G. BOLDUR

Some Aspects of the Cutting Stock Problem as Applied to Rectangular Elements

The cutting stock problem has two parts:

a) *The determination of cutting distinct modes.*

b) *The choosing of optimum cutting modes and of their number.*

One presents, in the present article, an algorithm for the solving of the first part of the problem.

I. GENERAL ASPECTS OF THE CUTTING STOCK PROBLEM

The cutting stock problem was first approached with mathematical means by L.V. Kantorovich [1], nearly thirty years ago, and, in recent years, it was taken again and studied thoroughly by P.C. Gilmore and R.E. Gomory [2], [3], [4].

If we refer to the case of material in form of bars (steel bars, pipes, boards, a.s.o.), the problem can be presented as follows:

Let a lot product be in the form of bars with identical cross sections, being, manufactured into several lengths: L_1, L_2, ..., L_k. The material must be cut, according to a certain order, in pieces of different lengths, namely:

N_1 pieces of l_1 length;

N_2 pieces of kl_2 length;

.....

N_m pieces of l_m length.

The fulfilment of the order is possible, if some j would exist, such as to get, for all i: $L_j \geqslant l_i$ ($j = 1, 2, ..., k$; $i = 1, 2, ..., m$).

To each one of the lenghs L_i there corresponds a cost $c_i (i = 1, 2, .., k)$, and the problem consists in carrying out the order with a minimal cost.

In general, lengths l_i which must be cut to dimension are considerably smaller than the manufacturing lengths L_i. Taking this into account, several manners of dimension cutting bars L, in order to obtain bar l, are possible. Thus, if we have to cut to dimension three bars with the lengths $l_1 = 8$ m, $l_2 = 5.25$ m and $l_3 = 2.5$ m, from a bar having $L = 14$ m length, the following cutting solutions are possible [6]:

— 1 type l_1 — bar + 1 type l_2 — bar;
— 1 type l_1 — bar + 2 type l_3 — bars;
— 2 type l_2 — bars + 1 type l_3 — bar;
— 1 type l_2 — bar + 3 type l_3 — bars;
— 5 type l_3 bars.

One can see that the problem has two parts, the first one consisting in finding all the n distinct solutions of cutting bars, and the second one, in determining the numbers $x_j (j = 1, 2, ..., n)$, which represent the number of times every distinct cutting solution is used, such as to satisfy the necessary stock $N_i (ix = 1, 2, ..., m)$ for each piece; and to let the total cost be minimal.

We suppose that we found, by some procedure, all the n distinct cutting solutions, and consequently the second part of the problem is mathematically modelled by means of the linear programming in integers:

$$\sum_{j=1}^{m} a_{ij} x_j \geqslant N_i \qquad (i = 1, 2, ..., m)$$

$$\sum_{j=1}^{n} c_j x_j : \textit{minimum}$$

$$x_1 x_2, ..., x_n \in \{0, 1, 2, 3, ...\}$$

(In the above relation, a_{ij} represents the number of bars of l_i length, which are cut according to the cutting solution j, and c_j is the cost of the bar, to which the respective cutting solution is applied).

Solving such types of problems means solving both the first and the second part of the problem. As concerns the first part of the problem, the difficulty consists in the very large number of possible solutions for cutting bars, a number which, in certain problems, reaches millions of solutions.

In the second part of the problem, the difficulty lies in the restriction regarding the obtaining of a solution in integers.

Gilmore and Gomory, while showing that it is possible to neglect the restriction concerning the obtaining of a solution in integers, without introducing important alterations, succeed in working out an algorithm, by which the difficulty of the very large number of solutions is overcome [2; 3].

A second category of cutting stock problems refers to bidimensional elements (metal sheets, glass or paper sheets, a.s.o.). The mathematical formulation of the problem is similar to the case of unidimensional elements — instead of bars there are bidimensional pieces. This category of problem is characterized by the increased difficulty of finding all distinct cutting solu-

tions. Gilmore and Gomory solved this problem for some particular cases of rectangular elements [4], but they dit not find a general solution of the problem.

An interesting approach of the problem of finding all distinct cutting solutions, in the case of bidimensional elements, is that of P. Julien, in his paper concerning the "puzzle" [5]. In this paper, the author considers a polyhedral area, called support (e.g., Fig. 1 a) and several polyhedral areas, called pieces (e. g. Fig. 1 b), characterized by the fact that each of the latter can be placed into several positions within the support.

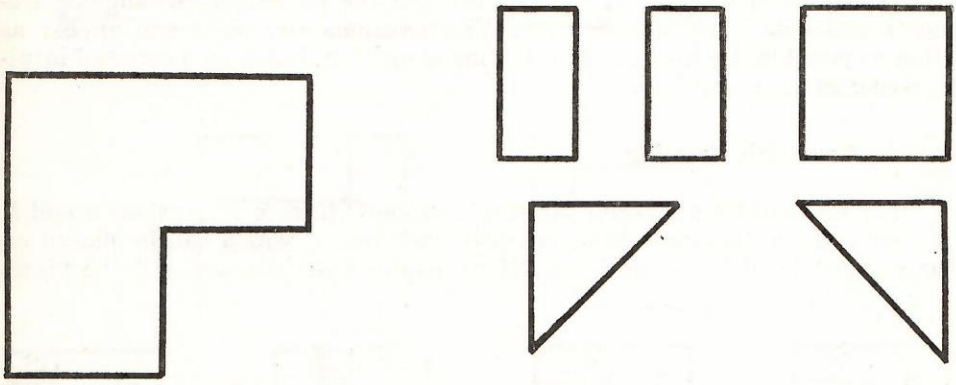


Fig. 1 a and b

From the infinity of positions, which each one of the 5 pieces can occupy, by locating them within the support, those positions of each separate piece will be taken into consideration, which allow the location of all the remaining 4 Pieces within the support. Each one of these positions is called placed-piece. A finite set of placed-pieces will be available for every Piece. Each one of these sets will be called Piece: Piece $A = \{A_1, A_2, ..., A_a\}$, Piece $B = \{B_1, B_2, ..., B_b\}$ Piece $C = \{C_1, C_2, ..., C_c\}$ a.s.o.

To find a solution of the puzzle, in the sense interesting us for the Cutting Stock Problem, means to form a set

$$P = \{A_h, B_i, C_j, D_k, E_l\}, \ (h \in \{1, 2, ..., a\}, \ i \in \{1, 2, ..., b\},$$

$$j \in 1, 2, ..., c\}, \ k \in \{1, 2, ..., d\}, \ l \in \{1, 2, ..., e\}),$$

including one placed-piece in each one of the sets A, B, C, D, E, in such a way, that the placed-pieces be compatible, that is to say, they do not overlap themselves upon the support.

Solving the puzzle is equivalent to finding all possible solutions.

The author points out two solving methods for this type of problem.

It is necessary, with both methods, to know the sets A, B, ... of placed-pieces, the finding of set P having to be done by successive approximations

on using a grapho-matrix algorithm with the first method, and successive attempts, with the second method.

It should be observed, in the case of Julien's problem, that, unlike the cutting stock problems occurring in practice, in each solution of the problem a Piece can appear only once.

II. DETERMINING THE POSSIBLE WAYS OF CUTTING RECTANGULAR ELEMENTS

In the cases when both the support and the pieces are rectangular elements and with each one of the cutting solutions any piece can appear as often as possible, we worked out a solving algorithm, based on a series of intuitive-geometric reasonings.

A. Geometric treating

Let us consider a rectangular sheet (support) having dimensions a and b, as well as s^* rectangular sheets (pieces), each one of which can be placed on the support in different positions. If we suppose the support and the pieces

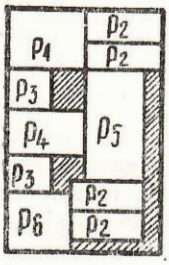


Fig. 2

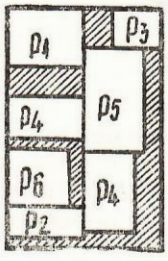


Fig. 3

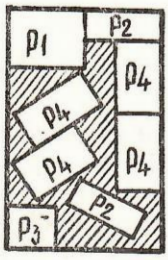


Fig. 4

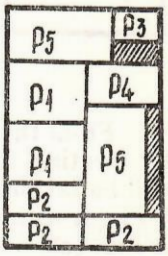


Fig. 5

scaled down, such as to be easily handled, there are infinitely more possibilities to place the pieces within the support (Figs. 2, 3, 4, 5).

We shall call each one of these possibilities cutting way or cutting-up. In the following, we shall deal only with cutting-ups of the types shown in Figs. 1 and 4, which are characterized by the fact that each piece can be placed only in one of the ways represented below (Figs. 6—9).

The cutting-ups in which all pieces are placed in one of the four ways shown above, and which do not allow any more placing of a piece other than those already placed, will be called complete cutting-ups. It is possible that two complete cutting-ups comprise the same pieces, but differently placed, as in Figs. 10 and 11.

Yet in the following we are interested merely in the complete cutting-ups differing one from another by at least one element. Two such complete cutting-ups will be called complete cutting-ups distinct between them, two by two.

Determining the possible ways of cutting rectangular elements can be expressed as follows: given a support S and s^* pieces, determine all complete cutting-ups, distinct between them, two by two.

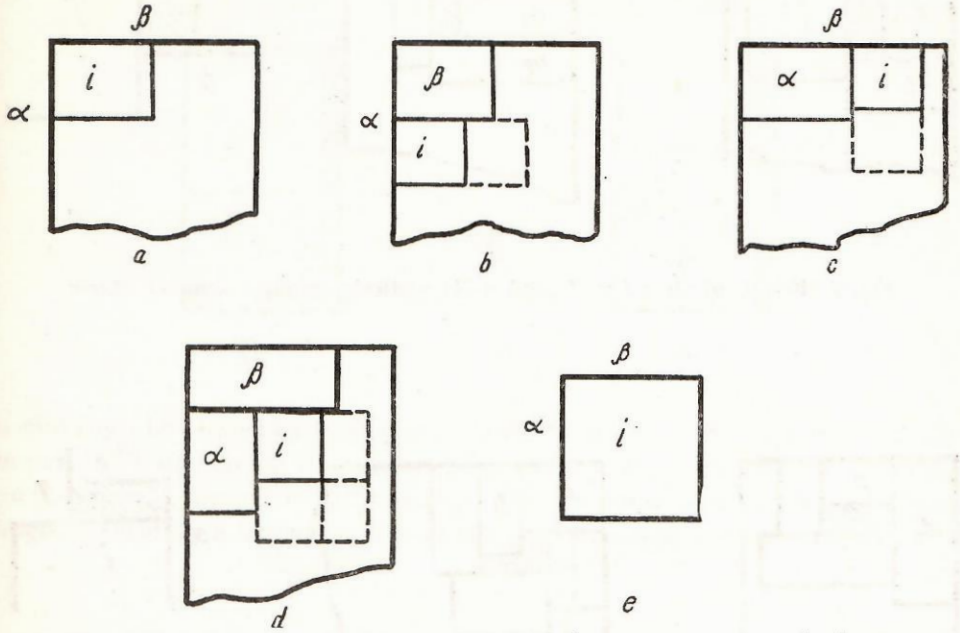


Fig. 6 Mode I of piece i settlement $a - d$: particular cases; c: general scheme

Determining methods of all complete cutting-ups distinct between them, two by two.

The working procedure is, in the main, as follows:

— build up the first complete cutting-up;

— build up, starting from the first complete cutting-up, a second cutting-up, distinct from the first one, then a third cuting-up, distinct from the first and the second ones, and so on, till it is no longer possible to build up any complete cutting-up, distinct from the other ones, previously built up.

Before discribing the way of building the first complete cutting-up, some preliminary operations are necessary:

a. We number the pieces (the number associated to a piece will be called "mark"), considering each of them twice, according to Fig. 10: once "standing", once "lying", and giving them different marks; it is obvious that square elements will be considered a single time.

In Fig. 12 we have:

$$AB = CD = a_i$$
$$AC = BD = b_i$$
$$BD = AC = a_{i+1} = b_i$$
$$BA = DC = b_{i+1} = a_i$$

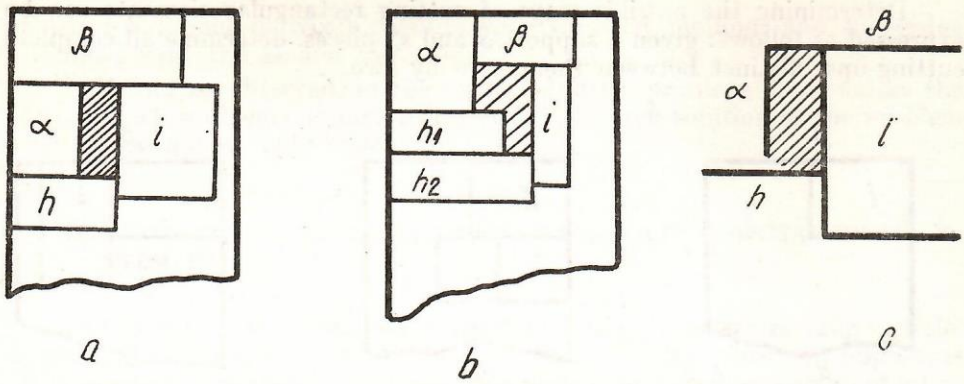


Fig. 7 Mode II of piece i settlement a—b: particular cases; c: general scheme

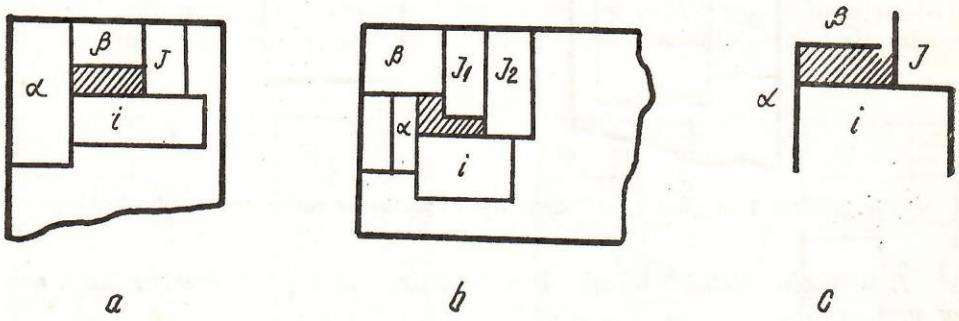


Fig. 8 Mode III of piece i settlement a — b: particular cases; c: general scheme

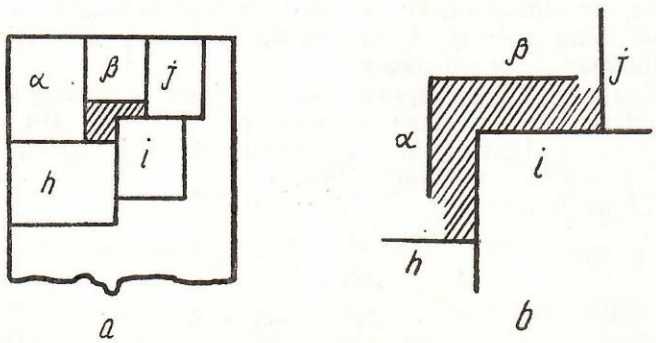


Fig. 9 Mode IV of piece i settlement a: particular cases; b: general scheme

We introduce the concepts of *useful angle* and of useful-angle *rank*. By useful angle, we understand the angle formed by two pieces α and β, such as the angle vertex be orientated towards N-W (Fig. 13 a); it is possible that the

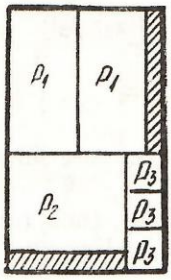


Fig. 10

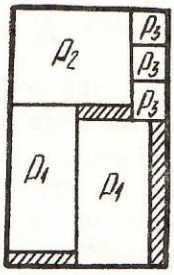


Fig. 11

useful angle be formed by a piece and one of the sides of the support (Fig. 13) or even by both sides of the support (Fig. 13 c); every piece i which is placed on the support forms two useful-angles (Fig. 12 b); we agree to call the useful angle, formed by a horizontal side of the piece i, a type I — useful angle, and

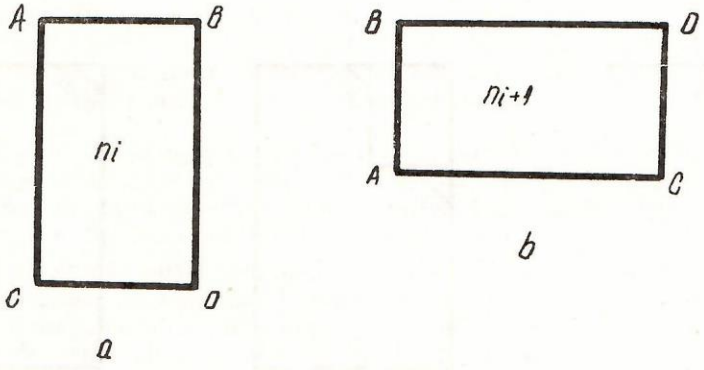


Fig. 12

the useful angle formed by a vertical side of the piece, a type II — useful angle.

Considering that placed-pieces are numbered in the succession of their placing, the useful angle will be denoted by the order numbers of the pieces constituting it, in this succession: I = order number of left-hand pieces; II = order number of top piece; thus, the useful angle in Fig. 12 a will be

denoted by (α, β). When the useful angle is formed by a piece side and a support side, the "order number" of the exterior will be considered to be 0 (see Figs. 13 b and 13 c).

The useful angle rank (α, β) will be determined by the formula

$$R(\alpha, \beta) = 1 + max \{R(\alpha_\alpha, \beta_\alpha), R(\alpha_\beta, \beta_\beta)\}$$

where $R(d_\alpha, \beta_\alpha)$ is the useful-angle rank of piece α
and $R(\alpha_\beta, \beta_\beta)$ is the useful-angle rank of piece β.

If the useful angle is formed with one of the support sides, the "exterior rank" will be considered 0 (zero).

Fig. 14 shows the way of denoting useful angles, their ranks, the pieces marks, as well as their order number in a complete cutting-up.

Building up the complete cutting-up with the order number 1

Select the piece marked 1, from the list of pieces numbered from 1 to s, and try to place it in the useful angle (0; 0), (see fig. 14); if this is not possible, try all pieces, one after another, till one is found which can be placed; this piece will have the order (placing) number $i = 1$.

Try to place, in the useful angle (0; 1), the piece having mark 1, and if this is not possible, the first of the s pieces for which placing is possible; its order number will be $i = 2$.

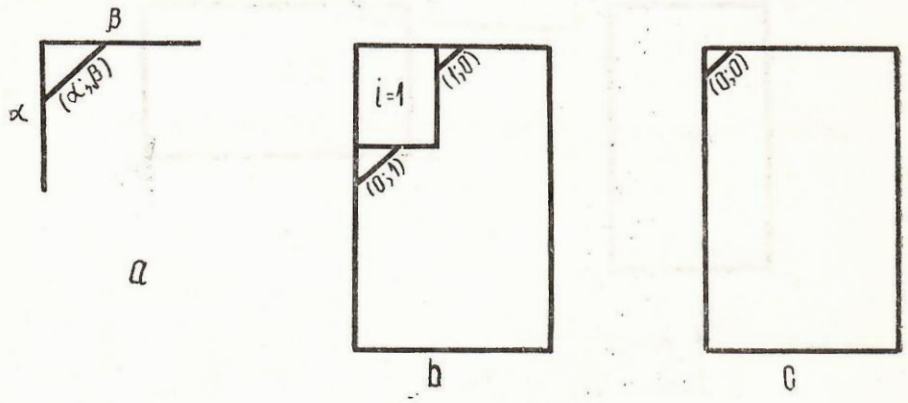


Fig. 13 a, b, c

Go on with this procedure, trying to complete new pieces, according as new useful angles appear; completion will be done in the increasing order of the useful-angle ranks, and if there are two or more useful angles with the same rank, their completion will be done in an arbitrary order, for instance, in the increasing order of element α of the useful angles; the introduction of

the elements into the useful angles will be done, according to case, in one of the 4 placing ways (Figs 6—9); thus, in the Fig. 7a the placing of piece i in the useful angle (α; β) is hindered by piece h, and consequently the placing will be done according to figure; it is obvious that when it is no more possible

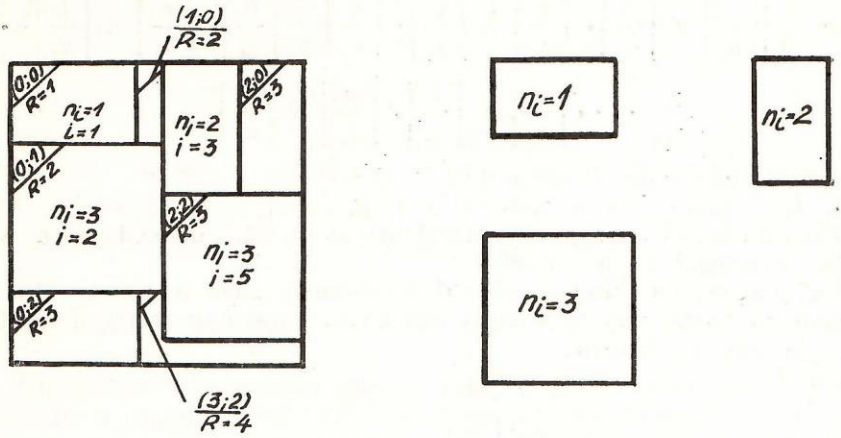


Fig. 14

to place some one of the pieces into a certain useful angle, one has to pass further to the next useful angle.

The cutting-up is complete when neither piece can any longer be placed on the support (in the 4 shown ways).

Building up of a complete cutting-up, with any order number k, distinct from all k-1 complete cutting-ups, previously built up

Start from the complete cutting-up with order number k-1 (pieces with the biggest order number); instead of it, try to place the first of the next pieces (as concerns the mark number), and, if this is not possible, try to place the second one, the third one, a.s.o, till a piece can be placed.

If no placing of any piece instead of the eliminated one is possible, one more piece should be eliminated (namely the second one, in the decreasing order of the mark number), attempting then, as above, to place the next piece in mark order, then the second one, the third one, a.s.o.

As many pieces of cutting-up k-1 should be eliminated, as are necessary in order to obtain a complete cutting-up, distinct from all k-1 previous complete cutting-ups, namely through replacing them by another piece and, if required, through adding of pieces.

Calculation of the number of distinct complete cutting-ups

The calculation of the number of distinct complete cutting-ups presents a particular interest. This number can be determined, with sufficient approximation, by using the formula

$$N^s_{m(p)} = C^p_0 N^s_{m-p} + C^1_p N^{s-1}_{m-p+1} + C^2_p N^{s-2}_{m-p+2} + \ldots + C^{p-1}_p N^{s-p+1}_{m-1} + C^p_p N^{s-p}_m \quad \text{(I)}$$

where:

N^s_m = number of distinct complete cutting-ups, for s pieces;

$$m = max \left\{ \left[\frac{a}{a_1}\right] \times \left[\frac{b}{b_1}\right], \left[\frac{a}{b_1}\right] \times \left[\frac{b}{a_1}\right], \left[\frac{a}{a_2}\right] \times \left[\frac{b}{b_2}\right], \left[\frac{a}{b_2}\right] \times \left[\frac{b}{a_2}\right], \ldots, \right.$$
$$\left. \ldots, \left[\frac{a}{a_s}\right] \times \left[\frac{b}{b_s}\right], \left[\frac{a}{b_s}\right] \times \left[\frac{b}{a_s}\right] \right\}$$

a, b = dimensions of support;

a_i, b_i = dimensions of pieces ($i = 1, 2, \ldots, s$).

The number of terms of the development of $N^s_{m(p)}$ depends on p, which will be an integer of the interval:

$1 \leqslant p \leqslant m$; its selection should be done in such a manner, that all development terms may be directly calculated; thus is possible, if the following relations are known:

$$N^s_0 = 1 \quad (s = 0)$$
$$N^s_1 = s$$
$$N^0_m = 0$$
$$N^1_m = 1$$
$$N^2_m = m + 1$$

For quickly establishing the number of complete cutting-ups, the Table in Annex 1, computed with the aid of above formula, can be used.

Each number entered in the Table can be obtained by adding the direct L.H. side number and the direct above number; thus for $m = 5$, $s = 5$ in the Table number 56 is found, which equals the sum of numbers 35 and 21, which are directly on the L.H. side and above, respectively. By applying this rule, the Table can be computed from close to close, for any m and s.

To fix up ideas, let us calculate the number of distinct complete cutting-ups, which can be built upon a support having the dimensions $a = 1.50$, $b = 1.00$, while the dimensions of the pieces are: 0.80×1.00; 0.60×1.20 and 0.90×0.90.

As was said above, each piece will be considered twice, first "standing", then "lying", excepting the third piece which, being a square, is taken only once; hence we shall get:

$$a_1 = 0.80; \qquad b_1 = 1.00$$
$$a_2 = 1.00; \qquad b_2 = 0.80$$
$$a_2 = 1.00; \qquad b_2 = 0.80$$
$$a_3 = 0.60; \qquad b_3 = 1.20$$
$$a_3 = 1.20; \qquad b_3 = 0.60$$
$$a_5 = 0.90; \qquad b_5 = 0.90$$
$$s = 5$$

We compute

$$m = max\left\{\left[\frac{1.50}{0.80}\right] \times \left[\frac{2.00}{1.00}\right], \left[\frac{1.50}{1.00}\right] \times \left[\frac{2.00}{0.80}\right], \left[\frac{1.50}{0.60}\right] \times \right.$$

$$\left. \times \left[\frac{2.00}{1.20}\right], \left[\frac{1.50}{2.10}\right] \times \left[\frac{2.00}{0.60}\right], \left[\frac{1.60}{0.90}\right] \times \left[\frac{2.00}{0.90}\right]\right\} = max\{1 \times 2{,}1 \times 2{,}2 \times$$

$$\times 1{,}1 \times 3{,}1 \times 2\} = 3$$

The value of $N^s_{m(p)}$ can be read directly from the Table:

$N^3_5 = 35$, or computed, taking $p \in \{1, 2, 3\}$. In all three cases the same result will be obtained:

$$N^5_{3(1)} = N^5_{3(2)} = N^5_{3(3)} = 35$$

B. Numerical algorithm

The geometric method of obtaining the possible ways of cutting rectangular elements implies great difficulties when applied in practice, as it makes necessary to scale down, on paper or cardboard sheets, both the support and the pieces, as well as their manual handling, in order to get complete cutting-ups. A numerical method of solving the problem is hence absolutely necessary, especially when the solving is done by electronic computers. Our purpose was to numerically transpose the whole geometric reasoning, mentioned before, by means of the algorithm we are presenting in the following.

The numerical equivalent of the cutting-up will be a Table with 15 columns and as many lines as pieces to be placed upon the support (Annex 2).

In the first column of the Table, enter the ordre number of the cutting-up;

in the second column, enter the order numbers of the pieces placed upon the support, within the considered cutting-up;

in the third column enter, in brackets, the pairs (α_i, β_i), which represent the useful angles to be used when placing pieces i;

in the fourth column enter the marks n_i of the pieces placed upon the suport.

The numbers X_{1i}, Y_{1i}, X_{2i} andc Y_{2i}, entered in the columns 5, 6, 7 and 8, respectively, represent abscissae and ordinates, respectively, of the corners of piece i, placed upon the support, in a coordinate system XOY, according to Fig. 15.

In the 9-th column enter, in brackets, the pairs (α_i, i), which represent the type X — useful angles of elements i;

In the 10-th column enter the ranks of angles (α_i, i);

in the 11-th column enter, in brackets, the pairs $(i; \beta_i)$; which represent the type II — useful angles of elements i;

in the 12-th column enter the ranks of the angles $(i; \beta_i)$;

in the 13-th column enter the areas of the elements placed upon the support;

in the 14-th column enter the cumulative areas;

in the last column, separately for each mark, enter the total number of pieces placed upon the support, within the considered complete cutting-up.

Complete tables correspond to the complete cutting-ups, and distinct complete tables correspond to the distinct complete cutting-ups.

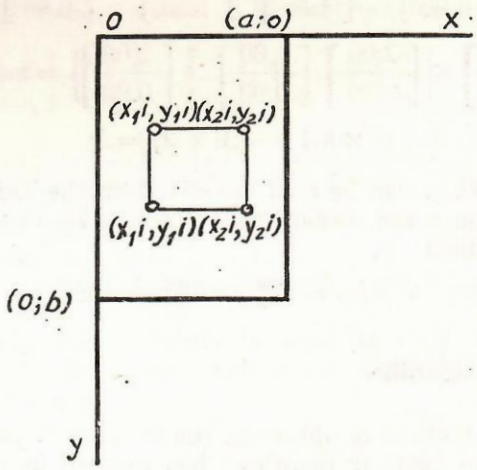


Fig. 15

Initial data

The data, to begin the calculation with, are as follows:
— support dimensions: a, b;
— dimensions of pieces:
— pieces with mark 1 : a_1, b_1;
— pieces with mark 2: a_2, b_2;
— pieces with mark s : a_s, b_s.

Building up the complete table having order number 1

1. The first line of the table is filled in according to Annex 2; number n_1, in the 4-th column, represents the first number in the sequence, 1, 2,..., s of the the marks of the pieces, for which the following relations are verified.

$$a_{n1} \leqslant a \tag{1}$$
$$b_{n1} \leqslant b$$

(we note that, excepting columns 13 and 14, for the filling in of which a multiplication was carried out, all columns are filled in only with data of the problem and with figures 0 and 1).

2. The second line in the table is filled in as follows:
2.1. Enter figure 1 in column 1.
2.2. Enter figure 2 in column 2.

2.3. Enter pair $(\alpha_2; \beta_2) = (0;1)$ from column 9, line 1, in column 3, but only if relations

$$\begin{matrix} b_{n1} + b_j \leqslant b \\ a_j \leqslant a \end{matrix} \qquad j \in \{1, 2, ..., s\} \tag{2}$$

are verified.

If relations (2) are not verified, instead of pair (0; 1) enter pair $(\alpha_2; \beta_2) =$ $= (0; 1)$ from column 11, line 1; in this case, relations

$$\begin{matrix} a_{n1} + a_j \leqslant a \\ b_j \leqslant b \end{matrix} \qquad j \in \{1, 2, ..., s\} \tag{3}$$

must be verified. If not even these relations are verified, the filling in of the table's second line is not possible, and consequently the figures entered in columns 1 and 2 of this line will be struck out; we shall get a complete table with a single line (geometrically, a cutting-up with a single piece upon the support).

2.4. Enter, in column 4, number n_2, representing the first number in the sequence 1, 2, ..., s of the pieces marks, for which relations (2) are veri, fied, if the pair (0; 1) appears in column 3, or which verify relations (3)- if the pair (1; 0) appears in column 3.

2.5. Enter value $X_{12} = 0$ in column 5, if pair (0; 1) appears in column 3, or value $X_{12} = a_{n1}$, if pair (1; 0) appears in column 3.

2.6. Enter value $Y_{12} = 0$ in column 6, if pair (1; 0) appears in column 3, or value $Y_{12} = b_{n1}$, if pair (0 ;1) appears in column 3.

2.7. Enter value $X_{22} = a_{n2}$ in column 7, if pair (0; 1) appears in column 3, or value $X_{22} = a_{n1} + a_{n2}$, if pair (1; 0) appears in column 3.

2.8. Enter value $Y_{22} = b_2$ in column 8, if pair (1; 0) appears in column 3, or value $Y_{22} = b_{n1} + b_{n2}$, if pair (0; 1) appears in column 3.

2.9. Enter pair $(\alpha_i; i) = (\alpha_i; 2)$ in column 9, only if one of the following relations is satisfied:

$$Y_{22} < b, \tag{4}$$

when pair (0; 1) appears in column 3
or

$$Y_{22} < b_{n1} \tag{4'}$$

when pair (1; 0) appears in column 3. When neither relation (4) nor (4') is satisfied, there should be no entry in column 9 (this will be marked by drawing a line).

2.10. If a pair is entered in column 9, figure 2 will be entered in column 10; if a line was drawn in column 9, another line will be drawn in column 10.

2.11. Enter pair $(i; \beta_i) = (2; \beta_2)$, in column 11, only if one of the following relations is satisfied

$$X_{22} < a, \tag{5}$$

when pair (1; 0) appears in column 3
or

$$X_{22} < a_{n1}, \tag{5'}$$

when pair (0; 1) appears in column 3; when neither relation is satisfied, draw a line in column 11.

2.12. If a pair was entered in column 11, enter figure 2 in column 12; if a line was drawn in column 11, draw a line in column 12, too.

2.13. Write product $A_2 = a_{n2} \times b_{n2}$ in column 13.

2.14. Write sum $A_{2c} = A_{1c} + a_{n2} \times b_{n2} = a_{n1} \times b_{n1} + a_{n2} \times b_{n2}$ in column 14.

2.15 Column 15 should be filled in as follows:

If a figure n_2 different from n_1 appeared in column 4, enter figure 1 in column 15; if $n_2 = n_1$, draw a line in column 16, line 2, and strike out 1 from column 16, line 1, and enter 2 instead.

3. Line having order number i should be filled in as follows:

3.1 Enter figure 1 in column 1.

3.2. Enter figure i in column 2.

3.3. Column 3 should be filled in like this:

Examine pairs $(\alpha_i; i)$ entered in column 9 and pairs $(i; \beta_i)$ entered in column 11, as well as their ranks; select, from the pairs in column 8 and 11, the pair with the minimal rank which does not occur in any of the first $i-1$ lines of column 3; if there are several pairs with minimal rank, select one of them, preferring one which entered in column 9; the pair selected in this way, let it be $(\alpha_i; \beta_i)$, will be entered in column 3.

3.4. Enter, in column 4, number n_i, which represents the first number in the sequence 1, 2,..., s of pieces mark, for which the following relations are verified:

$$X_{2\alpha i} + a_{ni} \leqslant a \tag{6}$$
$$Y_{2\beta i} + b_{ni} \leqslant b$$

if there is no number h in column 4 of the table ($h \in \{1, 2,..., i-1\}$), for which

$$X_{2h} > X_{2\alpha i} \tag{7}$$
$$Y_{2\beta i} < Y_{1h} < Y_{2\beta i} + b_{ni}$$

and also no number j, entered in the same column ($j \in \{1, 2,..., n-1\}$), for which

$$Y_{2j} > Y_{2\beta i} \tag{8}$$
$$X_{2\beta i} < X_{1i} < X_{2\alpha i} + a_{ni}$$

then one may enter in columns 5—8, respectively, the values

$$X_{1i} = X_{2\alpha i}$$
$$Y_{1i} = Y_{2\beta i} \tag{9}$$
$$X_{2i} = X_{1i} + a_{ni} = X_{2\alpha i} + a_{ni}$$
$$Y_{2i} = Y_{2i} + b_{ni} = Y_{2\beta i} + b_{ni}$$

If, in column 4, there are numbers h for which relations (7) are verified, but there are not numbers j for which relations (8) are verified, proceed as follows:

Let numbers $h_1, h_2, ..., h_p$, be, for which relations (7) are verified, and respectively $X_{2h1}, X_{2h2},..., X_{2hp}$, the corresponding values entered in column 7;

let h' be the number for which value $X_{2hk}(k = 1, 2, ..., p)$ is maximal; enter in column 4 the first number n_i of the sequence 1, 2, ..., s, for which relations

$$X_{2h'} + a_{ni} \leqslant a \qquad (10)$$
$$Y_{2\beta i} + b_{ni} \leqslant b$$

are verified.

In this case, fill in columns 5—8 with the following values:

$$X_{1i} = X_{2h'}$$
$$Y_{1i} = Y_{2\beta i} \qquad (11)$$
$$X_{2i} = X_{1i} + a_{ni} = X_{2h'} + a_{ni}$$
$$Y_{2i} = Y_{2i} + b_{ni} = Y_{2\beta i} + b_{ni}$$

If, in column 4, there are numbers j, for which relations (8) are verified, but there are not any numbers h for which relations (7) are verified, proceed as follows:

Let numbers $j_1, j_2, ..., j_n$ be, for which relations (8) are verified, and respectively Y_{2j}, Y_{2j2}, ,..., Y_{2jr} the corresponding values entered in column 8; let j' be the number for which value $Y_{2jm}(m = 1, 2, ..., r)$ is maximal; enter in column 8 the first number n_i in the sequence 1, 2, ..., s, for which relations

$$X_{2\alpha i} + a_{ni} \leqslant a \qquad (12)$$
$$Y_{2j'} + b_{ni} \leqslant b$$

are verified.

In this case, in columns 5—8 following values can be filled in:

$$X_{1i} = X_{2\alpha i}$$
$$Y_{1i} = Y_{2j'} \qquad (13)$$
$$X_{2i} = X_{1i} + a_{ni} = X_{2\alpha i} + a_{ni}$$
$$Y_{2i} = Y_{1i} + b_{ni} = Y_{2j'} + b_{ni}$$

If, in column 4, there are some numbers h for which relations (7) are verified and, in the same time, there are other numbers j for which relations (8) are verified, we act according to the above indications in order to determine h' and j'; enter in column 4 the first number n, in the sequence 1, 2, ..., 3, for which relations

$$X_{2h'} + a_{ni} \leqslant a \qquad (14)$$
$$Y_{2j'} + b_{ni} \leqslant b$$

are verified. In this case, in columns 5—8 the following values can be filled in:

$$X_{1i} = X_{2h'}$$
$$Y_{1i} = Y_{2j'} \qquad (15)$$
$$X_{2i} = X_{1i} + a_{ni} = X_{2h'} + a_{ni}$$
$$X_{2i} = Y_{1i} + b_{ni} = Y_{2i'} + b_{ni}$$

It is obvious that, if it would not be possible to enter in column 4, in the line with order number i, the first number in the sequence 1, 2, ..., s, in none of the ways established by formulas (9), (11), (13) or (15), one must take the second number in the sequence and, if necessary, the third one, a.s.o., acting then according to the above indications.

Also, if it is not possible, with the pair $(\alpha_i; \beta_i)$ in column 3, as determined according to the above indications, to enter any number from the sequence 1, 2, ..., s, one will enter in column 3, successively, in rank's order, as many pairs as required, till one is found for which a number n can be entered in column 3.

3.5. Enter pair $(\alpha_i; i)$, in column 9, only if unequality *)

$$Y_2 \alpha_i > Y_{2i} \tag{16}$$

is satisfied.

If this inequality is not satisfied, draw a line in column 9.

3.6. If in column 9 a pair was entered, enter in column 10 the value:

$$R(\alpha_i; i) = 1 + max\ [R(\alpha_{\alpha i}; \beta_{\alpha i}),\ R(\alpha_i; \beta_i)] \tag{17}$$

3.7. Enter in column 11 the pair $(i; \beta_i)$, only if unequality:

$$X_{2\beta i} > X_{2i} \tag{18}$$

is satisfied. If this inequality is not satisfied, draw a line in column 11.

3.8. If, in column 11, a pair was entered, enter in column 12 the value:

$$R(i; \beta_i) = 1 + max\ [R(\alpha_i; \beta_i),\ R(\alpha_{\beta i}; \beta_{\beta i})] \tag{19}$$

3.9. Enter in column 13 the product $a_{ni} \times b_{ni}$.

3.10. Enter in column 14 the value $A_{ic} = A_{i-1,c} + a_m \times b_n$.

3.11. Fill in column 15 in this way:

If in column 4, line i, a number n_i appeared, different from all numbers entered in this column up to line i-1 inclusively, enter figure 1 in column 15; if number n_i appeared also in other lines, draw a line in column 15, line i, increasing by a unit the number entered in this column, in the line where the cumulative number of marks n_i is entered.

The filling-in procedure of the table is continued till it is no longer possible to add a further line. At that moment the table with the order number i is completed, and subsequently one must proceed to the filling in of the complete table having order number i + 1.

Building up the complete table with order number k

Starting from the complete table with order number $k - 1$, try to build up the complete table with order number k, by altering line n (last line). To this end, proceed as follows:

— Strike out all numbers in line n, which thus becomes free and has to be filled in anew.

---

*) In this relation and in the following ones: whenever $Y_{2\alpha i} = Y_{20}$, its value will be taken equal to b, and whenever $X_{2\beta i} = X_{20}$, its value will be taken equal to a.

Enter number k in column 1 of line n.

Enter number n in column 2. Fill in the other columns, by applying the indications from points 3.3-3.11, aiming at obtaining a complete table differing* from the complete tables with order numbers 1, 2,..., $k-1$; it is understood that, with a view to obtaining a complete table, differing from the previous ones, one will try as many numbers following after n_n in the sequence 1, 2, ..., s, as required for it to be verified, namely, as the case stands, one of the above relation groups (6), (10), (12) or (14), try also, in the increasing order's rank, as many pairs $(\alpha_i; i)$ from column 9 and $(i; \beta_i)$ from column 11, as necessary in order to get a new complete table.

When the building up of the complete table with order number k is not possible, by applying above indications, and when it is not possible either to build up, by using these indications, a complete table, distinct from the tables with order numbers 1, 2, ..., $k-1$, try to alter 2, 3 or as many lines as required, starting from the last line of the table with order number $k-1$; hence the impossibility of building up a distinct table by altering the last $u-1$ lines compels us to try to alter the last u lines of the table; in this case, proceed as follows:

Strike out all figures on the last n lines.

Enter number k in column 1 of line $n-u+1$.

Enter number $n-u+1$ in column 2 of line $n-u+1$.

Fill in all the other columns of line $n-u+1$, as well as the remaining lines, according to the indications in the above paragraphs, aiming at obtaining a table, completely different from the preceding one.

After building up the complete table with order numbers k, enter number k in column 1, from top to bottom.

Building up all possible complete tables

Starting from the complete table having order number k, build up, if possible, and by the same methods, the complete table with order number $k+1$, different from all the previous tables, then the complete table with order number $k+2$, a.s.o., until the building up of any complete table, differing from those previously built up, is no more possible.

III. CONCLUSIONS

The algorithm used to determine all distinct ways of dimension-cutting of rectangular elements leads, as results from formula (I), to a large number of possible solutions, that is to say to many tables of the types presented in the algorithm. It follows that, beginning with this first part of the cutting stock problem, the use of electronic computers is necessary. After examination of Annex I a rather severe limiting results, as concerns the m and s problem

*) With a view to establishing whether two complete tables are different, compare columns 4 and 15 in the first table with columns 4 and 15 in the second table; proceed as follows:

Build set $\{n_1, d_1, n_2, d_2, ..., n_{q_2} d_{q_2}\}$ of all pairs in the first table, and set $\{n_1, d_1, n_2, d_2, ... n_{q_2} d_{q_2}\}$ of all pairs in the second table; if both sets do not contain the same elements, the tables are different.

parameters, even when using electronic computers. With the reserve of verifying some real cases, we appreciate, however, that there are, within the resolution limits of average-performance computers, numerous practical problems which can be optimally solved this way.

The first part of the cutting stock problem once solved, the approach of the second part, i.e. the establishing of the using number of each distinct dimension-cutting solution, meets with the difficulties, characteristic of solving a linear-programming problem with a great many variables. We appreciate that an average-performance computer can solve, without difficulties, a simplex problem having the dimensions which result from the solution of the problem's first part, by the same computer.

A particular problem is that of the optimum function.

In our country's conditions, where saving of materials is particularly important, especially saving of metal, we consider that the efficiency function must aim at minimizing wastes.

The efficiency function will be written, in this case, with the data of the presented algorithms, as follows:

$$\sum_{j=1}^{t} A_i \; x_i = maximum$$

where A_j $(j = 1, 2, ..., t)$ will represent the total cumulative area, which appears in the last line of the complete table j, column 14.

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CERTAINS ASPECTS DU DÉCOUPAGE DES ELEMENTS RECTANGULAIRES

La première partie de l'article comprend une exposition succinte du problème du découpage optime des matériaux. Après avoir consulté les travaux de L.V. Kantorovici (U.R.S.S.), P. C. Gilmore et R. E. Gomory (U.S.A.) une des manières de formuler ce problème devient la suivante:

Soit un produit de série ayant la forme de barre (barre d'acier, tuyaux, planches) ou de feuille rectangulaire (tôles, plaques de cerre, carton). De ce produit on doit débiter par découpage m éléments linéaires, respectivement rectangulaires. Si l'on considère qu'il existent n possibilités distinctes de débiter (couper) les éléments respectifs et qu'à chaque possibilité correspond un coût $C\,(j = 1, 2, ..., n)$ du matériel inutilisé, on doit déterminer le nom-

bre $X\,(j = 1,\ 2,\ ...,\ n)$ de fois qu'une autre modalité de découpage doit être employée, de telle façon que, tout en obtenant la quantité $N\ \ (i = 1,\ 2,\ ...,\ n)$ nécessaire de chaque élément devant être débité, le coût total du matériel inutilisé soit minimum.

Deux parties distinctes constituent le problème:

1) déterminer les n modalités distinctes de découpage

2) déterminer les nombres $X\ \ (j = 1,\ 2,\ ...,\ n)$ c'est-à-dire le nombre de fois qu'on utilise chaque façon distincte de découpage (problème de programmation linéaire en nombres entiers).

La seconde partie du problème traite en détail le premier point du problème dans le cas du découpage de feuilles rectangulaires en éléments rectangulaires aussi. Utilisant quelques idées contenues dans un article de P. Julien (France) concernant la théorie du jeu de puzzle, l'auteur élabore un algorithme pour résoudre le problème par des méthodes numériques.

Dans la partie finale on examine les possibilités d'appliquer l'algorithme aux problèmes pratiques, en vue de la programmation sur les ordinateurs numériques.

НЕКОТОРЫЕ АСПЕКТЫ ПРОБЛЕМЫ РАСКРОЯ ПРЯМОУГОЛЬНЫХ ЭЛЕМЕНТОВ

В первой части статьи кратко излагается вопрос оптимального раскроя материалов. На основе трудов Л. В. Канторовича (СССР), П. С. Гильмора и Р. Э. Гомори (США) одной из формулировок, которая может быть дана этому вопросе, является следующая:

Имеется изделие серийного производства в виде бруска (стальные брусья, трубы, доски) или в виде прямоугольного листа (листовая сталь, листовое стекло, картонные листы), из которых необходимо выкроить путем резки m линейных, соответственно прямоугольных элементов. Считая, что существует n отдельных возможностей резки (раскроя) данных элементов из куска рассматриваемого изделия и что каждой возможности соответствует стоимость c_j $(j = 1,\ 2,\ \ldots,\ n)$ неиспользованного материала, следует определить число $x_j (j = = 1,\ 2,\ \ldots,\ n)$, сколько раз необходимо применять каждый отдельный нетод раскроя таким образом, чтобы получить заданное количество $N_i (i = 1,\ 2,\ \ldots,\ m)$ из каждого подлежащего резке элемента, причем общая стоимость неиспользованных материалов должна быть наименьшей.

Задача имеет две отдельные части:

1. определение n отдельных методов раскроя;

2. определение чисел $x_j (j = 1,\ 2,\ \ldots,\ n)$, указывающих, сколько раз будет применен каждый отдельный метод раскроя (задача линейного программирования в целых числах).

Во второй части статьи подробно рассматривается первая часть задачи оптимального раскроя при резке прямоугольных листов на такие же прямоугольные элементы. На основе некоторых мыслей из статьи П. Жулиена (Франция) о теории игры „puzzlle" разрабатывается алгоритм решения задачи числовыми методами. В заключение рассматриваются возможности применения алгоритма к практическим задачам при программировании на цифровых калькуляторах.

Annex 1

s \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136
4	1	4	10	20	36	56	84	120	165	220	286	364	455	560	680	816
5	1	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380	3060	3876
6	1	6	21	56	126	252	462	792	1287	2002	3003	4368	6188	8568	11628	15504

Complete Table with Number 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	(O ;0)	n_1	0	0	a_{n_1}	b_{n_1}	(O; 1)	1	(1 ;0)	1	$a_{n_1} \times b_{n_1}$	$a_{n_1} \times b_{n_1}$	d_1
1	2	$(\alpha_2 ; \beta_2)$	n_2	X_{12}	Y_{12}	X_{22}	Y_{22}	$(\alpha_2 ; 2)$ or	2 or	$(2 ; \beta_2$ or	2 or	$a_{n_2} \times b_{n_2}$	$A_{1c} + a_{n_2} \times b_{n_2}$	d_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	i	$(\alpha_i ; \beta_i)$	n_i	$X_2\alpha_i$ or $X_{2h'}$	$Y_2\beta_i$ or $Y_{2j'}$	$X_2\alpha_i + a_{ni}$ or $X_2b + b_{ni}$	$Y_2\beta_i + b_{ni}$ or $Y_{2j'} + b_{ni}$	$(\alpha_i ; i)$ or	$R(\alpha_i ; i)$ or	$(i ; \beta)_i$ or	$R(i ; \beta_i)$ or	$a_{ni} \times b_{ni}$	$A_{i-1,\ c} + + a_{ni} \times \times b_{ni}$	d_i
⋮	⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	n	$(\alpha_n ; \beta_n)$	n_n	X_{1n}	Y_{1n}	X_{2n}	Y_{2n}	$(\alpha_n ;\ n)$	$R(\alpha_n ; n)$	$(n ; \beta_n)$	$R(n ; \beta_n$	$a_{n_n} \times b_{n_n}$	$A_{n-1,c} + a_{n_n} \times b_{n_n}$	d_n