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Stochastic Programming Models of Dynamic Planning Problems*

Each decision model has an underlying structural model.

1. INTRODUCTION

In this paper we formulate a series of quantitative planning models that are natural extensions of Tinbergen's classic economic policy schema [28, 29]. See also [3, 4, 14, 25, 26].

Underlying each decision problem is a structural model that relates endogenous, decision or instrument, exogenous and data variables. The single decision maker is assumed to have unique (most preferred) target values for the endogenous and decision variables in each of the T periods of his planning horizon. He has cardinal losses in the deviations of achieved values from the targets and his objective is to choose the feasible instrument policy that minimizes the sum of the expected losses over the T periods. [1]

Our main intent is to identify the level of complexity of the resulting decision model when differing assumptions are made on the underlying structural model. The first three models utilize a stationary linear model with known constant coefficients, where the equation errors are assumed random with known distributions. These models differ with respect to inclusion or omission of lagged endogenous variables, random exogenous variables, and bounds on the achieved values of the current period's endogenous variables.

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[1] We prefer to consider the deviation approach over an absolute variable approach in which the decision maker has preference over various variable realizations and maximizes the expected preferences over the T-periods. Essentially everything developed here for deviation models applies with minor changes to absolute models.

The solution of each of these models can, in theory, proceed via dynamic programming and each model is equivalent to a static nonlinear program.

In the last model the decision maker is allowed to utilize some of the information that he receives as the process develops. The underlying structural model is nonlinear with known constant parameters. The (additive) random elements in the model have distributions with unkonwn parameters and the decision maker has prior and sampling distributions for these unknown parameters and he revises his prior distributions in a Bayesian fashion. His problem is to choose an instrument selection policy to minimize the sum of the expected losses. If several additional assumptions are satisfied then the decision maker's problem can also be theoretically solved via dynamic programming and is equivalent to a static nonlinear program.

The last section of the paper contains a brief discussion of computational approaches for the solution of the decision problems developed. A conclusion is that reasonably general static problems can be solved but only very special dynamic problems are computationally feasible.

2. STATIONARY LINEAR MODELS WITHOUT LAGGED ENDOGENOUS VARIABLES

Suppose that the underlying structural model is

$$\beta_0 Y_t = \sum_{\tau=0}^{q} \beta_{\tau 2} X_{t-\tau} + \sum_{\tau=0}^{r} \beta_{\tau 3} Z_{t-\tau} + E_t, \tag{1}$$

where t denotes the present time period, X, Y, Z represent (column) vectors of decision, endogenous, and purely exogenous variables, respectively, E represents a (column) vector of random disturbances, and the β's are matrices of known coefficients. Assume that, at the beginning of a T period planning horizon, the single decision maker (D.M.) knows that the elements of each E_t have a joint multivariate cumulative distribution with mean zero and variance-covariance matrix Σ_t, i.e., $E_t \sim F_t(0, \Sigma_t)$ with F_t independent of F_τ, $t \neq \tau$ and that the Z_t are the constants $\bar{Z}_t$. The D.M. has (unique) preferred targets X_t^* and Y_t^* for the vectors X_t and Y_t in each period t. Let $x_t \equiv X_t - X_t^*$ and $y_t \equiv Y_t - Y_t^*$ refer to deviations away from these targes and f_t and g_t be realvalued (i. e., cardinal) preference functions that rank the deviations x_t and y_t, respectively. The D. M.'s problem is to chose a set of instruments X_1 from a constraint set J_1, them observe the random vector Y_1, then choose X_2 from J_2, observe Y_2, etc. His objective being to choose X_1 in such a way that he will be able to choose decision policies X_2,... X_T sequentially using relevant information concerning $(\hat{E}, Y_1)$,..., $(\hat{E}_{T-1}, Y_{T-1})$ so as to minimize the expected sum of the preference functions over $t = 1, ..., T$.

Suppose that[2] $J_t \equiv \{X_t \mid \alpha_t \leqq X_t \leqq \beta_t\}$, with $\alpha_t \leqslant \beta_t$ where the vector inequality $\leqslant$ means $\leqq$ for all components but $<$ for at least one com-

[2] This assumption, a standard one in economic policy problems, means that the instruments are independent and that some instruments may have their choice limited to some specified range.

ponent, and that[3] β_0 is square and non-singular. Then $X_t \in J_t$ is equivalent to $x_t \in K_t$, where $K_t \equiv \{x_t \mid \alpha_t - X_t^* \leqq x_t \leqq \beta_t - X_t^*\}$, and decision problem (d.p.) I is[4]

$$\textit{minimize } f_1(x_1) + E_{\psi_1}\{g_1(y_1) + \min_{x_2}[f_2(x_2) + E_{\psi_2}(g_2(y_2) \tag{2}$$
$$+ \min_{x_3}\{f_3(x_3) + \ldots + E_{\psi_T}[g_T(y_T)]\})]\},$$

subject to

$$\begin{array}{llll}
B_0x_1 & & + y_1 & = \psi_1, \\
B_1x_1 + B_0x_2 & & + y_2 & = \psi_2, \\
\vdots & & & \vdots \\
B_{q-1}x_1 + \ldots & + B_0x_q & + y_q & = \psi_q, \\
\vdots & & \vdots & \vdots \\
B_qx_{T-q} + \ldots + & B_0x_T & + y_T & = \psi_T, \\
x_1 \in K_1, \; x_2 \in K_2, \ldots & x_T \in K_T, & &
\end{array}$$

where E_{ψ_t} represents mathematical expectation with respect to ψ_t, with ψ_t

$$\sim G_t(\bar{\psi}_t, \Sigma_{\psi t}), \; \bar{\psi}_t \equiv \sum_{\tau=0}^{q} \beta_0^{-1}\beta_{\tau 2}X_{t-\tau}^* + \sum_{\tau=0}^{s} \beta_0^{-1}\beta_{\tau 3}\bar{Z}_{t-\tau} - \beta_0^{-1}Y_t^* + \beta_0^{-1}\bar{E}_t,$$

$$\Sigma_{\psi_t} \equiv (\beta_0^{-1})' \, \Sigma_t \beta_0^{-1}, \; t = 1, \ldots, T \text{ and } B_\tau \equiv -\beta_0^{-1}\beta_{\tau 2}, \; \tau = 0, \ldots, q.$$

Under the assumption the ψ_t are independent, (2) is equivalent to the $T - 1$ stage program

$$\textit{minimize}_{x_1} f_1(x_1) + E_{\psi_1}\{g_1(y_1) + \min_{x_2}[f_2(x_2) + E_{\psi_2}(g_2(y_2)) \tag{3}$$
$$+ \min_{x_3}\{f_3(x_3) + \ldots + \min_{x_{T-1}}[f_{T-1}(x_{T-1})$$
$$+ E_{\psi_{T-1}}\{g_{T-1}(y_{T-1}) + Q_{T-1}(x_{T-q+1}, \ldots, x_{T-1})\}]\})]\}$$

subject to

$$\begin{array}{llll}
B_0x_1 & & + y_1 & = \psi_1, \\
B_1x_1 + B_0x_2 & & + y_2 & = \psi_2, \\
\ddots & & \ddots & \vdots \\
B_qX_{T-q-1} + \ldots + B_0X_{T-1} & & + y_{T-1} & = \psi_{T-1} \\
x_1 \in K_1, \; x_2 \in K_2, \ldots, \; x_{T-1} \in K_{T-1}, & & &
\end{array}$$

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[3] This assumption, normally made in econometric estimation, means that there are as many independent explaining equations as variables to be explained.

[4] Note that for any $t < 0$, X_t^* and $\bar{Z}_t$ are defined to be their previously observed values.

where

$$Q_{T-1}(x_{T-q},\ldots, x_{T-1}) \equiv \min_{x_T \in K_T} [f_T(x_T) + E_{\psi_T}\{g_T(\psi_T - \sum_{\tau=0}^{q} B_\tau x_{T-\tau})\}].$$

Now, if f_T and g_T are convex[5] then so is Q_{T-1} since expectation is a non-negative linear operation, convex functions of linear functions are convex and minimization over x_T does not destsoy the convexity property with respect to x_τ, $\tau < T$. Continuing to T-2, T-3,..., one can establish.[6],[7]

Theorem 1 [11]: Assuming that all expectations and minima exist there is an equivalent deterministic program to (2) of the form: $\{minimize\ h_1(x_1) \mid x_1 \in K_1\}$. Furthermore, h_1 is convex if the f_t and g_t are convex.

The **Theorem** means that it is theoretically possible to "shrink" the problem (2) into an equivalent deterministic problem that involves only the first period deviation choice x_1. Intuitively, this is plausible since any choice of the x_1 followed by a sequence of optimal decision policies $x_2, \ldots, x_T$ has some total cost over the T periods, i.e., $h_1(x_1)$, and since the choice of x_1 does not influence the feasibility of any later x_τ, the feasible choices remain K_1.

Suppose that in addition to the constraints on the decision there are constraints on the outcomes, i.e., the Y_t. Generally, in the presence of the random vectors ψ_t one cannot guarantee that any bounding constraints will be satisfied. Suppose then that the D.M. considers a formulation [10] in which the bounds may be violated in some rare cases, i.e.,

$$P_r\{Y_{tj} \geqq \alpha_{3Tj}\} \geqq \delta_{3Tj};$$
$$P_r\{Y_{Tj} \geqq \alpha_{3Tj}\} \geqq \delta_{4Tj};\quad j = 1, \ldots, m,\ t = 1,\ldots, T, \tag{4}$$

where the α_{3Tj} and α_{4Tj} are the lower and upper bounds, respectively, the δ's are numbers between zero and one, and a subscripted j refers to the jth row of the vector (or matrix) subscripted.

In the last period, $t = T$, these constraints become

$$P_r\left\{\left(\sum_{r=0}^{q} B_r x_{T-r}\right)_j \leqq \psi_{Tj} + Y^*_{Tj} - \alpha_{3Tj}\right\} \geqq \delta_{3Tj},$$

$$P_r\left\{-\left(\sum_{r=0}^{q} B_r x_{T-r}\right)_j \leqq -\psi_{Tj} - Y^*_{Tj} + \alpha_{4Tj}\right\} \geqq \delta_{4Tj},\quad j = 1,\ldots, m. \tag{5}$$

These probabilistic constraints can be converted into an equivalent set of deterministic constraints by employing

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[5] Convexity is a natural assumption since it implies that penalties are non-decreasing in the absolute magnitude of the deviations (unless some directions are linear).

[6] By a deterministic equivalent we mean a deterministic program whose set of optimal solutions is the same as the set of optimal solutions of the random problem.

[7] In this and several other Theorems we will utilize Bellman's Principle of Optimality [6], "an optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision," to demonstrate the optimality of the period-by-period optimizations.

Theorem 2 [8]: Suppose that D and x are real-valued fixed vectors, that ζ is a random variable with cumulative distribution function F, and that δ is a number between zero and one. Then

a) $Pr\ \{Dx \leqq \zeta\} \geqq \delta$ if and only if $Dx \leqq \eta$,
where $\eta \equiv \{max\ \eta_1 \mid F(\eta_1) \leqq 1 - \delta\}$; and

b) if F is strictly monotonic then $\eta = F^{-1}(1 - \delta)$ where F^{-1} is the inverse function of F.

Hence, the constraints (5) are equivalent to

$$\theta_{Tj} \leqq \left(\sum_{r=0}^{q} B_r x_{T-r}\right)_j \leqq \eta_{Tj},$$

where $\eta_{Tj} \equiv F^{-1}_{\psi_{Tj}+Y^*_{Tj}-\alpha_{3Tj}}(1 - \delta_{3Tj})$, $\theta_{Tj} \equiv -F^{-1}_{-\psi_{Tj}-y^*+\alpha_{4Tj}(1-\delta_{4Tj})}$.

$$j = 1,\ldots, m$$

In matrix notation these constraints are

$$\theta_T \leqq \sum_{\tau=0}^{q} B_\tau x_{T-r} \leqq \eta_T. \tag{6}$$

The constraints (6) impose a linkage between the choice vectors $x_{T-q},\ldots, x_T$. Continuing in this manner for each t one obtains the T sets of constraints

$$\theta_t \leqq \sum_{r=0}^{q} B_r x_{t-r} \leqq \eta_t, \quad t = 1,\ldots, T. \tag{7}$$

Hence, problem (2) becomes d.p. I

$$\underset{x_1}{minimize}\ f_1(x_1) + E_{\psi_2}\{g_1(y_1) + \underset{x_2}{min}\,[f_2(x_2) + E_{\psi_2}(g_2(y_2) + \underset{x_3}{min}\,\{f_3(x_3) + \ldots +$$

$$+ E_{\psi_T}[g_T(y_T)]\})]\}, \tag{8}$$

subject to

$$B_0 x_1 + y_1 = \psi_1,$$

$$B_0 x_1 \leqq \eta_1 - \sum_{r=1}^{q} B_r x_{1-r},$$

$$-B_0 x_1 \leqq -\theta_1 + \sum_{r=1}^{q} B\, x_{1-r},$$

$$
\begin{aligned}
&B_1x_1 + B_0x_2 && + y_2 && = \psi_2,\\
&B_1x_1 + B_0x_2 && && \leqq \eta_2 - \sum_{\tau=2}^{q} B_\tau x_{2-\tau},\\
&-B_1x_1 - B_0x_2 && && \leqq -\theta_2 + \sum_{\tau=2}^{q} B_\tau x_{2-\tau},\\
&\vdots && \ddots && \vdots\\
&B_{q-1}x_1 + \ldots + B_0x_q && + y_q && = \psi_q,\\
&B_{q-1}x_1 + \ldots + B_0x_q && && \leqq \eta_q - B_qX_0,\\
&-B_{q-1}x_1 - \ldots - B_0x_q && && \leqq -\theta_q + B_qX_0,\\
&\quad\ddots && \ddots && \vdots\\
&\quad B_qx_{T-q} + \ldots + B_0x_T && + y_T && = \psi_T,\\
&\quad B_qx_{T-q} + \ldots + B_0x_T && && \leqq \eta_T,\\
&\quad -B_qx_{T-q} - \ldots - B_0x_T && && \leqq -\theta_T,\\
&x_1 \in K_1,\ x_2 \in K_2, \ldots && x_T \in K_T.
\end{aligned}
$$

Now, if one attempts to „shrink" (8), as we did (2), the situation is slightly more complicated because the choice vectors are now linked by (7) and, hence, Theorem 1 does not apply. However, the following modified Theorem does apply.

Theorem 3 [11]: Assuming that all expectations and minima exist, there is an equivalent deterministic program to (8) of the form: {*minimize* $i_1(x_1 \mid \mid x_1 \in D_1\}$, where $D_1 \in K_1$ is a convex polytype.

Further i_1 is convex if the f_t and g_t are convex.

The set of feasible solutions D_1 and x_1 may, of course, be empty if the constraints on the Y_t are too tight. The construction of D_1 is not a simple task; however, one may proceed using some of the constructions in [30].

3. STATIONARY LINEAR MODELS WITH LAGGED ENDOGENOUS VARIABLES

Let

$$\beta_0 Y_t = \sum_{\tau=1}^{r} \beta_{\tau 1} Y_{t-\tau} + \sum_{\tau=0}^{q} \beta_{\tau 2} X_{t-\tau} + \sum_{\tau=0}^{s} \beta_{\tau 3} Z_{t-\tau} + E_t, \tag{9}$$

be the underlying structural model and make the same assumptions as in the first model of Section 2. The reduced form of (9) is

$$Y_t = \sum_{\tau=1}^{r} A_\tau Y_{t-\tau} + \sum_{\tau=0}^{q} B_\tau X_{t-\tau} + \sum_{\tau=0}^{s} C_\tau Z_{t-\tau} + \mathcal{C}_t, \tag{10}$$

where

$$\mathcal{T}_t \equiv \beta_0^{-1}E_t,\ \mathcal{T}_t \sim H_t(0, \Sigma_R),\ \Sigma_R \equiv (\beta_0^{-1})'\,\Sigma_t\beta_0^{-1},\ A_\tau \equiv \beta_0^{-1}\beta_{\tau 1},$$
$$B_\tau \equiv \beta_0^{-1}\beta_{\tau 2} \text{ and } C_\tau \equiv \beta_0^{-1}\beta_{\tau 3}.$$

Enlarge the system (10) by r vector identities giving

$$\begin{aligned} Y_{t-r+1} &\qquad\qquad Y_{t-r+1}, \\ &\vdots \qquad\quad \vdots \\ Y_{t-1} &\qquad Y_{t-1}, \\ Y_t &= A_1Y_{t-1} + \ldots + A_rY_{t-r} \quad + B_0x_t + \ldots + B_qx_{t-q} + \\ &\qquad\qquad\qquad\qquad + C_0Z_t + \ldots + C_sZ_{t-s} + \mathcal{T}_t, \end{aligned}$$

which is

$$\hat{Y}_t = \hat{A}\hat{Y}_{t-1} + \hat{B}\hat{X}_t + \hat{C}\hat{Z}_t + \hat{\mathcal{T}}_t, \tag{11}$$

where $\hat{Y}_t \equiv \begin{pmatrix} Y_{t-r+1} \\ \vdots \\ Y_t \end{pmatrix}$, $\hat{X}_t \equiv \begin{pmatrix} X_{t-q} \\ \vdots \\ X_t \end{pmatrix}$, $\hat{Z}_t \equiv \begin{pmatrix} Z_{t-s} \\ \vdots \\ Z_t \end{pmatrix}$, $\hat{\mathcal{T}}_t \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathcal{T}_t \end{pmatrix}$,

$$\hat{A} \equiv \begin{pmatrix} 0 & 0 & \ldots & 0 & I & 0 \\ 0 & 0 & \ldots & I & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & I & \ldots & 0 & 0 & 0 \\ I & 0 & \ldots & 0 & 0 & 0 \\ A_r & A_{r-1} & \ldots & A_3 & A_2 & A_1 \end{pmatrix}, \qquad \hat{B} \equiv \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \\ 0 & \ldots & 0 \\ B_q & \ldots & B_0 \end{pmatrix},$$

and

$$\hat{C} \equiv \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \\ 0 & \ldots & 0 \\ C_s & \ldots & C_0 \end{pmatrix}.$$

Now, assuming that a stability condition[8] is satisfied, the final form [27] of the econometric model is[9]

$$\mathcal{Y}_t = \hat{A}^{r+1}\,\hat{Y}_{t-r+1} + \sum_{r=0}^{r} \hat{A}^r\,\hat{B}\hat{X}_{tr} + \sum_{r=0}^{r} \hat{A}^r\hat{C}\hat{Z}_{t-r} + \sum_{r=0}^{r} \hat{A}^r\mathcal{T}_{t-r}; \tag{12}$$

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[8] A necessary and sufficient condition for stability is that the maximum absolute value of the characteristic roots of $A^* < 1$. For more discussion see [5, Ch. 15—16] or [27, pp. 142-4].

[9] The expressions on the righthand side of (12) represent the impact of decision and data variables in the present and previous periods on the present values of the target variables, and the coefficients represent multipliers of the magnitude of these variables. For an analysis of these multiplier effects in the context of the Klein-Goldberger model see [17].

At this point we allow the vectors Z_t to be random by assuming that each $Z_t \sim I_t(\bar{Z}_t, \Sigma_{z_t})$ independently of X_t and E_t. We assume, however, that the D. M. observes Z_t with certainty at the end of period t and, hence, Z_t is known when the decision x_{t+1} is made. Now we would like to again consider a T period decision problem and wish to express the vectors Y_t, $t = 1,...,$ T as a function of the decision vectors x_τ, $1 \leqq \tau \leqq t$ random variables Y_r, Z_t, and constants. Let $m.$ refer to the last m rows of a matrix. The bottom m rows of (12) are then

$$Y_t = (\hat{A}^{r+1})\, m.\, \hat{Y}_{t-r+1} + \sum_{r=1}^{r} (\hat{A}^r \hat{B})\, m.\, \hat{X}_{t-r} + (\hat{A}^r \hat{C})\, m.\, \hat{Z}_{t-r}$$

$$+ (\hat{A}^r)\, m.\, \hat{\mathcal{C}}_{t-r} + \sum_{r=0}^{q} B_r X_{t-\tau} + \sum_{r=0}^{s} C_r Z_{t-r} + \mathcal{C}_t, \tag{13}$$

or

$$Y_t = B_0 X_t + ... + B_{t-1} X_1 + \Lambda_t, \tag{14}$$

where $\Lambda_t \equiv (\hat{A}^{r+1})\, m.\, Y_{t-r+1} + \sum_{r=1}^{r} \{(\hat{A}^r \hat{B})\, m.\hat{X}_{t-r} + (\hat{A}^r \hat{C})\, m.\, \hat{Z}_{t-r}$

$$(\hat{A}^\tau) m._{\,t-\tau}\} + B_t X_0 + ... + B_q X_{t-q} + \sum_{\tau=0}^{s} C_\tau Z_{t-\tau} + \mathcal{C}_t.$$

Now, putting (14) into deviation form utilizing $x_t \equiv X_t - X_t^*$, and $y_t \equiv$ $\equiv Y_t - Y_t^*$, with the*'s again denoting target values, gives

$$y_t = B_0 x_t + ... + B_{t-1} x_1 - \xi_t, \tag{15}$$

where $\xi_t \sim J_t(\bar{\xi}_t, \Sigma_{\xi_t})$,

$$\bar{\xi}_t \equiv Y_t^* - \sum_{\tau=0}^{t-1} B_\tau X_{t-\tau}^* - \bar{\Lambda}_t,$$

$$\Sigma_{\xi_t} \equiv \Sigma_R + C_0' \Sigma_{z_t} C_0, \qquad t = 1,..., T.$$

Utilizing (15) gives d.p. III

$$\underset{x_1}{minimize}\ f_1(x_1) + E_{\xi_1}\{g_1(y_1) + \underset{x_2}{min}\,[f_2(x_2) + E_{\xi_2}(g_2(y_2) + \underset{x_3}{min}\,\{f_3(x_3) + ...$$

$$+ E_{\xi_T}[g_T(y_T)]\})]\}, \tag{16}$$

subject to

$$
\begin{aligned}
&B_0x_1 && + y_1 && = \xi_1,\\
&B_1x_1 + B_0x_2 && + y_2 && = \xi_2,\\
&\vdots && \ddots && \vdots\\
&B_{q-1}x_1 + \ldots + B_0x_q && + y_q && = \xi_q,\\
&\quad \ddots && \ddots && \vdots\\
&\quad B_qx_{T-q} + \ldots + B_0x_T && + y_T && = \xi_T,
\end{aligned}
$$

$$x_1 \in K_1,\ x_2 \in K_2, \ldots,\ x_T \in K_T.$$

The program (16) is formally identical to (2) except that the ξ_t are not independent.[10] However, a Theorem similar to Theorem 1 does apply to (16). The development is as follows:

Let $F(\xi_1, \ldots, \xi_T)$ be the joint cumulative distribution function of the random m-vectors $\xi_1, \ldots, \xi_T$, $F_j(\xi_j)$ be the marginal cumulative distribution function of ξ_j, and $\bar{F}_j(\xi_j \mid \xi_1, \ldots, \xi_{j-1})$ be the marginal conditional cumulative distribution function of ξ_j that is conditional on the past events $\xi_1 \ldots, \xi_{j-1}$.

Upon substitution of the last set of equations in (16) into the functional, that program is

$$
\begin{aligned}
\underset{x_1}{\text{minimize}}\ & f_1(x_1) + E_{\xi_1}\{g_1(y_1) + \min_{x_2}[f_2(x_2) + E_{\xi_2}(g_2(y_2) + \min_{x_3}[f_3(x^3) + \ldots + \\
& + \min_{x_{T-1}}\{f_{T-1}(x_{T-1}) + E_{\xi_{T-1}}[g_{T-1}(y_{T-1}) + \min_{x_T}(f_T(x_T) + E_{\xi_T}\{g_T[\xi_T - \\
& - B_qx_{T-q} - \ldots - B_0x_T]\})]\}])]\},
\end{aligned}
$$

subject to (17)

$$
\begin{aligned}
&B_0x_1 && + y_1 && = \xi_1,\\
&B_1x_1 + B_0x_2 && + y_2 && = \xi_2,\\
&\vdots && \ddots && \vdots\\
&B_{q-1}x_1 + \ldots + B_0x_q && + y_q && = \xi_q,\\
&\quad \ddots && \ddots && \vdots\\
&\quad B_qx_{T-q-1} + \ldots + B_0x_{T-1} && + y_{T-1} && = \xi_{T-1},
\end{aligned}
$$

$$x_1 \in K_1,\ x_2 \in K_2, \ldots,\ x_T \in K_T.$$

Now let

$$
\begin{aligned}
\bar{Q}_T(x_{T-q}, \ldots, x_T) &\equiv \bar{E}_{\xi_T}[g_T(\xi_T - B_qx_{T-q} - \ldots - B_0x_T)]\\
&= \int \cdots \int g_T(\xi_T - B_qx_{T-q} - \ldots - B_0x_T)\\
&\qquad dF(\xi_T),
\end{aligned}
$$

[10] Since the essential new feature of this model is that the ξ_t are dependent, the result below (Theorem 4) also applies to situations when the random disturbances of the underlying model are serially correlated.

be the conditional expectation of $g_T(.)$ with respect to ξ_T given $\xi_1, ..., \xi_{T-1}$ and $x_{T-q}, ..., x_T$. Hence, the objective in (17) becomes (18)

$$\underset{x_1}{minimize}\ f_1(x_1) + E_{\xi_1}\{g_1(y_1) + \underset{x_2}{min}\ [f_2(x_2) + E_{\xi_2}(g_2(y_2) + \underset{x_3}{min}\ \{f_3(x_3) + \ldots + \underset{x_{T-1}}{min}[f_{T-1}(x_{T-1}) + E_{\xi_{T-1}}\{g_{T-1}(y_{T-1}) + \underset{x_T}{min}\ (f_T(x_T) + \bar{Q}_T(x_{T-q}, ..., x_T))\}]\})]\}.$$

Now perform the minimization over $x_T \in K_T$ of $\{f_T(x_T) + \bar{Q}_T(x_{T-q}, ..., x_T)\}$ and define

$$R_{T-1}(x_{T-q}, ..., x_{T-1}) \equiv min\ \{f_T(x_T) + \bar{Q}_T(x_{T-q}, ..., x_T) \mid x_T \in K_T\}.$$

Hence, an equivalent problem to (17) is

$$\underset{x_1}{minimize}\ f_1(x_1) + E_{\xi_1}\{g_1(y_1) + \underset{x_2}{min}\ [f_2(x_2) + E_{\xi_2}(g_2(y_2) + \underset{x_3}{min}\ \{f_3(x_3) + \ldots + \underset{x_{T-1}}{min}\ [f_{T-1}(x_{T-1}) + E_{\xi_{T-1}}\{g_{T-1}(y_{T-1}) + R_{T-1}(x_{T-q}, ..., x_{T-1})\}]\})]\}, \quad (19)$$

subject to

$$\begin{aligned}
&B_0 x_1 && + y_1 && && = \xi_1,\\
&B_1 x_1 + B_0 x_2 && && + y_2 && = \xi_2,\\
&\vdots && && \ddots && \vdots\\
&B_{q-1} x_1 + \ldots + B_0 x_q && && + y_2 && = \xi_q,\\
&\ddots && && && \vdots\\
&\quad B_q x_{T-q-1} + \ldots + B_0 x_{T-1} && && && + y_{T-1} = \xi_{T-1},\\
&x_1 \in K_1,\ x_2 \in K_2, \ldots, && x_{T-1} \in K_{T-1}
\end{aligned}$$

Continuing in a like manner, one may substitute the last set of equations of (19) into the functional and let

$$\bar{Q}_{T-1}(x_{T-q-1}, ..., x_{T-1}) \equiv \bar{E}_{\xi_{T-1}}[g_{T-1}(\xi_{T-1} - B_q x_{T-q-1} - \ldots - B_0 x_{T-1}) + R_{T-1}(x_{T-q}, ..., x_{T-1}),$$

be the conditional expectation of $\{g_{T-1}(\cdot) + R_{T-1}(\cdot)\}$ with respect to ξ_{T-1} given $\xi_1, ..., \xi_{T-2}$ and $x_{T-q-1}, ..., x_{T-1}$. One may then optimize with respect to x_{T-1} and then repeat. Following this process, one may "shrink" the problem (16) into a problem of the form $\{min\ e_1(x_1) \mid x_1 \in K_1\}$ and establish.

Theorem 4: Assuming that all conditional expectations and minima exist, there is an equivalent deterministic program to (16) of the form: $\{min\ e_1(x_1) \mid \mid x_1 K_1\}$. Furthermore, e_1 is convex if the f_t and g_t are convex.

If one allows bounding relations on the Y_t, as in Section 2, then, clearly, one will obtain a result analogous to Theorem 3 with the only change being that conditional expectations replace unconditional expectations.

4. A BAYESIAN ADAPTIVE MODEL

In the models considered above it was assumed that the D.M. knew the probability distribution of each random variable. More generally, the D.M. does not know these distributions but as time proceeds he learns more about them. We will assume that the D.M. learns in a Bayesian fashion, that is, he has certain prior beliefs concerning the unknown factors in his environment and he changes his beliefs in the light of observed information via Bayes Theorem.

We proceed to develop a model as follows. Let

$$Y^h \equiv \{Y_{t-1},\ldots, Y_{t-r}\},\ X_t^h \equiv \{X_{t-1},\ldots, X_{t-q}\},$$

$$Z_t^h \equiv \{Z_{t-1},\ldots, Z_{t-s}\},$$

and

$$\zeta_t^h \equiv \{\zeta_{t-1},\ldots, \zeta_{t-w}\},$$

refer to the historical observations, which are assumed to be observed without error, of the endogenous, decision, exogenous, and error in equation vectors, respectively. That is, once X_t is chosen and Y_t and Z_t are observed, ζ_t is such that the equations hold. Suppose that the underlying structural model is

$$l_i(Y_t,\ Y_t^h,\ X_t,\ X_t^h,\ Z_t,\ Z_t^h,\ \zeta_t,\ \zeta_t^h,\ \beta) = 0,\ i = 1,\ldots, m. \tag{20}$$

where β is a vector of known parameters, and there are as many equations as variables to be explained, i.e., m. Again suppose that the D.M. has a T period planning horizon and has preference functions f_t and g_t that rank the deviations x_t and y_t of achieved values X_t and Y_t from target variables X_t^* and Y_t^*, respectively, $t=1,\ldots, T$. His objective is to choose the $x_t \in K_t$ so as to minimize the sum of the expected costs.

Let us combine the random vectors Z_t and ζ_t as $\varepsilon_t \equiv (Z_t',\ \zeta_t')'$ and let $q(\varepsilon_1,\ldots, \varepsilon_T \mid \theta)$ be the joint p.d.f. of the ε_t and $r(\theta)$ be the prior p.d.f. of the vector θ. Then, by Bayes Theorem, the posterior p.d.f. of θ for period t is

$$h_t(\theta \mid \varepsilon_t^h) = \frac{r(\theta)\ \bar{q}\ (\varepsilon_t^h \mid \theta)}{\bar{q}(\varepsilon_t^h)}, \tag{21}$$

where $\bar{q}(\varepsilon_t^h) \equiv \bar{q}(Z_t^h,\ \zeta_t^h) = \int\cdots\int q(\varepsilon_1, \ldots,\ \varepsilon_T \mid \theta)\ r(\theta) \mathrm{d}\theta \mathrm{d}\ \varepsilon_t \ldots\ \mathrm{d}\varepsilon_T,$

and $\bar{q}(\varepsilon_t^h \mid \theta) \equiv \bar{q}(Z_t^h,\ \zeta_t^h \mid \theta) = \int\cdots\int q(\varepsilon_1,\ldots,\ \varepsilon_T)\ \mathrm{d}\varepsilon_t \ldots \mathrm{d}\varepsilon_T,$

are, respectively, the joint marginal p.d.f. of ε_t^h and the joint marginal p.d.f. of ε_t^h conditional on θ. The posterior p.d.f. of θ will be used to update the p.d.f of the $\in_t$ in each period. Using conditional probabilities gives

$$\bar{q}(\varepsilon_t \mid \theta,\ \varepsilon_t^h) = \frac{\bar{q}(\varepsilon_t,\ \varepsilon_t^h \mid \theta)}{\bar{q}(\varepsilon_t^h \mid \theta)} \equiv \frac{\bar{q}(\varepsilon_{t+1}^h \mid \theta)}{\bar{q}(\varepsilon_t^h \mid \theta)},\ \text{and} \tag{22}$$

$$\bar{q}(\varepsilon_t \mid \varepsilon_t^h) = \int\cdots\int \bar{q}(\varepsilon_t \mid \theta,\ \varepsilon_t^h) r(\theta)\ d\theta = \int\cdots\int \frac{\bar{q}(\varepsilon_t,\ \varepsilon_t^h \mid \theta)}{\bar{q}(\varepsilon_t^h \mid \theta)}\, r(\theta)\, \mathrm{d}\theta. \tag{23}$$

Now (23), the marginal p.d.f. of ε_t conditional on the history, ε_t^h, is the relevant p.d.f. for decision making purposes in period t.

Consider then d.p. IV

$$\underset{x_1}{Minimize}\, f_1(x_1) + E_{\varepsilon_2}\{g_1(y_1) + \underset{x_2}{min}\,[f_2(x_2) + E_{\varepsilon_2}(g_2(y_2) + \underset{x_3}{min}\{f_3(x_3) + \ldots + E_{\varepsilon_T}[g_T(y_T)]\})]\}, \tag{24}$$

subject to

$$\begin{gathered} l(y_1 - Y_1^*,\ Y_1^h,\ x_1 - X_1^*,\ X_1^h,\ \varepsilon_1,\ \varepsilon_1^h,\ \beta) = 0, \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ l(y_T - Y_T^*,\ Y_T^h,\ x_T - X_T^*,\ X_T^h,\ \varepsilon_T,\ \varepsilon_T^h,\ \beta) = 0, \\ x_1 \varepsilon K_1,\ x_2 \varepsilon K_2,\ \ldots,\ x_T\ \varepsilon K_T. \end{gathered}$$

The model (24), although not the most general model imaginable, [11] includes a number of the features that one would desire in a sequential plannings schema based on an underlying structural model.
We now give a sketch of a solution procedure via dynamic programming.

Suppose that (20) is invertable in the sense that there exists a unique vector valued function[12] l^{-1} such that

$$y_t = l^{-1}(Y_t^*,\ Y_t^h,\ x_t - X_t^*,\ X_t^h,\ \varepsilon_t,\ \varepsilon_t^h,\ \beta). \tag{25}$$

Inserting (25) with $t = T$ into the functional of (24) gives

$$\underset{x_1}{minimize}\, f_1(x_1) + E_{\varepsilon_1}\{g_1(y_1) + \underset{x_2}{min}\,[f_2(x_2) + E_{\varepsilon_2}(g_2(y_2) + \underset{x_3}{min}\{f_3(x_3) + \ldots + E_{\varepsilon_{T-1}}(g_{T-1}(y_{T-1}) + \underset{x_T}{min}[f_T(x_T) + E_{\varepsilon_T}(g_T\{l^{-1}(Y_T^*,\ Y_T^h,\ x_T - X_T^*,\ X_T^h,\ \varepsilon_T,\ \varepsilon_T^h,\ \beta)\})])\})]\}. \tag{26}$$

---

[11] The following are several assumptions that could be relaxed. One could allow the D.M. to revise his preference functions, targets, and planning horizon in the light of accumulated information. The vector β could be unknown. The information itself might be observed with some noise. The D.M. might not know the class of distributions that he is sampling from and...

The model does allow for adaptive targets Y_t^* if these targets are linear functions, perhaps subject to additive errors, of variables known at the start of period t. Also the preference functions f_t and g_t can be altered by allowing the D.M., at the start of period t, to choose specified functional forms for the preference functions. Unfortunately, this latter feature will greatly increase the computational effort involved in the dynamic programming solution procedure.

[12] Conditions on l that imply the existance of a unique l^{-1} are implied by the global univalence results of Gale and Nikaidô [16]. In particular if the Jacobian matrix of l with respect to y is a P-matrix (i. e. all principal submatrices have positive determinants) and l is defined on a rectangular region then a unique l^{-1} exists.

To begin the backward induction, one must evaluate

$$E\varepsilon_T\{l^{-1}(Y_T^*, Y_T^h, x_T - X_T^*, \varepsilon_T, \varepsilon_T^h, \beta) = \int \cdots \int l^{-1}(Y_T^*, Y_T^h, x_T -$$

$$- X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta)\, \bar{q}(\varepsilon_T \mid \varepsilon_T^h)\, d\varepsilon_T = \int \cdots \int l^{-1}(Y_T^*, Y_T^h, x_T -$$

$$- X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta) \left\{ \cdots \frac{\bar{q}(\varepsilon_T, \varepsilon_T^h \mid \theta)}{\bar{q}(\varepsilon_T^h \mid \theta)} r(\theta) d\theta \right\} d\varepsilon_T = \int \cdots \int l^{-1}(Y_T^*, Y_T^h, x_T -$$

$$- X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta) \left\{ \frac{q(\varepsilon_1, \ldots, \varepsilon_T \mid \theta)}{\int \cdots \int \bar{q}(\varepsilon_1, \ldots, \varepsilon_T)\, d\varepsilon_T} r(\theta)\, d\theta \right\} d\varepsilon_T. \qquad (27)$$

Now suppose that the distribution of ε_T possesses sufficient statistics [18] for each of its unknown parameters, i.e., the θ_{Ti}. This means that all the information concerning the p.d.f's parametrs is contained in certain functions of the observed data. Hence, a knowledge of certain functions of the previously observed and chosen values is tantamount to knowing the values (i.e., the estimates) of the θ_{Ti} that will be used in period T, and the calculation of the optimal decision function in each period is greatly simplified.

If one specifies θ_T to be, say, $\hat{\theta}_T$, then the marginal joint p.d.f. of ε_T is

$$\bar{q}(\varepsilon_T \mid \hat{\theta}) = \int \cdots \int q(\varepsilon_1, \ldots, \varepsilon_T \mid \hat{\theta}_T)\, d\varepsilon_1, \ldots, d\varepsilon_{T-1}, \qquad (28)$$

which is a known distribution. Substituting (28) into (27) gives

$$E\varepsilon_T[g_T\{l^{-1}(Y_T^*, Y_T^h, x_T - X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta)\}] = \int \cdots \int g_T\{l^{-1}(Y_T^*, Y_T^h, x_T -$$

$$- X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta)\}\, \bar{q}(\varepsilon_T \mid \theta_T)\, d\varepsilon_T = \int \cdots \int g_T\, \{l^{-1}(Y_T^*, Y_T^h, x_T -$$

$$- X_T^*, X_T^h, \varepsilon_T, \varepsilon_T^h, \beta)\} \int \cdots \int q(\varepsilon_1, \ldots, \varepsilon_T \mid \hat{\theta}_T)\, d\varepsilon_1 \ldots d\varepsilon_{T-1}\}\, d\varepsilon_T =$$

$$= S_T(Y_T^h, x_T, X_T^h, \hat{\theta}_T), \qquad (29)$$

which is a function of previously observed values, T^{th} stage decision variables, and $\hat{\theta}_T$. The constants Y_T^*, X_T^* and β have been suppressed. Utilizing (29), the T^{th} stage minimization is

$$U_{T-1}(Y_T^h, X_T^h, \hat{\theta}_T) \equiv Min\{f_T(x_T) + S_T(Y_T^h, X_T, X_T^h, \hat{\theta}_T) \mid x_T \varepsilon K_T\}. \qquad (30)$$

It is clear that the usual dynamic programming reduction develops and in each period there are the additional "state" variables $\hat{\theta}_t$ assuming that the functional form of the posterior distribution is the same for all t (i.e.the prior and sampling distributions are conjugates, see [22]).[13] One obtains in

[13] The idea that we have sketched has been suggested by Bellman and Kalaba. See [7, Ch. 16] and the references cited there. A more elementary but extremely lucid exposition of these ideas is given by [18]

each period t expressions analogous to (29) and (30), which include terms from $t+1, \ldots, T$.

The functions S_t will be convex if l^{-1} is convex [14] and g_t is convex and non-decreasing (proof [9, p. 191]), or if $g_t(y_t) = \tilde{g}_t(y_t^+, y_t^-)$, where $y_t \equiv y_t^+ - y_t^-$, $y_t^+ \geq 0, y_t^- \geq 0, y_t^{+\prime} y_t^- = 0$ and g_t is convex and non-decreasing. This latter condition being proposed to aleviate the problem that for negative deviations the convexity and non-decreasing properties are contradictory[15]. Hence, one may "shrink" the program (26), as was done with d.p.'s I, II, and III, to establish

Theorem 5: Assuming that all conditional expectations and minima exist, that p.d.f.'s r and q are conjugate, θ has sufficient statistics, and that his invertible, there is an equivalent deterministic program to (26) of the form: $\{\min m_1(x_1) | x_1 \in K_1, n_1(x_1) = 0\}$. Furthermore, m_1 is convex if the f_t and the inverse function l^{-1} are convex and the g_t are convex and non-decreasing.

5. ALGORITHMS

This section consists of a brief summary of some algorithms that may be utilized to solve d.p.'s I — IV.

5a. *Static Algorithms*

A static algorithm is a systematic procedure that generates a sequence of decision points $\{x_1^k\}$ such that $\lim_{k\to\infty} x_1^k \to x_1^*$, the optimal solution, of a one period policy model based on a known underlying economic model. If x_1^* does not exist the algorithm indicates so.

In the static case d.p. I is

$$\text{minimize } f_1(x_1) = E_{\psi_1}\{g_1(y_1)\}, \tag{31}$$

subject to $$B_0 x_1 + y_1 = \psi_1,$$

$$x_1 \in K_1.$$

Substituting the equations into the functional gives

$$\text{minimize } f_1(x_1) + E_{\psi_1}\{g_1(\psi_1 - B_0 x_1)\}, \tag{32}$$

subject to $x_1 \in K_1$.

---

[14] The question of when l^{-1} is convex has been considered by Warburton and Ziemba [29a] who present necessary and sufficient conditions for this convexity given the existance of l^{-1}. The following Theorem gives conditions on l such that l^{-1} exists and is convex and continuously differentiable.

Theorem [29 a]: Suppose l: $H \times D \to R^m$, where l is a continuously differentiable quasi-concave mapping, H is a rectangular subset of R^m and D is an open convex subset of R^k. Suppose further that the Jacobian matrix of l with respect to y is a P-matrix of Leontieff type on $H \times S$ ($S \equiv \{y \in H: l(y, u) = 0$ for some $u \in D\}$, then l^{-1} exists is convex and is continuously differentiable on S if $S \neq \varnothing$

[15] However if one does take this route the y_t^+ and y_t^- are now variables that must be optimized on. The condition $y_t^{+\prime} y_t^- = 0$ is superfluous if g_t is convex and non-negative.

The objective of (32) is an implicit function of x_1. One may proceed by integrating out ψ_1 to give the deterministic program[16]

$$\begin{aligned} &\textit{minimize} \quad t_1(x_1), \qquad (33)\\ &\text{subject to} \quad x_1 \in K_1, \end{aligned}$$

where $t_1(x_1) \equiv E_{\psi_1}\{g_1(\psi_1 - B_0 x_1)\}$, which is a nonlinear program with linear constraints, and one may apply any of a number of algorithms, e.g., [12, 13, 15, 31, 32, 37]. Generally speaking, these algorithms converge[17] if t_1 is continuously differentiable and pseudo-convex.[18] However, for each algorithm, convergence is dependent also on minor assumptions on either K_1 or the sequence $\{x_1^k\}$.

Unfortunately, in many cases it is difficult or impossible to explicitly[19] calculate $t_1(x_1)$. Often, in these cases, one is able to calculate for a given feasible point the numerical value of the gradient and the algorith modifications presented in [35] may be used. Again the same convergence remarks apply.

For d.p. II, the static assumption leads to the problem

$$\begin{aligned} \text{minimize} \quad & f_1(x_1) + E_{\psi_1}\{g_1(y_1)\}, \qquad (34)\\ \text{subject to} \quad & B_0 x_1 + y_1 = \psi_1,\\ & x_1 \in K_1 \cap H_1, \end{aligned}$$

where $H_1 \equiv \{x_1 | \theta \leqq B_0 x_1 \leqq \eta_1\}$, which is a convex polytope. Hence, in the static case, d.p. II is formally identical to d.p. I. Also, d.p. III is formally identical to d.p.I, with ξ_1 replacing ψ_1, where it is recalled that ξ_1 contains the lagged terms involving y_0, ... Y_{1-r} as well as lagged terms involving x_0, ..., x_{1-q}, $Z_1, \ldots, Z_{1-s}$, and the residual $\mathcal{C}_1$. Thus, the remarks concerning the solution of d.p.I also apply to the solution of d.p.'s II and III.[20]

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[16] See [34] for the calculation of $t_1(x_1)$ in the special case when g is separable, quadratic, and asymmetric (about the origin) and the elements of ψ_1 are either uniform, exponential, or normal. In the separable case one has only to consider the marginal distributions of the random variables and, hence, the calculations are greatly simplified. An approach to the case when g_1 is non- separable, quadratic, and asymmetric and ψ_1 has a joint normal distribution also appears in [34]. Unfortunately, the results are not very satisfactory unless ψ_1 has very few elements.

[17] For a rigorous discussion of a theory of algorithmic convergence and the convergence proofs of some of the listed algorithms see [32].

[18] Let $\nabla t(x) \equiv (\partial t(x)/\partial x_1, \ldots, \partial t(x)/\partial x_n)$ denote the gradient of t at the point x. Pseudo-convex functions [21] are differentiable functions that satisfy: $\nabla t(x^1)' (x^2 - x^1) \geqq 0 \Rightarrow t(x^2) \geqq t(x^1)$ for all x^1, $x^2 \in K$, the domain of t (i.e., when a directional derivative points up the function increases in that direction). These functions have the property that local minima are global minima. All differentiable convex functions are pseudo-convex but not conversely.

[19] Note, however, that one can always explicitly calculate $t_1(x_1)$ if ψ_1 is discrete. To verify this suppose $\psi_1 = \psi^l$ with probability $p^l \geqq 0$, where $l = 1, \ldots, L < \infty \sum_{l=1}^{L} p^l = 1$. Then $t_1(x_1) = f_1(x_1) + \sum_{l=1}^{L} p^l g_1(p_1^l - B_0 x_1)$.

[20] In [33] we have discussed in more detail the static versions of several decision models including d.p. 's I and II.

In the static case d.p. IV is

$$\textit{minimize} \quad f_1(x_1) + E_{\psi_1}\{g_1(y_1)\}, \tag{35}$$

$$\text{subject to} \quad l_i(y_1, x_1, \psi_1) = 0, \; i = 1, \ldots, m,$$

$$x_1 \in K_1,$$

where in each l_1 we have utilized the deviation form and have suppressed all constants. The known joint p.d.f. of ψ_1, given the unknown (vector) parameter θ, is $q(\psi_1 | \theta)$, and $r(\theta)$ is the prior p.d.f. of θ, which generally depends upon the history ψ_1^h. The p.d.f. of ψ_1 relevant for decision making purposes is

$$\bar{q}(\psi_1) = \int \cdots \int q(\psi_1 | \theta)\, r(\theta)\, d\theta_1 \ldots d\theta_L. \tag{36}$$

Now assume that the system $l_1(\cdot) = 0$, $i = 1, \ldots, m$, can be uniquely inverted to give

$$y_{1_j} = l_i^{-1}(x_1, \psi_1), \; i = 1, \ldots, m. \tag{37}$$

Substituting (37) into (35) gives

$$\textit{minimize} \quad Z_1(x_1) = f_1(x_1) + E_{\psi_1}\{g_1[l_1^{-1}(x_1, \psi_1), \ldots, l_m^{-1}(x_1, \psi_1)]\}, \tag{38}$$

$$\text{subject to } x_1 \in K_1.$$

The program (38) is essentially a complicated version of (32). In some cases one may proceed to solve (38) by: finding the l_1^{-1} and $\bar{q}(\psi_1)$, then integrating the second term of the functional with respect to ψ_1 to obtain a deterministic program in x_1, whose solution may be sought via a non-linear programming algorithm. In other cases one may employ the gradient methods of [35] when the gradient of Z_1 can be evaluated for any feasible x_1. Under certain assumptions (see, e. g., [34]) one can interchange expectation and differentiation and, hence, the gradient of the second term of Z_1 can be found via the chain rule if g_1 and the l_1^{-1} are differentiable.

5b. *Dynamic Algorithms*

A dynamic algorithm is a systematic procedure that determines a first period decision $x^*(\psi_1^h, Y_T^h, X_1^h)$ and a sequence of policies $x_2^*(\psi_2^h, Y_2^h, X_2^h, \ldots, x_T^*(\psi_T^h, Y_T^h, X_T^h)$, such that the expected loss that will be incurred over the T periods is lower under strategy $x_1^*(\cdot), \ldots, x_T^*(\cdot)$ than under any other feasible strategy $x_1(\cdot), \ldots, x_T(\cdot)$

If the constraints K_t are present and are non-trivial then there are no general algorithms (for any of the d.p.'s). Hence, one is led to consider special cases in which the dynamic programming [21] approach is manageable or the problem reduces to a simpler equivalent problem.

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[21] In this paper the dynamic programming approach has been emphasized. One could also proceed via a stochastic discrete minimum principle. Under this approach one would obtain a system of equations and inequalities whose solutions are either (depending on the assumptions made) candidates for the optimal solution or actual optimal solutions.

The classic special case occurs when the functionals f_t and g_t are quadratic and the constraints K_t are absent for all t. In this case, if one performs the backward induction on, say, d.p. I, one finds in (3) that $E_{\psi_T}\{g_T(\psi_T - \sum_{\tau=0}^{q} B_\tau x_{T-\tau})\} = g_T(\bar{\psi}_T - \sum_{\tau=0}^{q} B_\tau x_{T-\tau}) + \Lambda_T$ where Λ_T is a constant. Then the minimization in x_T of $[f_T(x_T) + g_T(\bar{\psi}_T - \sum_{\tau=0}^{q} B_\tau x_{T-\tau})]$, gives a quadratic function in $x_{T-q}, \ldots, x_{T-1}$. Then one continues and at each step $\bar{\psi}_t$ is a certainty equivalent for ψ_t and the objective remains quadratic. Hence, one is able to determine $x_T^*(\cdot), \ldots, x_1^*(\cdot)$. To guarantee the existence of these $x_T^*(\cdot)$, one may require that the f_t are strictly convex and the g_t are convex and that the variances and covariances of the elements of ψ_t are finite. To perform the expectations backward from period to period, it is necessary that ψ_t be distributed independently of x_τ if $t \geqq \tau$ for $t = 1, \ldots, T$. Clearly , this result applies also to d.p. III and does not apply to d.p. II or d.p. IV. The result is due to Simon [23] and Theil [24]. A detailed presentation, along with a proof different than that sketched here, appears in [26, Chap. 4] in the context of d.p. III.

We will not describe other special cases. Certain special cases (and more general formulations in other contexts than we have described) appear in the literature of stochastic optimal control, see, e.g., [1, 2], and deal in particular with Markovian systems. In [36] we have presented other special cases that lead to relatively simple optimization problems when the state vectors occur linearly in the preference functions and constraints.

The result survey by Larson [20] discusses many special cases and approximation schemes including polynomial approximation, quantizing the state and decision variables into levels, search procedures, use of analytic results and necessary optimality conditions, use of Lagrange multipliers, quasilinearization, and approximation in function and policy space.

REFERENCES

1. M. AOKI: *Optimal Control of Partially Observable Markovian Systems.* J. Franklin Institute, CCLXXX (1965), pp. 367–386
2. M. AOKI: *Optimization of Stochastic Systems.* New York, Academic Press, 1967
3. K. J. ARROW: *Statistics and Economic Policy.* Econometrica, XXVI (1957) pp. 523–531
4. K. J. ARROW: *Tinbergen on Economic Policy.* J. Amer. Stat. Assoc., LIII (1958), 89–97
5. W. J. BAUMOL, *Economic Dynamics.* New York, The Macmillan Co., 2-nd Ed., 1959
6. R. BELLMAN, *Dynamic Programming.* Princeton, Princeton University Press, 1957
7. R. BELLMAN: *Adaptive Control Processes. A Guided Tour.* Princeton, Princeton University Press, 1961
8. A. BEN-ISRAEL: *On Some Problems of Mathematical Programming.* Unpublished Ph. D. Thesis in Engineering Science, Evanston, Illinois, Northwestern University, 1962
9. C. BERGE: *Topological Spaces.* New York, The Macmillan Co, 1963
10. A. CHARNES and M. J. L. KIRBY: *Optimal Decision Rules for the E-Model of Chance Constrained Programming.* Cahiers du Centre d'Etudes de Recherche Opérationnelle, VIII (1966), 5–44
11. G. B. DANTZIG: *Linear Programming Under Uncertainty.* Management Science, I (1955), 197–206

12. P. FAURE and P. HUARD: *Resolution de programmes mathematiques a fonction nonlineaire par la méthode du gradient réduit.* Revue Française de Recherche Opérationnelle, XXXVI (1965), 167, 206

13. A. V. FIACCO and G. P. MCCORMICK: *The Sequential Unconstrained Minimization Technique for Nonlinear Programming. A Primal-Dual Method.* Management Science, X (1964), 360—366

14. K. A. FOX, J. K. SENGUPTA and E. THORBECKE: *The Theory of Quantitative Economic Policy.* Chicago, Rand McNally and Co, 1966

15. M. FRANK and P. WOLFE: *An Algorithm for Quadratic Programming.* Naval Research Logistics Quarterly, III (1956), 95—110

16. D. GALE and H. NIKAIDO: *The Jacobian Matrix and Global Univalence of Mappings.* Mathematics Annalen, CLIX (1965), 81—93

17. A. S. GOLDBERGER: *Impact Multipliers and Dynamic Properties of the Klein-Goldberger Model.* Amsterdam, North-Holland Publishing Co, 1959

18. R. V. HOGG and A. T. CRAIG: *Introduction to Mathematical Statistics.* New York, Macmillan, Co, 2nd Ed., 1965

19. O. L. R. JACOBS: *An Introduction to Dynamic Programming.* London, Chapman and Hall, Ltd, 1967

20. R. E. LARSON: *A Survey of Dynamic Programming Computational Procedures.* IEEE Transactions on Automatic Control, AC-12, No. 6 (1967), 764—774

21. O. L. MANGASARIAN: *Pseudo-convex Functions.* SIAM J. Control, III (1965), 281—290

22. H. RAIFFA and R. SCHLAIFER: *Applied Statistical Decision Theory.* Cambridge, Division of Research, Graduate School of Business Administration, Harvard University, 1961

23. H. A. SIMON: *Dynamic Programming under Uncertainty with a Quadratic Criterion Function.* Econometrica, XXIV (1956).

24. H. THEIL: *A Note on Certainty Equivalence in Dynamic Planning.* Econometrica, XXV (1957), 346—349

25. H. THEIL: *Economic Forecasts and Policy.* Amsterdam, North-Holland Publishing Co., 2nd Ed., 1965

26. H. THEIL: *Optimal Decision Rules for Government and Industry.* Chicago, Rand McNally and Co, 1964

27. H. THEIL and J.C.G. BOOT: *The Final Form of Econometric Equation Systems.* Review of the International Statistical Institute, XXX (1963), 136—152

28. J. TINBERGEN: *On the Theory of Economic Policy.* Amsterdam, North Holland Publishing Co., 2nd Ed., 1966

29. J. TINBERGEN: *Economic Policy. Principles and Design.* Amsterdam, North-Holland Publishing Co., 2nd Ed., 1967

30. A. WARBURTON and W.T.ZIEMBA: *Convex Inversion.* Journal of Mathematical Analysis and Applications.

31. R. WETS: *Programming Under Uncertainty. The Solution Set.* J. SIAM, XIV (1966), 1143—1151

32. W. I. ZANGWILL: *The Convex Simplex Method.* Management Science, XIV (1967), 221—228

33. W. I. ZANGWILL: *Nonlinear Programming. A Unified Approach.* Englewood Cliffs, New Jersey: Prentice-Hall 1969

34. W. ZIEMBA: *Static Economic Policy.*

35. W. ZIEMBA: *Nonlinear Stochastic Programming with Simple Recourse.*

36. W. ZIEMBA: *Computational Algorithms for Convex Stochastic Programs with Simple Recourse.* Operations Research.

37. W. ZIEMBA: *Transforming Stochastic Dynamic Programs into Nonlinear Programs.* Management Science.

38. G. ZOUTENDIJK: *Methods of Feasible Directions.* Amsterdam, Elsevier Publishing Company, 1966.

DES MODELES DE PROGRAMMATION STOCHASTIQUE DES PROBLEMES DE PLANIFICATION DYNAMIQUE

Dans cet article on expose plusieurs modèles de décision dynamique, suggérés par le schéma de stratégie économique de Tinbergen. Chaque modèle de décision a pour fondement un modèle structurel. On suppose que le décident désire choisir une décision initiale (vecteur) et une séries de règles de décision au moyen desquelles il puisse apprendre le minimum de la somme des coûts attendus, produits par les écarts des valeurs réalisées des variables, par rapport à leurs valeurs préférées.

Les modèles de décision diffèrent par les suppositions faites sur le modèle structurel et selon la forme du problème, stochastique ou adaptive. On essaie de déterminer quand, au point de vue formel, les divers modèles de décision sont équivalents. On fait des suppositions qui puissent mener à des procédés de résolution théorique par la programmation dynamique. Un chapitre sur les méthodes de résolution algorithmique achève l'article.

МОДЕЛИ СТОХАСТИЧЕСКОГО ПРОГРАММИРОВАНИЯ ЗАДАЧ ДИНАМИЧЕСКОГО ПЛАНИРОВАНИЯ

В этой статье формулируются отдельные модели динамического решения, вытекающие из схемы экономической стратегии Тинбергера. Каждая модель решения имеет в своей основе структурную модель. Предполагается, что принимающий решение желает выбрать первоначальное решение (вектор) и ряд правил принятия решений, при помощи которых находится минимум предполагаемой суммы расходов, обусловленных отклонениями осуществленных величин переменных от их наиболее предпочитительных величин.

Модели решений различаются предположениями относительно лежащей в основе структурной модели, а также в зависимости от того, является ли задача стохастической или адаптивной. Делается попытка определить, когда различные модели решений эквивелентны в формальном отношении. Делаются предположения, ведущие к способам теоретического разрешения путем динамического программирования. Статья заканчивается разделом о методах алгоритмического разрешения.