corresponding modification, the following optimum solution is reached after two iterations:

(Optimum variant)

Table 9

Plants \ Users		1	2	3	4	5	6	7	Production capacity
1	*A*	2 000		833		2 000	2 667	2 500	10 000
2	*A'*			2 667					2 667
3	*B*	1 000			4 000				5 000
4	*B'*	2 000							2 000
5	*D*		3 000				5 333		8 333
Necessary quantity		5 000	3 000	3 500	4 000	2 000	8 000	2 500	

According to the optimum variant in Table 9, plant A should be expanded up to a capacity of $10,000 + 2,667 = 12,667$ t yearly, plant B up to a capacity of $5,000 + 2,000 = 7,000$ t yearly, and plant D should have a capacity of 8,333 t yearly.

The investment cost would be:

$$8,333 \times 460 + 2,667 \times 400 + 2,000 \times 350 = 5,600,000$$

i.e. equal to the investment fund.

The total cost (cost price + investment costs + Ee_i) corresponding to the optimum variant in Table 9 is of 6,705,840 lei, i.e. 30,840 (6,705,840 — — 6,675,000) larger than in the optimum variant in Table 6.

There results in conclusion that the optimum solution of the problem stated is the one in Table 6.

In other words, in order to have minimum production and transportation costs, the 5,600,000 lei investment fund should be used as follows: $2,000 \times 400 = 800,000$ lei for expanding plant A, $1,500 \times 350 = 525,000$ lei for expanding plant B and $9,500 \times 450 = 4,275,000$ lei for plant C.

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UN ALGORITHME POUR LE SOLUTIONNEMENT DE CERTAINS PROBLÈMES RELIÉS À L'UTILISATION OPTIMALE DES FONDS D'INVESTISSEMENTS DANS UNE BRANCHE INDUSTRIELLE

Les organismes de planification ont souvent à résoudre des problèmes concernant le mode de répartition des fonds d'investissements entre les entreprises d'un secteur.

L'étude propose un modèle mathématique dont la solution permet de déterminer les proportions auxquelles il convient de développer les centres existents, ainsi que le nombre, le lieu de localisation et la capacité des nouveaux objectifs, en connaissant pour une certaine branche: les centre de production existants et les capacités maximales auxquelles ils peuvent être développés; les lieux dans lesquels on peut placer les nouveaux objectifs et les projets existants pour chaque place; les investissements spécifiques pour chaque centre de production; le coefficient d'efficacité des investissements dans la branche considérée; le volume des fonds d'investissements alloués; la position et le nécessaire des bénéficiaires (en dynamique); le prix de revient pour chaque entreprise y compris les nouvelles entreprises).

Pour le modèle construit, l'auteur propose aussi un algorithme original de solutionnement.

АЛГОРИТМ ДЛЯ РЕШЕНИЯ ЗАДАЧ ПО ОПТИМАЛЬНОМУ ИСПОЛЬЗОВАНИЮ ФОНДА КАПИТАЛЬНЫХ ВЛОЖЕНИЙ В ОТДЕЛЬНОЙ ОТРАСЛИ ПРОМЫШЛЕННОСТИ

Перед плановыми органами часто ставятся вопросы распределения фонда капитальных вложений между предприятиями отдельного сектора.

В статье предлагается математическая модель, путем решения которой можно определять величину, до которой следует развивать существующие центры, а также число, место размещения и мощность новых объектов, зная по определенной отрасли следующее: существующие производственные центры и наибольшие мощности, до которых они могут быть развиты; места, где могут быть размещены новые объекты, и имеющиеся по каждому объекту проекты; удельные капитальные вложения для каждого производственного центра; коэффициент эффективности капитальных вложений в рассматриваемой отрасли; объём выделенного фонда капитальных вложений; местонахождение и потребности заказчиков (в динамике); себестоимость на каждом предприятии (включая новые).

Для построенной модели предлагается и оригинальный алгоритм решения.