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Project Scheduling with Resource Allocation

The total duration of the project and the terms of beginning of all activities are determined by scheduling

I. THE PUTTING OF THE PROBLEM

Scheduling is a method with the aid of which the projects to be carried out are divided into working operations and stages with a technological interdependence between them; it establishes and controls the succession of the decisive phases for spacing out work, the necessary period of time and resources for the carrying out of the projects.

The carrying out of a project implies the consumption of several financial, labour, material and other resources which, generally, are not unlimited but are available in certain precisely defined quantities.

By scheduling with the allocation of the resources, we mean the establishment of a working programme for the project to be carried out with the available resources, within a limited period of time. The essential reason in this scheduling is to avoid the surpassing of the date established for the completion of the project, that is within a limited period of time (longer or equal to the length of the critical path), the activities will be scheduled between the minimum term of beginning and the maximum term of completion, so that the necessary resources should not surpass the available quantities. If an inevitable surpassing of the available quantity by the necessary resources appears, this should be minimum and most flattened.

a) *Feed-in Information*

The following information is necessary for a certain activity (a) of a project (P):

1. The nodes of recognition i and j, which are the extremes of the arc correspondingly pointed to the activity (a); the nodes can be numbered arbitrarily.

2. Duration $d(a)$ of the execution of the activity (a).

3. The activities preceding and succeeding the activity (a). This information represents the graph-network which intuitively renders the structure of the project.

4. The minimum term of beginning $P(a)$ and the maximum term of completion $Q(a)$ between which we have the relation

$$Q(a) - P(a) \geqslant d(a)$$

The activity (a) must be executed in a period of length $d(a)$ included in the period $[P(a), Q(a))$, of a length equal to $L(a) = Q(a) - P(a)$. The difference $L(a) - d(a)$ represents the total time reserve for the activity (a).

Observation 1

If (a) and (b) are two consecutive activities with the same time reserves, that is the end node of the activity (a) is the same with the start node of the activity (b), we have

$$[P(a), Q(a)) \cap [P(b), Q(b)) \neq \varnothing.$$

If (a) and (b) are critical activities, this intersection is void. The terms $P(a)$ and $Q(a)$ are obtained by computation of time analysis through the Critical Path Method.

5. The intensity of each resource (s) necessary for the execution of the activity (a) denoted with $r_s(a)$, that is the quantity of resource per time unit. For the execution of the activity (a) are necessary $g_s(a) = r_s(a) \cdot d(a)$ for the unit of the resource (s).

The following information is given for each resource:

6. An index of recognition (s), a code or a name.

7. The maximum number R_s of units of the resource (s) available for the execution of the project P, in each time unit. Consider R_s constant for the whole duration of the execution of the project P.

b) *Conditions to be fulfilled*

F_1. If (a) and (b) are two consecutive activities, that is the end node of the activity (a) is the same with the start node of the activity (b)

$$q(a) <p\ (b)$$

where $p(a)$ and $p(b)$ are the terms for the beginning of the activities (a) and (b) respectively, with $p(a) \in [P(a),\ Q(a) - d(a)]$, $p(b) \in [P(b),\ Q(b) - d(b)]$,

and $q(a)$ and $q(b)$ are the terms for the completion of the activities (a) and (b) respectively, with $q(a) \in [P(a) + d(a),\ Q(a)]$, $q(b) \in [P(b) + d(b),\ Q(b)]$. Consequently, the condition F_1 demands that the activities (a) and (b) should be scheduled in the periods $[p(a),\ q(a))$, $[p(b),\ q(b))$ disjoint between them.

$$F_2 \cdot p(a) \geqslant P(a) \text{ and } q(a) \leqslant Q(a)$$

where $d(a) = q(a) - p(a)$ and the activity (a) is scheduled to be carried out in this period of time $[(p(a),\ q(a))$.

The conditions F_1 and F_2 are imposed by the technology of the project.

F_3. If the duration of execution of the project P is given this must be greater or equal to the length of the critical path $T_c (T \geqslant T_c)$. Taking this condition into consideration, the terms $P(a)$ and $Q(a)$ are computed by the Critical Path Method.

Take

$$x_s(t) = \sum_{a \in U} r_s(a) \qquad s = 1, 2, ..., s_0,\ t = 0, 1, ...,\ T-1,$$

where $U = \{a \,|\, p(a) = t \text{ or } p(a) < t \text{ and } q(a) \geqslant t\}$, which means that between the moments t and $t + 1$, $x_s(t)$ units of the resource (s) are consummed for the execution of the activities for which $p(a) < t$ and $q(a) \geqslant t$ and which start at the moment $t(p(a) = t)$.

We note

$$y_s(t) = x_s(t) - R_s, \qquad s = 1,\ 2, ...,\ s_0,\ t = 0,1,\ ...,\ T-1.$$

If $y_s(t) > 0$ for the resource (s), between the moments t and $t + 1$, $y_s(t)$ units of the resource (s) are necessary in addition above the available R_s.

F_4. $$E = \sum_{s=1}^{s_0} \sum_{t=0}^{T-1} z_s(t) \to min$$

where

$$z_s(t) = \begin{cases} y_s(t), & \text{if } y_s(t) > 0; \\ 0, & \text{if } y_s(t) \leqslant 0. \end{cases} \tag{1}$$

The problem which is put is to find the terms $p(a)$ and $q(a)$ so that for each activity (a), in the conditions F_1, F_2, F_3 and a system of consumptions $x_s(t)$ $s = 1, 2, ..., s.$ $t = 0, 1 ..., T-1$, the expression E should be minimum in the sense of F_4.

If $E_{min} = 0$, we have a solution, consequently an acceptable planning of the terms for activities with the integration of the necessary resources into the available assets. In the event that there are several solutions for the same E_{min}, one condition F_5 will be introduced in § 3.

Observation 2

The following situations can be studied in keeping with the way of execution of the project P:

a) if $E_{min} \neq 0$, increasing the duration of execution T of the project P we may have a more convenient integration of the necessary resources into the available assets;

b) if $E_{min} = 0$, in the event that the duration of execution T of the project P is $T > T_c$, we may reduce this duration as long as E_{min} remains zero or we may reduce the available of all or part of the resources.

Only economic considerations can decide on the examination of these situations.

III. THE SOLVING METHOD

Suppose the activities of the project P sorted topologically in the technological order of the project and the activities with the same node of beginning in the growing order of the time reserve. We establish for each activity (a) an admissible interval $I(a) = [m_1(a)]\ m_2(a))$ so that if (a) and (b) are two consecutive activities, $m_1(b) = m_2(a)$.

Consider the activity (a) of the project P is scheduled to be execude in the interval $[l_1,\ l_2) \subset [m_1(a),\ m_2(a))$ and compute

$$y_s(t) := y_s(t) - r_s(a), \quad \text{for} \quad t \in [l_1,\ l_2). \tag{2}$$

Take

$$w_s(a) = \sum_{t=n_1}^{n_2-1} z_s(t)$$

where $z_s(t)$ are computed with the formula (1), $y_s(t)$ computed with the formula (2) and $[n_1\ n_2)$ is a certain interval included in $[m_1(a),\ m_2(a))$ for which $n_2 - n_1 = d(a)$.

The method is based on the analysis of each activity (a) taken separately, displacing it in the interval $[m_1(a),\ m_2(a))$.
For this reason, the criterium F_4 is replaced by F_4'.

$$E(a) = \sum_{s=1}^{s_0} w_s(a) \to min.$$

where $[n_1,\ n_2)$ is displaced to $[m_1(a),\ m_2(a))$.

Consequently, the problem in this case becomes: to find an interval $[n_1,\ n_2)$ for the activity (a) in the conditions F_1, F_2, F_3 so that $F(a)$ should be minimum in the sense of F_4'.

Observation 3

F_4' implies F_4 because $[n_1,\ n_2) \subset [0,\ T)$ and

$$E = E(a) + \sum_{s=1}^{s_0} \sum_{\substack{t \in [0,\ T) \\ t \in [n_1,\ n_2)}} z_s(t)$$

$E(a) \to min$ results that also $E \to min$.

Observation 4

As the minimum can be attained for several intervals $[n_1, n_2) \subset [m_1(a), m_2(a))$ which we note with $[n_{1i}, n_{2i})$ $i = 1, 2, ..., k$, a new requirement is introduced:

F_5. The activity (a) is scheduled in the interval $[n_1, n_2)$ for which

$$n_1 = \min_i n_{1i}$$

which means that the activity (a) is scheduled as soon as possible.

If it results that the activity is scheduled in the interval $[n_1, n_2)$ we must add for each resource to $y_s(t)$ the consumptions of the resource (s) for the activity (a).

$$y_s(t) := y_s(t) + r_s(a) \qquad \text{for } t \in [n_1, n_2).$$

The same method is applied for all the activities of the project P. If, covering all the activities of the project, no change has occured in the interval of execution of any of the activities, that is $n_1 = l_1$, $n_2 = l_2$, the solution to the problem was obtained for each activity; if not, this method is resumed for all the activities of the project. The solution to the problem consists in the determination of the terms n_1 and n_2 for each activity (a), which is nothing but the planning of the activities of the project P in the imposed interval $[0, T)$. In keeping with this planning of the activities, for each resource (s) we have the necessary computed with the relation

$$x_s(t) = y_s(t) + R_s, \qquad s = 1, 2, ..., s_0, \; t = 0, 1, ..., T - 1.$$

Taking into consideration the necessities concerning the utilization of the electronic computer for finding the interval $[n_1, n_2)$ of programming of the activities (a), the criterium of optimization F_4' was changed into another criterium F_4'', which is expounded below.

A certain interval $[n_1, n_2)$, where the activity (a) might be carried out is, as regards the interval $[l_1, l_2)$, in the following five cases:

a) $n_2 < l_1$

b) $l_1 < n_2 < l_2$

c) $n_1 = l_1$

d) $l_1 < n_1 < l_2$

e) $n_1 > l_2$

The only way of finding an interval of execution for the activity (a), so as to have the conditions F_4' and F_5 satisfied, is to move $[n_1, n_2)$ in $[m_1(a) m_2(a))$. For a most rational utilization of time on the computer, we shall not compute $y_s(t)$ with the aid of the formula (2), after which we move $[n_1, n_2)$ to $[m_1(a), m_2(a))$, but will do the following: consider $n_1 = m_1(a)$ and then we increase n_1 by another unit, until n_2 becomes equal to $m_2(a)$ and, each time we try to verify the criterium F_4''.

For $n_1 = m_1(a)$, we shall have one of the cases a), b), c); as the intervals $[n_1 n_2)$ and $[l_1, l_2)$ may have common points in which no changes occur, if we schedule the execution of the activity (a) in the interval $[n_1, n_2)$ and not in the interval $[l_1, l_2)$ we shall compute the size $H_1(a)$ and $H_2(s)$ $s = 1, 2 \ldots s_0$, for the case which appears as follows:

a) $$H_1(s) = \sum_{t=l_1}^{l_1} z_s(t)$$

$$H_2(s) = \sum_{t=n_1}^{n_1} z_s(t) \qquad s = 1, 2, \ldots, s_0$$

b) $$H_1(s) = \sum_{t=n_1+1}^{l_2} z_s(t)$$

$$H_2(s) = \sum_{t=n_1}^{l_1-1} z_s(t) \qquad s = 1, 2, \ldots, s_0$$

c) $$H_1(s) = H_2(s) = 0$$

where $z_s(t)$ are given by the formula (1).

The displacement to the right of the interval $[n_1, n_2)$ by one time unit in the interval $[m_1(a), m_2(a))$, that is $n_1 := n_1 + 1$, $n_2 := n_2 + 1$, implies the following change in the size $H_1(s)$ and $H_2(s)$, in one of the above five cases:

a) $H_2(s) := H_2(s) - z_s(n_1 - 1) + z_s(n_2)$

b) $H_1(s) := H_1(s) - z_s(n_2)$

$H_2(s) := H_2(s) - z_s(n_1 - 1)$

c) $H_1(s) := H_2(s) = 0$

d) $H_1(s) := H_1(s) + z_s(n_1 - 1)$

$H_2(s) := H_2(s) + z_s(n_2)$

e) $H_2(s) := H_2(s) - z_s(n_1 - 1) + z_s(n_2)$

for each $s = 1, 2, \ldots s_0$

Thus, for every moment, we have the size $H_1(s)$ and $H_2(s)$ defined for each $s = 1, 2, \ldots s_0$, size which gives the surpassing over the available of the resources in the planned periods $[l_1, l_2)$ and $[n_1, n_2)$. With this computed size, the requirement F_4' becomes:

F_4''. The activity (a) is scheduled to be executed in the interval $[n_1, n_2) \subset [m_1(a), m_2(a))$ for which the expression

$$E(n_1) = \sum_{\substack{r_s(a) \neq 0 \\ s \in [1, s_0]}} (H_1(s) - H_2(s)) \to max.$$

If $H_1(s) - H_2(s) < 0$, for the resource (s), then the activity (a) is scheduled in the interval $[l_1, l_2)$ and not in the interval $[n_1, n_2)$ because the surpas-

sing over the available is smaller; otherwise, for $H_1(s) - H_2(s) > 0$ the interval $[n_1, n_2)$ and not the interval $[l_1, l_2)$ is preferred, because the surpassing over the available is smaller. Having s_0 resources, we try to schedule the activity (a) in the interval $[n_1, n_2)$ for which $\sum_{r_s(a) \neq 0} (H_1(s) - H_2(s))$ should have a maximum positive value, that is to have the greatest possible diminishing of the surpassing above the available asset of each resource; consequently, the criteria F_4' and F_4'' are equivalent.

If we find an interval $[n_1, n_2)$ which meets the conditions F_4'' and F_5, the following computation is made:

$$y_s(t) := y_s(t) - r_s(a), \quad \text{if} \quad t \in [l_1, \ l_2);$$

$$y_s(t) := y_s(t) + r_s(a), \quad \text{if} \quad t \in [n_1, \ n_2).$$

IV. THE DISTRIBUTION OF TIME RESERVES ON ACTIVITIES

Consider that for each activity (a), a period of time is established, in which $[T_1(a), T_2(a))$ can be executed, where, for the beginning $T_1(a) = Q(a) - d(a)$, $T_2(a) = Q(a)$. If according to the computation method, it was found that the activity (a) should be scheduled to be executed in the interval $[n_1, n_2,)$ then $T_1(a) = n_1$, $T_2(a) = n_2$.

When the activity (a), with the extreme nodes i and j, is analysed, then

$$m_1(a) = \min_{b \in U_i} T_1(b)$$

where U_i is the multitude of all activities (b) with the same initial node i and

$$m_2(a) = \max_{c \in V_j} T_2(c)$$

where V_j is the multitude of the activities (c) which have the same fin al node j

Observation 5

Regardless of the activity (a)

$$P(a) \leqslant m_1(a) \leqslant m_2(a) \leqslant Q(a)$$

because $P(a)$ represents the minimum possible term of beginning for the event i of the project and $Q(a)$ the maximum possible term of completion for the event j of the project; any activity which has the node i as an initial node and the node j as a final node can be scheduled only between these terms. In this way, the condition F_2 is fulfilled because

$$P(a) \leqslant m_1(a) \leqslant p(a) \leqslant q(a) \leqslant m_2(a) \leqslant Q(a).$$

Suppose the activity (a) was scheduled in the interval $[p(a), q(a))$ with $q(a) \leqslant m_2(a)$. If the activity (b) has the initial node equal to the final node of the activity (a), then

$$m_1(b) = \min\left(\min_{d \in U_j} T_1(d), \max_{e \in V_j} T_2(e)\right)$$
$$m_2(b) = \max_{e \in V_k} T_2(c) \tag{3}$$

where i and j are the nodes of the activity (a) and j and k the nodes of the activity (b).

The interval $[m_1(b), m_2(b))$ established with the aid of the relations (3), responds also to the condition F_1, because

$$[T_1(a), T_2(a)) \cap [m_1(b), m_2(b)) = \varnothing$$
$$q(a) = T_2(a)$$
$$p(b) \in [m_1(b), m_2(b))$$

V. EXAMPLE

Suppose a project P with a structure represented by the graph-network of figure 1. Two resources LC and SP are used for the execution of the project. The resource LC has 14 units available and the resource SP has 6 units available. The duration and intensity of resources for the corresponding activity are written on each event.

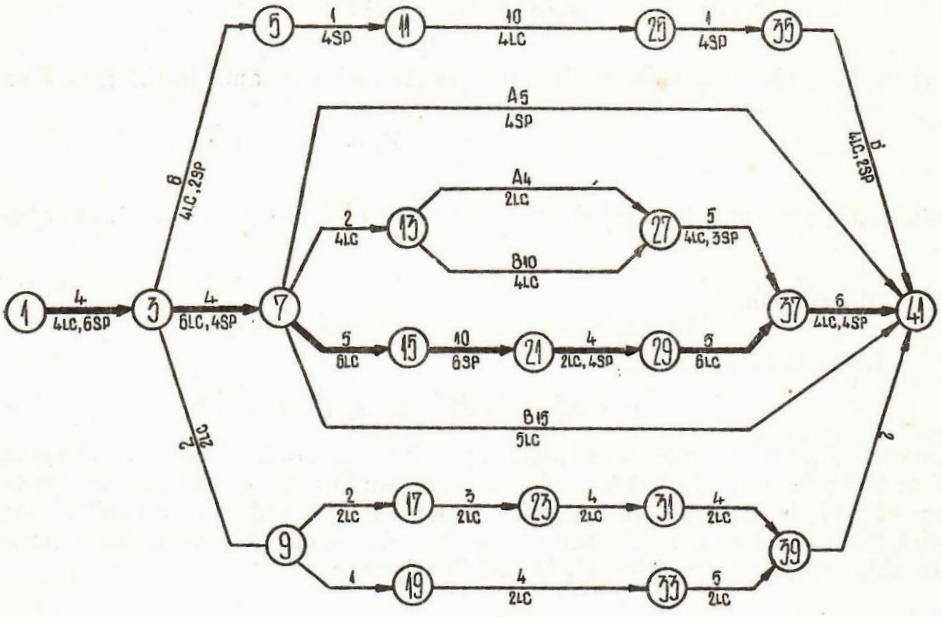

Fig. 1

Figures 2 and 3 represent the profiles (the values $x_s(t)$, where $t = 0, 1, ..., T$ and $s = 1, 2$) of the two resources, as follows: with an interrupted line — in programming the activities at the minimum term of beginning $P(a)$ — and with a continuous line — in programming the activities at the maximum term of beginning, found by using the above-mentioned method.

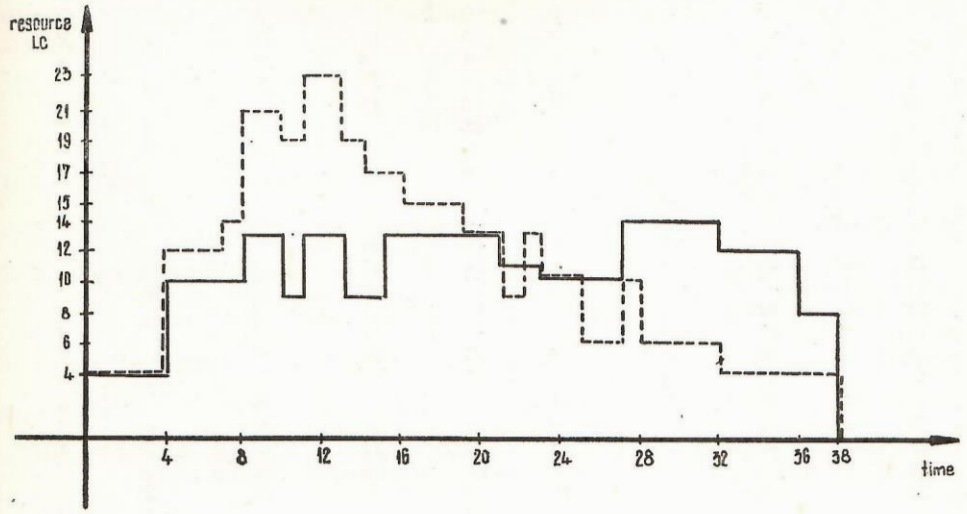

Fig. 2

It is noticed that the best solution does not surpass the available given through the straight lines $D1 = 14$, $D2 = 6$ only for the second resource in one time unit. In keeping with these best profiles, the programming of the best terms (see figure 4) which are framed by the minimum terms of beginning and the maximum terms of completion, was made.

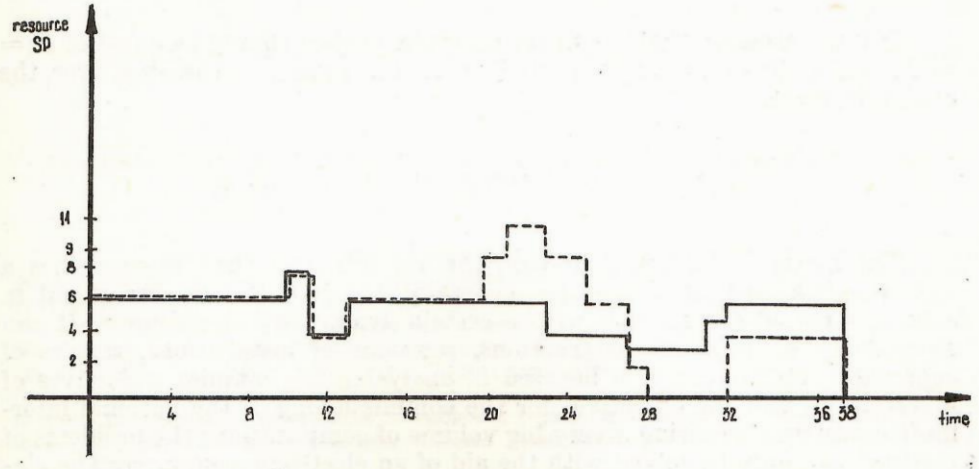

Fig. 3

Number	Preceding Event	Successive Event	Duration	Minim Start	Progr. Start	Progr. Final	Max. Final	Resources *LC*	Resources *SP*
1	1	3	4	0	0	4	4	4	6
2	3	5	6	4	4	10	20	4	2
3	3	7	4	4	4	8	8	6	4
4	3	9	2	4	21	23	23	2	
5	5	11	1	10	10	11	21		4
6	7	13	2	8	15	17	17	4	
7	7	15	5	8	8	13	13	4	
8	7	41A	5	8	8	13	38		4
9	7	41B	15	8	8	23	38	5	
10	9	17	2	6	23	25	25	2	
11	9	19	1	6	23	24	27		
12	11	25	10	11	11	21	31	4	
13	13	27A	4	10	23	27	27	2	
14	13	27B	10	10	17	27	27	4	
15	15	21	10	13	13	23	23		6
16	17	23	3	8	25	28	28	2	
17	19	33	4	7	27	31	31	2	
18	25	35	1	21	31	32	32		2
19	27	37	5	20	27	32	32	4	3
20	21	39	4	23	23	27	27	2	4
21	23	31	4	11	28	32	32	2	
22	33	39	5	11	31	36	36	2	
23	35	41	6	22	32	38	38	4	2
24	29	37	5	27	27	32	32	6	
25	31	39	4	15	32	36	36	2	
26	37	41	6	32	32	38	38	4	4
27	39	41	2	19	36	38	38		

Figure 4. List of the activities programmed at various terms

If it is necessary that the duration of the project should be equal to $T = T_c + 1 = 38 + 1 = 39$, we shall no longer have a surpassing over the available asset.

VI. CONCLUSIONS

The method of scheduling with the allocation of the resources has a wide practical utilization because, generally, the projects are carried out in a limited period of time and with a certain availability of resources. It can successfully be used in constructions, revisions of installations, repairs of aggregates, etc. It can also be used in analysing the complex objectives of investments, and the objectives for the commissioning of the national information system. Requiring a very big volume of computations the problems of this sort can only be solved with the aid of an electronic computer. The elaborated algorithm was checked by the author on the computer IBM 360/40 of

the Centre of Economic Computation of the Department of Economic Cibernetics of the Academy of Economic Sciences, for graph-networks which may have up to 2500 activities and up to 20 resources can be allocated. The programme has the possibility of allocating a certain number of resources and of uniting other resources to the terms resulting from the allocation. The utilization of the programme and the reports which can be supplied are described in [6].

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L'ORDONNANCEMENT DES PROJETS EN AFFECTANT LES RESSOURCES

Pour un projet représenté par un graphique-réseau on considère données: la durée et les ressources par unité de temps pour chaque activité et le disponible (la limite) pour chaque ressource. Dans l'article on définit et on donne la résolution pour le suivant problème: établir un plan d'exécution du projet pour que dans un temps limité (plus grand ou égal à la longueur du chemin critique) les activités soient programmées entre le terme minime de commencement et le terme maximum final, de telle manière que le nécessaire de ressources ne dépasse pas la quantité disponible; dans le cas ou il existe un dépassement de la quantité disponible par le nécessaire de ressources, ces dépassements, s'ils ne peuvent pas être évités, doivent être minimes et uniformisés.

L'algorithme a été programmé pour l'ordinateur IBM 360/40 au Centre de Calcul appartenant à la Section de Cybernétique économique de l'Académie des Etudes Economiques.

О СОСТАВЛЕНИИ ДОКУМЕНТОВ НА ОПЛАТУ ВЫПОЛНЕНИЯ ПРОЕКТОВ С ВЫДЕЛЕНИЕМ РЕСУРСОВ

Для проекта, представленного сетевым графиком, считаются заданными продолжительность и ресурсы на единицу времени по каждому виду деятельности, а не располагаемое количество (лимит) по каждому ресурсу. В статье получает свое определение и разрешение следующая задача: разработать план выполнения проекта таким образом, чтобы в ограниченный промежуток времени (больший или равный длине критического пути) различные виды деятельности были запрограммированы между наименьшим сроком начала и наибольшим сроком окончания с тем, чтобы потребность в ресурсах не превышала располагаемого количества; в случае, если потребность в ресурсах превышает располагаемое количество, это превышение — если оно является неизбежным — должно быть минимальным и выравненным.

Алгоритм был запрограммирован для калькулятора ИБМ 360/40 в центре расчета при кафедре экономической кибернетики Академии экономических наук.