Prof. Dr. L. DOBRESCU

Some Mathematical Models to Feature the Technological Processes for Mineral Dressing

"One can prove anything, but the actual difficulty lies in the knowledge of what one means to prove".

Emile Chartier Alain

The technological processes in view of mineral dressing, play an important part in turning ores to good account. The possibilities of subsequent working (smelting, coking, manufacturing of ceramic materials, etc.) and the costs of finished products (metals, coke, etc.) depend to a great extent on the efficacious development of these processes.

Because of their importance, the dressing processes have been investigated in numerous respects with the purpose of finding the laws to which they are subject [1], [2], [3], [4].

The aim of this article is to ascertain some general mathematical models able to represent correctly the development of the basic processes employed for mineral dressing.

It is well-known that the main dressing processes are:

— crushing and grinding;

— classification;

— beneficiation (flotation, gravitational/magnetic concentration, dewatering, etc.)

Notwithstanding the essential differences existing between these processes, they meet certain characteristics enabling the operation of the same laws and the working out of certain mathematical models in order to picture the way of their development.

The main characteristic of the dressing processes is the statistical composition of the system. Irrespectively of the process under survey, in its development participates a system made up of an enormous number of mineral grains which pass in time from a state into another (from run-of-mine ore into the product). The reasons which cause the passing from a state into another are numerous and, in most cases, only partially known. Therefore, the sole right way for finding mathematical relationships in order to picture

truthfully such processes lies in the application of the statistical laws and of the calculus of probability.

In its most general and simplified form, a dressing process can be pictured as follows: a lot N of grains is characterized by a well-defined property, which underlies the process development (for instance, friability for crushing, size for classification, flotability for flotation). By means of such a property we can tell if a grain belongs to the respective lot or not. Such a lot is finite but not steady, since some other grains may go into it (through continous feeding) and, on the other hand, some grains may go out of it by their driving out of the process under the form of products (through normal dressing phenomena as a result of accidental occurrences). The grains pertaining to lot N are liable to assume different incompatible states n, each grain assuming at one time t solely one of the states, which is marked with indexes 1, 2, ..., n.

Consider N_1, N_2, ..., N_n to be the n groups into which the grains of lot N are divided at one time t. One of these groups is associated to the phenomenon of driving out of the system, or say, it is the product intended.

The fundamental functions of the process are:

1. The probability for a grain to appear in lot N or in one of the groups N_i, coming from without at one time or during a certain time interval.

2. The probability for a grain to go out of lot N at one time t, or during a time interval (or say, to integrate the product).

3. The probability for a grain to be found at one time (or during a time interval) in a group N_1.

The stages prior to our survey are not of interest in processes of this type. The probability for the process to assume at one time t_2 a certain state depends only on the state it had assumed before (at time t_1) and is independent of the stages assumed up to time t_1. The probabilities for passing from one state into another being constant, the chain is homogeneous.

It often occurs that certain properties of the grains may alter during the dressing process. The time development of the process depends in this instance both on the initial and the intermediate states, under which the initial characteristics of the grains making up the number had been altered. The statistical phenomenon of the dressing process considered in successive stages observe the laws of a full-connection chain. The ascertainment of the probable development of the dressing processes of this type may be included in the general problem for statistics carried out in respect of the variable characteristics.

The classification and concentration (densimetric, magnetic, electrostatic, flotation, a.s.o.) processes in which one or more products are obtained, without any re-cycling, can be pictured by two types of mathematical models, depending on the conditions to be considered in respect of the driving out of finished products:

I) The resulting products are driven out of the system as the effect of one reason;

II) The resulting products are removed as a result of two reasons: first, as the effect of the dressing phenomena involved by the characteristic properties (like in case I), and the second, as the effect of some addi-

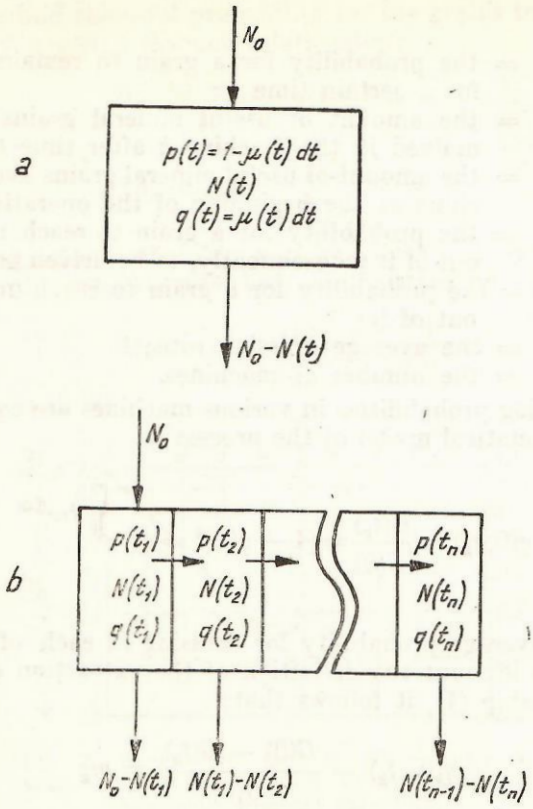


Fig. 1

tional phenomena (mechanical, physical, physical-chemical, operational failures of apparatus) with permanent action.

THE MATHEMATICAL MODEL FOR PROCESSES OF TYPE I

The dressing processes of this category can be pictured by simple circuits of the type shown in Fig. 1.

No matter with the feeding manner (continuous or discontinuous), the ore is being processed for a certain time in the machine, resulting in one or two products that are driven out. Mathematically, this process can be assimilated with a statistical Markov process on which a single elimination reason works.

For actual dressing processes, in which the processed ores are made up by groups of grains of certain sizes, beneficiation capacities, etc., the mathematical model can be expressed by relationship [3]:

$$p(0,\ t_k) = \frac{G(t_k)}{G(0)} = \prod_{i=1}^{i=k} [1 - q(t_{i-1},\ t_i)] = 1 - q(0,\ t_k) = e^{-\int_0^t \mu_m(t)\,dt} \qquad (1)$$

where:

$p(0, t_k)$ = the probability for a grain to remain in the machine for a certain time t_k;

$G(t_k)$ = the amount of useful mineral grains which have remained in the machine k after time t_k;

$G(0)$ = the amount of useful mineral grains available in the machine at the beginning of the operation;

$q(0, t_k)$ = the probability for a grain to reach machine k and go out of it (consequently, to be driven out of machine k);

$q(t_{i-1}, t_i)$ = the probability for a grain to reach machine i and go out of it;

$\mu_m(t)$ = the average dressing rate;

$i = 1 \to k$ = the number of machines.

If the dressing probabilities in various machines are equal among themselves, the mathematical model of the process is:

$$1 - q(0, t_k) = \frac{G(t_k)}{G(0)} = [1 - q(t)]^k = e^{-\int_0^t \mu_m(t)\,dt} \tag{2}$$

where:

$q(t)$ = the average probability for dressing in each of the k machines.

Taking into account the definition of the extraction of useful mineral (m_k) and relationship (1), it follows that:

$$q(0, t_k) = \frac{G(0) - G(t_k)}{G(0)} = m_k \tag{3}$$

hence, the mathematical model of actual processes developing in agreement with a simple Markov chain (processes of type I) is:

$$m_k = 1 - e^{-\int_0^t \mu_m(t)\,dt} \tag{4}$$

To apply this model practically, we must know the value of the average dressing rate $\mu_m(t)$. In [3] we have described the way to follow in order to find the value of the average dressing rate, obtaining relationships:

$$q(0, t_k) = 1 - exp\,[A(a^t - 1) + Bt] = m_k. \tag{5}$$

The average probability for the entire amount of ore to be dressed can be applied without restraint both for mono- and poly-disperse ores.

In case the dressing probabilities in various machines are quite distinct, instead of an average probability it should be advisable to ascertain the dressing probabilities for each machine during the time interval it is operating (for instance, t_i and t_j), with the aid of relationship:

$$q_{t_j} q_{t_i} = 1n G(t_j) - 1n G(t_1) = 1 - exp\,[A a^{(t_j - t_i)} + B(t_j + t_2)] \tag{6}$$

and subsequently find the total probability for the grains to go out in the instance of k machines $q(t_k)$ through relationship:

$$q(0, t_k) = {}_0q_1 \cdot {}_1q_2 \cdots {}_{k-1}q_k \tag{7}$$

or

$$1n\, q(0, t_k) = \sum_{i=0,\ j=1}^{k-1,\ k} 1n\ {}_jq_i \tag{8}$$

The dressing total probability can be plotted easily; the values of k are noted on the abscissa, whereas the values corresponding to function:

$$A(a^t - 1) + Bt = 1n \frac{G(t)}{G(0)} \tag{9}$$

are noted on the ordinate.

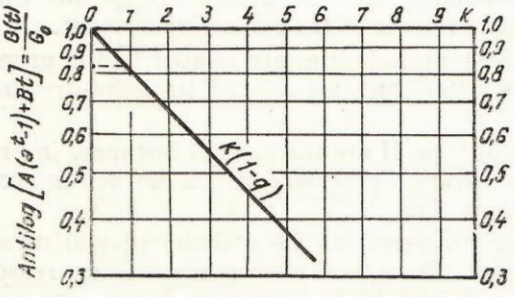


Fig. 2. Variation of function K[1 — q(t)] as against K and log $G(t)$ $G(0)$ in semilogarithmic co-ordinates (decimal logarithms)

If the ordinate is divided logarithmically, function $[1 - q(t)]^k$ has a linear variation depending on $\frac{G(T)}{G(0)}$ (Fig. 2), because probability $q(t)$ is constant on the whole interval.

THE MATHEMATICAL MODEL FOR PROCESSES OF TYPE II

The processes of type II give rise to a more complex diagram. In such an instance, three statistical occurrences incompatible between themselves may interfere, that is to say grains may be found in one of these positions:

R = the grain remains in the dressing machine;

C = the grain is turned into a product thanks to its characteristic properties and the separation phenomena;

E = the grains are driven out owing to arbitrary reasons working constantly.

On noting $P_{\alpha\beta}$ the probability for a grain possessing a certain property x, found in state α, to be in state β after a time t has elapsed, we may have to do with the following probabilities;

$P_{RP} = p_x =$ the probability for the grain available in the machine to be there also after a time t;

$P_{RC} = q_x =$ the probability for the grain to be driven out of the machine due to normal reasons;

$P_{RE} = s_x =$ the probability for the grain to be driven out owing to arbitrary reasons;

$P_{CA} = 0 =$ the probability for the grain turned into product due to normal reasons to come back to the machine;

$P_{CC} = 1 =$ the probability for the grains driven out owing to normal reasons to be found in the respective product;

$P_{EA} = \gamma_x =$ the probability for the grains driven out due to arbitrary reasons to be brought back into the circuit (in case of a re-processing of intermediate products);

$P_{EC} = c_x =$ the probability for the grains driven out owing to arbitrary reasons to be turned into the finished product;

$P_{EE} = t_x =$ the probability for the grains driven out owing to arbitrary reasons to remain in the respective product.

The former three probabilities are related to the primary dressing processes, whereas the latter three are related to the re-dressing of intermediate products.

The processes of type II are the current instances in practice for the classification and concentration processes, which develop in accordance with diagrams like the one in Fig. 3.

These dressing processes can be assimilated statistically to a variable phenomenon F under the action of two elimination reasons, incompatible between themselves, A and B. Phenomenon F concerns a number of grains fed continuously. Because of the double action, the probabilities for a grain to be driven out of the machine after a time t are of two kinds. In order to make them obvious, we shall first consider reason A. The probabilitiy q_x for a grain to be driven out by reason A, when both reasons A and B are working, is termed relative probability of reason A, and proabability q_x for the grain to be driven out if only reason A is working is termed absolute probability of reason A. The action of reason B is left out of the calculus, admitting that a grain driven out by reason B is replaced by another.

The probabilities of reason B(s_x and s'_x) are termed in a similar manner.

Each driving out probability is characterized by an occurrence intensity of the respective phenomenon ($f(t) = f'(t)$ and $g(t) = g'(t)$).

If the amount of grains initially available in the machine is noted N_0, and the average number of grains available in the machine after a certain time t has elapsed since the beginning of the process, when both driving out reasons are working, is noted $N(t)$, we can write the following relationship:

$$N(t) = N_0 \, exp\left[-\int_0^t f(t)\, dt \right] exp\left[-\int_0^t g(t)\, dt \right] \qquad (10)$$

where $f(t)$ and $g(t)$ are the grains driving intensities due to the two reasons

If we note $N_A(t)$, and respectively $N_B(t)$, the amount of grains in the lot, which remain in the machine after time t, when only reason A (and, respect-

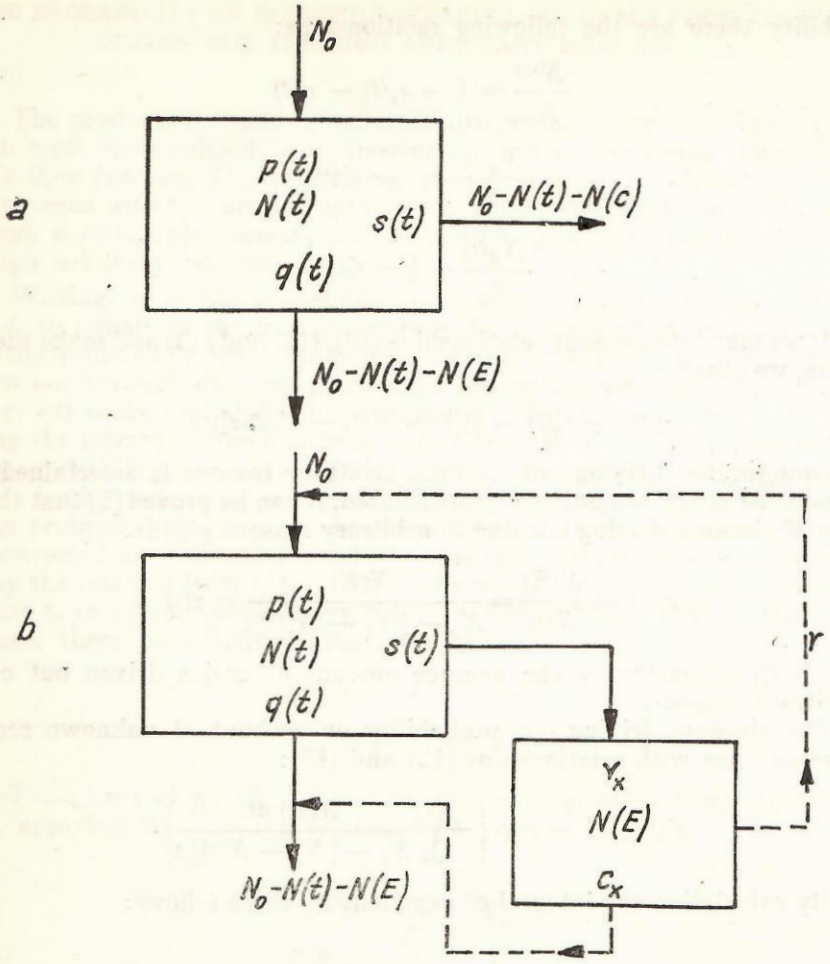


Fig. 3

ively, only reason *B*) is working, the following relationships can be ascertained:

$$N_A(t) = N_0\, e^{-\int_0^t f(t)\,dt} \tag{11}$$

and

$$N_B(t) = N_0\, e^{-\int_0^t g(t)\,dt} \tag{12}$$

Applying the previously used symbols, it is obvious that between the amount of grains which have remained in the machine and the driving out

probability there are the following relationships:

$$\frac{N(t)}{N_0} = 1 - q_x(t) - s_x(t) \tag{13}$$

$$\frac{N_A(t)}{N_0} = 1 - q'_x(t) \tag{14}$$

$$\frac{N_B(t)}{N_0} = 1 - s'_x(t) \tag{15}$$

If we take into account relationships (10), (11) and (12) and make the calculation, we obtain:

$$q_x(t) + s_x(t) = q'_x(t) + s'_x(t) - q'_x(t)\, s'_x(t) \tag{16}$$

Assuming the driving out through arbitrary reasons is ascertained and the newly fed grains are uniformly distributed, it can be proved [5] that the intensity of absolute driving out due to arbitrary reasons $g'(t)$ is:

$$g'(t) = \frac{N(E)}{N(t)} = \frac{N(E)}{N_0 - [N_0 - N(t)]\, t} = g(t) \tag{17}$$

where $N(E)$ = constant = the average amount of grains driven out owing to arbitrary reasons.

The absolute driving out probability on account of unknown reasons is in accordance with relationships (12) and (15):

$$s'(t) = 1 - exp\left\{-\int_0^t \frac{N(E)\, dt}{N_0 - [N_0 - N(t)]\, t}\right\}. \tag{18}$$

By calculating the integral of exponent E, there follows:

$$s'(t) = 1 - \left[\frac{N(t)}{N_0}\right]^{\frac{N(E)}{N_0 - N(t)}} \tag{19}$$

The relative driving out probability on account of arbitrary reasons is deduced from the absolute probability [5] and is equal to:

$$s(t) = \left[1 - \frac{q'(t)}{2}\right] s'(t) \tag{20}$$

where $q'(t)$ is the absolute probability for a grain to be driven out of the machine through normal reasons (due to dressing phenomena).

The relative and absolute probabilities of the driving out process through normal reasons, due to dressing phenomena, are found in agreement with the methods described in connection with type I processes, taking into account also relationships (10), (11) and (12).

THE PROBABILITY OF RECOVERING USEFUL MINERALS FROM PRODUCTS DRIVEN OUT THROUGH ARBITRARY REASONS

The products containing materials driven out owing to arbitrary reasons are in most cases subject to re-dressing in view of recovering the useful minerals they contain. The re-dressing process can be considered a stochastic phenomenon with two driving out reasons: a. conversion into finished products through dressing phenomena, and b. re-cycling of the material driven out through arbitrary reasons.

Writing: p_t^f = the probability for a grain available in the machine at time t_0 to remain in the machine during the interval extending from t_0 to t: q_t^f = the probability for a grain available at time t_0 in the machine to be driven out through dressing phenomena during the interval extending from t_0 to t; $s(t)$ = the probability for a grain available at time t_0 to be driven out during the interval extending from t_0 to t because of arbitrary reasons; p_t^a = = the probability for a grain driven out by arbitrary reasons at time t_0 to remain in the re-dressing machine during the time interval from t_0 to t; q_t^a = = the probability for a grain driven out by arbitrary reasons at time t_0 to be converted into finished products through normal dressing phenomena during the interval from t_0 to t; $r(t)$ = the probability for a grain driven out at time t_0 to be re-cycled during the time interval from t_0 to t; there exist, between these probabilities, the equalities:

$$p_t^f + q_t^f + s(t) = 1 \tag{21}$$

$$p_t^a + q_t^a + r(t) = 1 \tag{22}$$

The values of q_t^f, q_t^a, $s(t)$ and $r(t)$ can be ascertained from experimental data, applying the follwing relationships;

$$q_t^f = \frac{N_D}{N_A + \frac{N_G}{2}} \tag{23}$$

$$q_t^a = \frac{N_B}{N_B - \frac{N_R}{2}} \tag{24}$$

$$s(t) = \frac{N_I}{N_A + \frac{N_G}{2}} \tag{25}$$

where:

$$r(t) = \frac{N_R}{N_B - \frac{N_R}{2}} \tag{26}$$

N_A = the amount of grains to be found in the machine for primary dressing at moment t_0;

N_B = the amount of grains to be found in the re-dressing machine at moment t_0;

N_G = the amount of grains entering the machine for primary dressing during time t:

N_I = the amount of grains driven out during time t because of arbitrary reasons;

N_D = the amount of grains driven out during time t through dressing phenomena;

N_R = the amount of grains brought back for re-cycling (from those available in the re-cycling machine);

N_S = the amount of grains converted into finished product (from those available in the re-dressing machine).

FINDING THE MATHEMATICAL MODELS FOR DRESSING PROCESSES DERIVED FROM THE LAWS GOVERNING A FULL-CONNECTIONS CHAIN

This category includes the dressing processes derived from complex diagrams including the re-dressing and re-cycling of intermediate products. Likewise, the crushing and grinding processes can be described by means of the laws governing the full-connections chaining.

A dressing diagram of this type is given in Fig. 4.

A dressing process resulting in accordance with the laws governing a full-connections simple chain can be described as follows: in the dressing machine there is a lot of grains N, which may assume one of the states $A_i (i = 1, 2, 3, ..., n)$. Considering a grain in state A_i at time t_j, its distribution density for passing from state A_i into state A_j after a period dt has elapsed is $p_{ij}(t_j, t)$. There arises the problem of finding the amount of grains of each state at a moment t, or, as defined by Galbrun [6] and Taucer [7], the ascertainment of the composition of the groups depending on time.

The probability of passing from a group to another (p_{ij}) depends on the time spent by the grain before entering the respective group (t_j) and on the time of its remaining in the group, (t). Unlike the Markov type processes, in full-connections processes each group is characterized by a variable property developing in time, and the ascertainment of the probable evolution of grains distribution in each is a general problem for the statistics drawn up in relation to the variable characteristics.

The simplest case met with in this dressing processes group is the one where a finite number of states s is considered and one sole previous state is taken into account when ascertaining the composition of each group. Let us assume that at one moment t, in a dressing machine there is a lot of grains which can be classified on the basis of some properties (for instance, grain size, flotability) found in n partial groups $A_i (i = 1, 2, ..., n)$.

Noting $f_{ij}(t, \tau)\, dt$ the probability for a grain which had entered group A_i at moment τ to pass from there to group A_j after a time extending from t to $t + dt$ has elapsed, then the probability for the same grain to pass to all

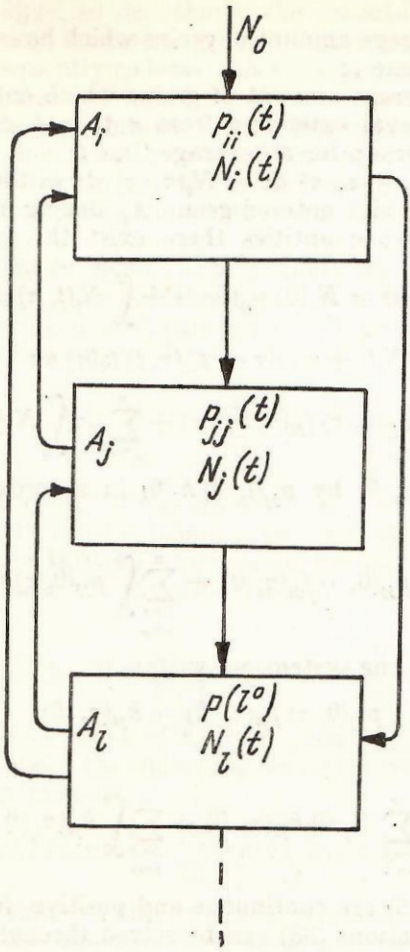


Fig. 4

the groups and get out of the lot after a time extending along the interval t to $t + dt$ is:

$$F_i(t, \tau)\, dt = \sum_{j=i}^{n} f_{ij}(t, \tau)\, dt. \tag{27}$$

It follows from this relationship that the probability for a grain initially present in group N_i at time τ to remain continuously in the same group up to time t is:

$$p_{ij}(\tau, t) = e^{-\int_{\tau}^{t} F_i(t, \tau)\, dt} \tag{28}$$

By means of this probability we can find the compositions of the various groups N_i of the system, by making the following calculations:

We write:

$N_i(t)$ = the average amount of grains which have remained in group A_i for a time t;

$N_i(t, \tau)d\tau$ = the average amount of grains which entered group A_i during an interval extending from τ to $\tau + d\tau$ and have remained in the group for an average time t;

$h_i(\tau)d\tau$ = $\lim N_i(\tau + \varepsilon, \tau)\, d\tau = N_i(\tau, \tau)\, d\tau$ = the average amount of grains which entered group A_i during interval τ to $\tau + d\tau$.

Between these two quantities there exist the following relationships:

$$N_i(t) = N_i(0)\, p_{ii}(0, t) + \int_0^t N_i(t, \tau)\, d\tau \tag{29}$$

$$N_i(t + \tau)\, d\tau = p_{ii}(\tau, t)\, h_i(\tau)\, d\tau \tag{30}$$

$$h_i(\tau)\, d\tau = \sum_{j=1}^{n} N_j(0)\, p_{jj}(0, \tau)\, f_{ji}(\tau, 0)\, d\tau) + \sum_{j=1}^{n} d\tau \int_0^\tau N_j(\tau, \theta)\, f_{ij}(\tau, \theta)\, d\theta \tag{31}$$

Substituting $N_j(t, \theta)$ by $p_{jj}(t, \theta)\, h_j(\theta)$ in accordance with relationship (30), there follows:

$$h_i(\tau) = \sum_{\substack{j=1 \\ i=1}}^{n} N_j(0)\, p_{jj}(0, t)\, f_{ji}(\tau, 0) + \sum_{\substack{j=1 \\ i=1}}^{n} \int_0^t p_{jj}(\theta, \tau)\, f_{ij}(\tau, \theta)\, h_i(\theta)\, d\theta. \tag{32}$$

In order to solve the system, we write:

$$p_{jj}(\theta, \tau)\, f_{ji}(\tau, \theta) = k_{ji}(\tau, \theta)$$

obtaining equation:

$$h_i(\tau) = \sum_{\substack{j=1 \\ i=1}}^{n} N_j(0)\, k_{ji}(\tau, 0) + \sum_{\substack{j=1 \\ i=1}}^{n} \int_0^\tau k_{ji}(\tau, 0)\, h_i(\theta)\, d\theta \tag{33}$$

The terms $k_{ji}(\tau, \theta)$ are continuous and positive functions, so that the system of integral equations (33) can be solved through iteration.

For the successive terms k_{ji}, we can write the relationships:

$$k_{ji}^{(1)}(t, \theta) = k_{ji}(t, \theta)$$

$$k_{ji}^{(\tau)}(t, \theta) = \sum_{l=1}^{n} \int_0^t k_{jl}^{(s)}(\theta, \sigma)\, k_{li}^{(\tau - s)}(\theta, \sigma)\, d\sigma.$$

The terms of the iteration stand for various probabilites:

$$k_{ji}^{(1)}(t, \theta)\, dt = p_{jj}(\theta, t)\, f_{ji}(t, \theta)\, dt$$

and show the probability for a grain which entered group A_j at time θ to enter group A_j at a time comprised between τ and $\tau = d\tau$.

In the following factor:

$$k_{ji}^{(2)}(\tau, \theta)\, d\tau = \sum_{l=1}^{n} d\tau \int_0^t k_{jl}^{(1)}(\sigma, \theta)\, k_{li}^{(1)}(\tau, \sigma)\, d\sigma$$

expression $k_{jl}(\sigma, \theta)\ k_{li}^{(1)}(\tau, \sigma)\ d\sigma$ shows the probability for a grain which entered group A_j at time θ to enter group A_1 at a time comprised between σ and $\sigma + d\sigma$ and subsequently to leave this state and enter group A_i at a time comprised between τ and $\tau + \tau d$.

By totalling the probabilities for all the values comprised between θ and τ, and for the values of N between 1 and n, it follows that $k_{ji}^{2}(\tau, \theta)\, d\tau$ denotes the probabilty for a grain which entered group A_j at time θ to enter group A_i at a time comprised between τ and $\tau + d\tau$, after having passed through another group.

The term $k_{ji}^{(\nu)}(\tau, \theta)\, d\tau$ shows the probability for a grain which was available at time θ in group A_i to enter group A_j at a time comprised between τ and $\nu + d\tau$, after having passed through $n - 1$ groups. If all the terms of the iteration are totalled up, we have:

$$S_{ji}(\tau, \theta)\, d\tau = \sum_{\substack{r=1 \\ i=1}}^{\infty} k_{ji}^{(\nu)}(\tau, \theta)\ d\tau$$

which shows the probability for a grain which entered group A_i at time θ to enter group A_j at a time comprised between τ and $\tau + d\tau$, irrespectively of the number of groups through which it has passed in the mean time.

In accordance with the iteration theory, it follows that system (33) is equal to:

$$h_i(\tau) = \sum_{j=1}^{n} N_j(0)[h_{ji}^{(1)}(\tau,\ 0)] + \sum_{j=1}^{n} \int_0^{\tau} s_{ji}(\tau,\ \theta) \times \sum_{h=1}^{n} N_h(0)\ k_{hj}^{(1)}(\theta,\ 0)\ d\theta \qquad (34)$$

If we introduce this value of $h_i(\tau)$ into relation (29) and perform some simplifications, we obtain the following equation yielding the amount of grains in group A_i at moment t:

$$N_i(t) = N_i(0)\, p_{ii}(0,\ t) + \sum_{j=1}^{n} N_j(0) \int_0^{\tau} p_{ij}(\tau, t)\, S_{ji}(\tau,\ 0)\ d\tau \qquad (35)$$

A more general example than the one just presented is offfered by the dressing processes where the transition to a new state depends on several previous states. To make things simpler, we shall examine an example where only two previous states are considered [8].

Consider a lot of N grains available in the dressing machine at one moment t. Assume we have to ascertain the amount of grains $N_j(t)$ available in a group A_j at time t, after having remained in another previous group for a time extending from t_1 to t_2.

The probability for a grain entering group $A_i (i = 1, 2, 3, ..., n)$ at time t_1 and group $A_j (j \neq i)$ at time t_2, to pass directly to group $A_k (k \neq j)$ at a time extending from t to $t + dt$ is written $p_{ijk}\ (t_1 t_2,\ t)\ dt$, and the probability for the same grain to remain in group A_j continuously for a time equal to the interval from t_2 to t is written $P_{ij}(t_1,\ t_2,\ t)$.

Between these two probabilities we find relationship:

$$P_{ij,j}(t_1,\ t_2;\ t) = e^{-\int_{t_2}^{t} \sum_{k=1}^{n} p_{ijk}(t_1, t_2;\, t)\, dt} \qquad (36)$$

The amount of grains available at time t in group A_j is:

$$N_j(t) = N_j(t_0)\, P_{jj,j}(t_0, t_0; t) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t} \sum_{i=1}^{n} P_{ij,j}(t_1, t_2; t)\, \varphi_{ij}(t_1, t_2)\, dt_1 \quad (37)$$

where:

$N_j(t)\, P_{jj,j}(t_0, t_0; t)$ = the amount of grains available in group A_j at moment t_0 and not driven out of the group before time t;

$P_{ij,j}(t_1, t_2, t)\, \varphi_{ij}(t_1, t_2)\, dt_1 dt_2$ = the amount of grains which entered group A_i during the interval extending from t_1 and $t_1 + dt_1$, and for the last time in A_j during the time interval extending from t_2 and $t_2 + dt_2$;

$\varphi_{ij}(t_1, t_2) dt_1\, dt_2$ = the amount of grains which entered group A_i during the time interval extending from t_1 and dt_1, and went out at the time extending from t_2 and dt_2, passing to group A_j.

The solution of equation (37) requires intricate calculations, the value of functions $\varphi(t_1, t_2)$ being found by solving a system of transgressive linear integral equations with variable limits [5], [8].

Equation (37) describes the most complete model for dressing operations. From this operation we can pass directly to any of the models described before.

If relationship (37) is compared to (35), it becomes obvious that they are similar, differing in the value of the probabilities of continuously remaining in the group considered (A_i, respectively A_j), and in the amount of grains entering the group during the interval surveyed.

The mathematical model for the Markov type (I and II) processes can be inferred from the general relationship (37) if the driving of grains out of the machine (that is, their dressing) is considered a transition from one group to another, and no previously dressed grains enter the initial group. The two types of models (I and II) are different from each other by the fact that, whereas the dressing intensity in the former is $\mu(t)$, the product of the driving out intensities due to normal and arbitrary reasons is $[g(t) \cdot f(t)]$.

PRACTICAL APPLICATIONS OF THE GENERAL MATHEMATICAL MODELS FOR THE DRESSING PROCESSES

Dressing processes carried out in accordance with the Markov chains laws — Type I.

The basic equations of the mathematical model used for processes of this type are:

$$1 - \frac{G(t)}{G_0} = q(t) = 1 - e^{-\int_0^t \mu(t)\, dt} \quad (38)$$

$$1 - \frac{G(t)}{G_0} = 1 - exp\, \frac{A(a^t - 1) + Bt}{k} \quad (39)$$

where the notations have the meanings pointed out before.

Applications to flotation

Research [3] has evidenced that in flotation, equation (39) yields results very close to those obtained experimentally with cells operating in a discontinuous manner. The differences between the experimental data and the computed ones do not go over 0.5 — 2 per cent.

The relationships ascertained have direct application in flotation. The more important aspects are: estimation of the flotation selectivity; ascertainment of the effect brought about by various changes in the flotation conditions; ascertainment of the effect brought about by an irregular operation of machines, or by using different flotation machines; scientific ascertainment of a correlation between the laboratory results and the results achieved in practice.

Estimation of the flotation selectivity.

The flotation of a certain mineral is perfectly characterized by its flotation probability; it follows that, for flotating two different minerals, X and Y, selectivity can be estimated through the ratio:

$$S = \frac{q_x}{q_y} \tag{40}$$

where q_x and q_y are the flotation probabilities of the two minerals (along the entire flotation interval, or along certain intervals if required).

In some instances, it is more convenient to estimate flotation selectivity through the ratio:

$$S_1 = \frac{1 - q_x}{1 - q_y} \tag{41}$$

The values of the flotation probabilities can be ascertained analytically using one of the formulas (6), (7) and (8), or graphically, following the method described above (Fig. 2).

For an analytical ascertainment of the flotation selectivity it is necesary to carry out systematical tests for finding all the elements required for calculation. The method involves in this respect certain difficulties, but the results are exact.

For technological determinations of minor precision the graphical method can be used [3].

If the flotation probabilities for minerals vary during the process, this shall be reckoned with in plotting [3].

Finding the effect brought about by the alteration of flotation conditions.

If the various flotation conditions are altered, the equation of the flotation probability and, consequently, the curves of the flotation probability will be altered. In order to find out the effect brought about by the alteration of the flotation conditions, the curves of the flotation probability according to the described proceeding are found first, and subsequently the selectivity index for each of the working conditions is calculated:

$$S'_1 = \frac{1 - q_{x_1}}{1 - q_{y_1}}, \quad S'_2 = \frac{1 - q_x}{1 - q_y}, \quad S'_3 = \frac{1 - q_x}{1 - q_y} \tag{42}$$

By comparing the selectivity indexes we can define the optimal flotation conditions.

Finding the effect brought about by an irregular operation of the flotation machines or by the use of machines out of order.

The flotation probability of a mineral depends on the nature of its surface (superficial tension of mineral and of the aqueous solution, the amount and size of the grains, etc.) and on the manner in which stirring and ventilation are performed, hence on the way the machine is working. It can then be assumed that the flotation probability of a mineral X is the product of two independent factors:

$$q_x = q_m \cdot q_\alpha \tag{43}$$

where q_m is a factor depending on the construction and operation of the machine, and q_α is a factor depending on the nature of the mineral and on the physical-chemical working conditions.

When the flotation machine is operating in a faulty way, or different machines are in use, distinct values for q_m will be obtained, since factor q_x is variable.

Finding a correlation between the laboratory results and the industrial results.

When an ore made up of two minerals, X and Y, is flotated in the laboratory and in an industrial plant, the following relationships will be found:

Laboratory	Plant	
$q_{x_1} = q_{m_1} q_\alpha$	$q_x = q_m q_\alpha$	(44)
$q_{y_1} = q_{m_1} q_\beta$	$q_y = q_m q_\beta$	

It follows that:

$$\frac{q_{m_1}}{q_m} = \frac{q_{x_1}}{q_x} \text{ and } \frac{q_{m_1}}{q_m} = \frac{q_{y_1}}{q_y}. \tag{45}$$

If another ore, containing components Z and W, is flotated in the laboratory, the flotation probabilities will be:

$$q_{z_1} = q_{m_1} q_\gamma \quad \text{and} \quad q_{w_1} = q_{m_1} q_\delta. \tag{46}$$

The unknown flotation probabilities of this ore in the industrial plant can be computed by means of relationships:

$$q_z = q_{z_1} \frac{q_n}{q_{m_1}} = q_{z_1} \frac{q_x}{q_{x_1}} \tag{47}$$

$$q_w = q_{w_1} \cdot \frac{q_m}{q_{m_1}} = q_{w_1} \frac{q_x}{q_{x_1}} \tag{48}$$

In relationships (44), (45) we wrote q_m and q_{m_1} for the mechanical probability factors in the industrial plant and in the laboratory, q_α, q_β, q_γ and q_δ fo

the factors dependeing on the nature of the ore, q_x, q_y, q_z and q_w for the flotation probabilities in the industrial plant, and q_{x_1}, q_{y_1}, q_{z_1} and q_{w_1} for the flotation probabilities in the laboratory.

The use of the flotation probability diagrams for checking the plants.

Under such a circumstance the grades of the useful mineral available in the ore will be taken into account.

If we consider an ore containing two useful minerals, X and Y, available in the following proportions: $X = 20$ per cent, and $Y = 30$ per cent, in the diagram we shall pay attention to the flotation probabilities of these two minerals (Fig. 5) depending on their percentage grade. In the given instance, both minerals start flotating from the first cell, with different probabilities. In the first four cells where concentrate X is collected, this mineral flotates in 85 per cent (remaining 3 per cent in the pulp), and mineral Y flotates in 10 per cent (remaining 27 per cent in the pulp). In the cells 5 to 0, from mineral X,5 per cent flotates (2 per cent remaining in the wastes), and from mineral Y,76 · 7 per cent flotates (4 per cent remaining in the wastes).

In case of a faulty operation of the plant, the flotation probability diagrams will show this concretely. For instance, the dotted lines in Fig. 5 show that a bigger amount of Y passes into concentrate X and wastes, and a bigger quantity of X is lost in concentrate Y and wastes.

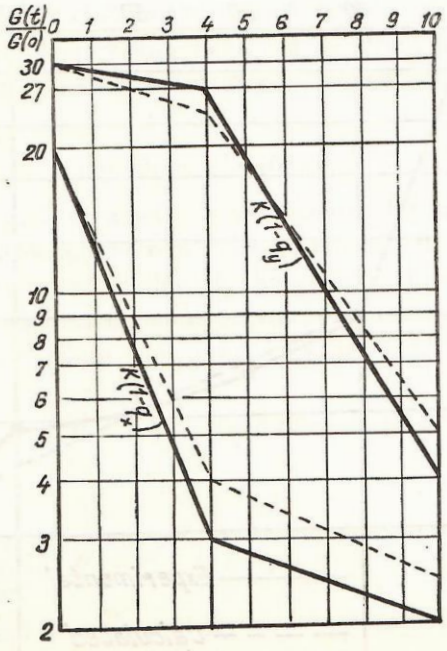


Fig. 5. Diagram to exemplify the way of using the flotation probability in checking the flotation plant operation

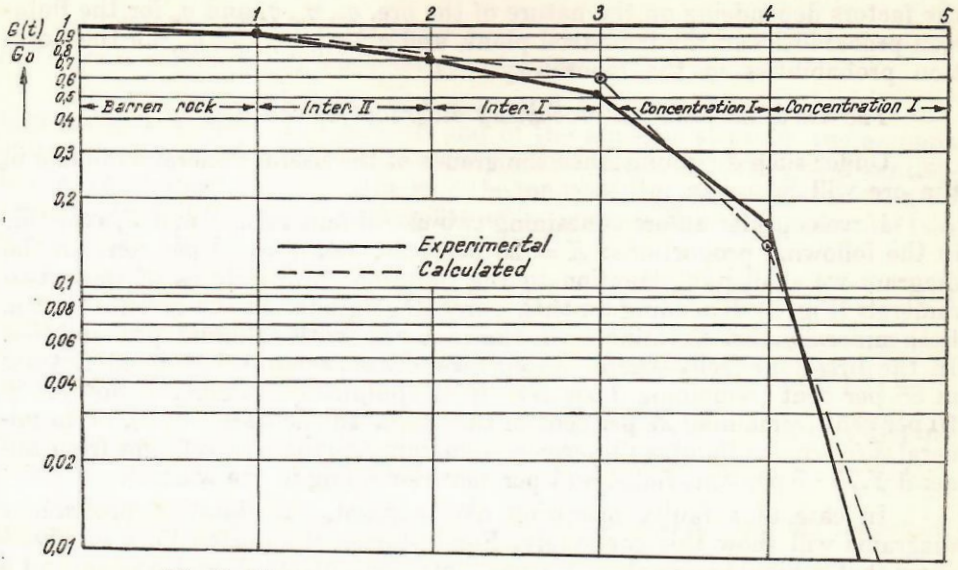


Fig. 6. Diagram of jigging concentration of a copper ore [10]

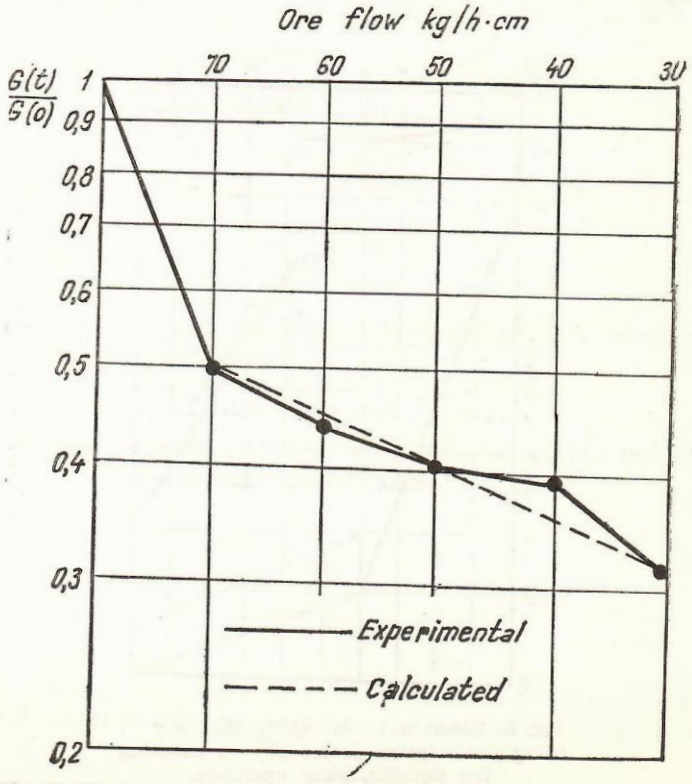


Fig. 7. Diagram of magnetic concentration of an iron ore depending on the feeding flow

Applications to gravitational concentration and magnetic concentration

Relationship (39) can be used also with other concentration methods in the same way as for flotation, no matter whether the occurrence intensity of the process varies in time or not. If the intensity does not vary, constancy A in relationship (39) will be equal to nil.

A comparison between the calculation made and the results achieved for magnetic concentration and gravitational concentration reveals a good consistency, as shown in Fig. 6 and 7.

Application to volumetric classification

In classification, relationship (38) is used, where $\mu(t)$ is replaced by the classification process intensity [9], [10]:

$$\mu = \alpha \frac{a - d}{a + b}$$

where α is a coefficient lesser than 1 (owing to the grains direction); a is the size of screen meshes; b is the diameter of the wire which makes up the meshes; and d is the diameter of the grains.

By means of the relationships ascertained,we can find the granulometric distribution of the materials classified in agreement with the screening time and the efficiency of the classification performed during a certain time interval.

By comparing the results achieved in practice with those achieved through calculation, we shall notice a good consistency, as shown in Table I and Fig. 8.

Applications of gravitational classification

The intensity of volumetric classification depends on the limit falling speed, but no mathematical relationship defining it has been found so far.

As a mathematical model of the classification operation we can use relationship (39), where t depends on the limit falling speed. The percentage of material having a falling limit v_i will be yielded by relationship:

$$\frac{G(v_0 i)}{G_0(v_0 i)} = 1 - exp\,[A(b^{v_{0i}} - 1) + Cv_{0i}] \tag{49}$$

where $b = a^{1/H}$, $C = \frac{B}{H}$ and H = the height of the overfall sill of the classifier.

Constants A, b and C are found by means of experimental data.

Fig. 9 shows the data obtained in practice, as against the experimental ones.

Table I

Granulometric distribution of a material after having passed through the Hummer screen

Size of material (mm)		13.2 3.3	3.3— 2.36	2.36 0.83	0.83— 0.295	0.295— 0.147	0.147 0
Feeding (per cent)		43.7	7	25.9	11.4	4.9	7.2
Refuse (per cent)	Experimentally	72.7	12	10.1	1.0	0.7	3.5
	Calculated	77.0	11.5	10.1	1.05	0.1	0.2
Screening intensity		0	0.0015	0.19	0.39	0.45	0.51
Longest time for remaining on the screen (*s*)		3	5	8	8	10	10
Probability of remaining on the screen		100	93	22	5	1	1

— Size of the screening surface: 1.25 × 1.50 m

— Size of the screen mesh (a): 2.56 mm

— Diameter of the wire which makes up the screen (b): 1.38 mm

— Screening capacity: 10 t/h

— Time for grains of a maximum diameter to remain on the screen: > 1 second

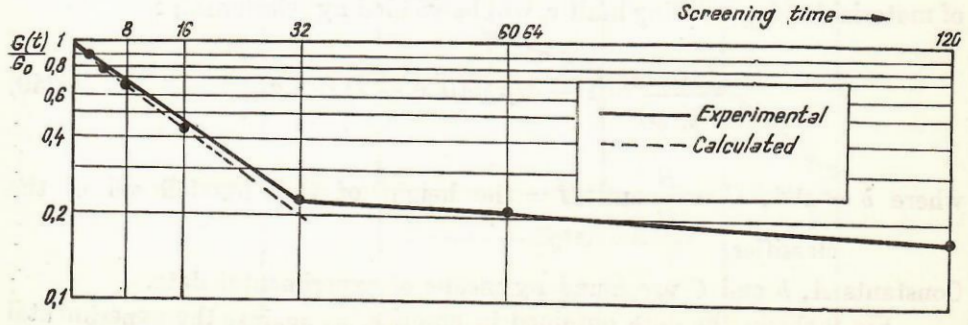


Fig. 8. Diagram of screening depending on time (according to Taggart [10])

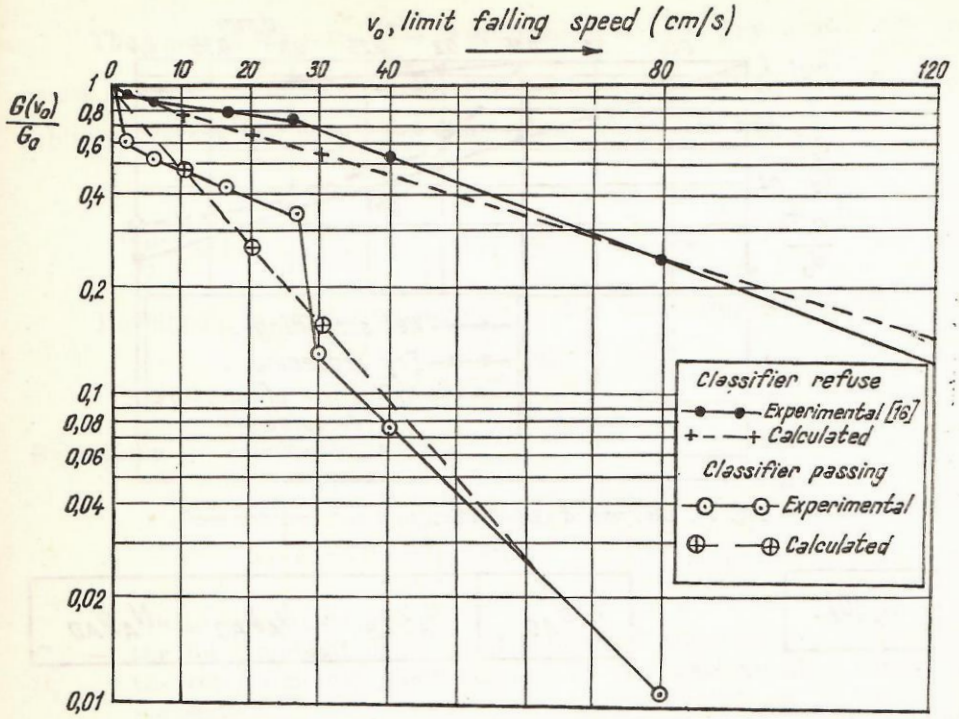


Fig. 9. Diagram of gravitational classification

Type II

The mathematical model of the processes of this type is based on the following equations proceeding from relationships (10) to (15), by replacing the amount of grains by their mass:

$$\frac{G(t)}{G_0} = e^{-\int_0^t [(t)+g(t)]\,dt} = 1 - q(t) - s(t) \tag{50}$$

$$\frac{G_F(t)}{G_0} = e^{-\int_0^t f(t)\,dt} = exp\ [A(a^t - 1) + Bt] \tag{51}$$

$$\frac{G_A(t)}{G_0} = e^{-\int_0^t g(t)\,dt}$$

$$\frac{G(t)}{G_0} = \frac{G_F(t)}{G_0} \cdot \frac{G_A(t)}{G_0}. \tag{52}$$

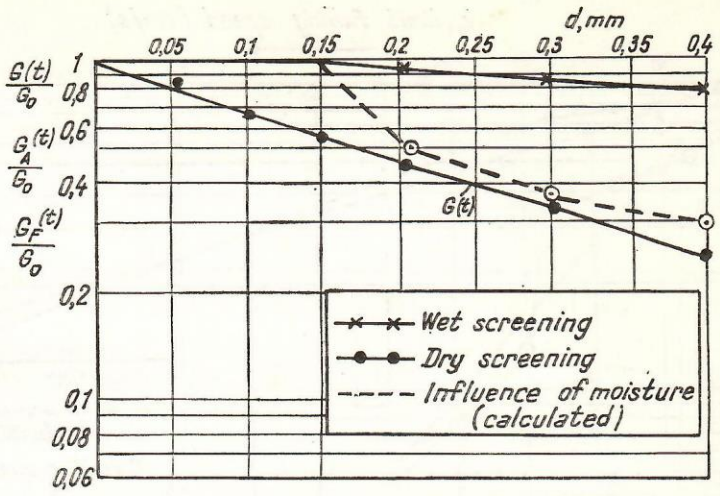


Fig. 10. Diagram of classification of a wet and dry ore

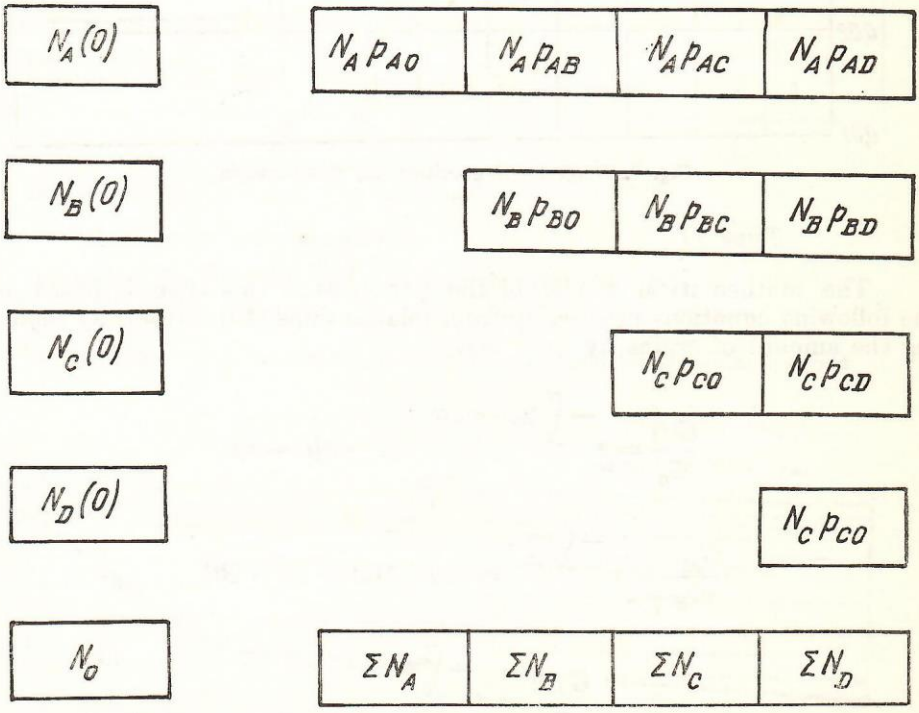


Fig. 11. Representation of the crushing process, according to Szantho [14]

N_i = quantities of fractions A, B, C, D in feeding
P_{ij} = crushing probability from i to j
N_i = quantity of fraction i after crushing

The value of the ratio $\frac{G_F(t)}{G_0}$ is ascertained in the way shown for processes of type I. The value of the percentage of the material driven out by arbitrary reasons $\frac{G_A(t)}{G_0}$ is inferred from relationships (15) and (20)

$$\frac{G_A(t)}{G_0} = 1 - s'(t) = 1 - \left[1 - \left(\frac{G(t)}{G_0}\right)^{\frac{G_A(t)}{G_0 - G(t)}}\right] = \left[\frac{G(t)}{G_0}\right]^{\frac{G_A(t)}{G_0 - G(t)}} \quad (53)$$

Developing in series and taking the first two terms, it follows:

$$\frac{G_A(t)}{G_0} = 1 + \frac{G_A(t)}{G_0 - G(t)} 1n \frac{G(t)}{G_0}.$$

By making the calculation, we find:

$$\frac{G_A}{G_0} = \frac{G_0 - G(t)}{G_0\left[i - 1n \frac{G(t)}{G_0}\right] - G(t)} = \frac{G_0 - G(t)}{G_0\, 1n \frac{2 \cdot 72\, G}{G(t)} - G(t)} \quad (54)$$

In these relationships:

G_0 = the total amount of material subject to dressing;

G_t = the total amount which remains in the machine after time t has elapsed;

G_F = the amount of material which would remain in the machine if only dressing phenomena were operating; it is found through a test from where arbitrary reasons are excluded, or it is inferred from relat onships (52) and (54);

G_A = the amount of material which would remain in the machine if solely arbitrary reasons were operating; it is found by means of relationships (54) or (52), if the value of G can be ascertained experimentally.

Since arbitrary reasons modifying the results often occurin practice, the relationships found for type II model are quite significant. The calculation made starting from experimental results obtained in classifying a dry and moist material has corroborated the validity of the relationships pointed to, as we can see in Fig. 10.

This method can be applied also to analytic studies of the two-phase flotation process, used by Arbiter [2] and Fuerstenau [4].

Applications of the mathematical models to the dressing processes which develop in agreement with the full-connections Markov chains laws.

The most complex problems which arise in the dressing processes can be solved by means of the models drawn up on the basis of the full-connections chaining laws.

In this article we analyze its application to crushing which, although it seems to be a simple process, its development in time is complex [11], [12], [13].

For systematical reasons we shall use diagram in Fig. 11, suggested by Szantho and co-workers [14].

In accordance with notations in Fig. 11, for the material of sizes lesser than $d = D$, we can use relationship:

$$N_i(t) = N_i(0)\, p_{ii}(0,\ t) + \sum_{\substack{j=1 \\ i=1}}^{n} N_j(0) \int_0^t p_{ij}(\tau,\ t)\, S_{ji}(\tau,\ 0)\, d\tau \qquad (55)$$

where:

$$N_i(0) = N_D(0);$$

$$p_{ii}(0,\ t) = e^{-\mu t}$$

(where μ is the instantaneous probability for the material of $d = D$ size to crush during time t);

$$N_j(0) = N_A(0),\ \ N_B(0),\ \ N_C(0)\ \ N_D(0);$$

$$p_{ij}(t,\ \tau) = e^{-\mu} AB^t;\ \ e^{-\mu} BC^t;\ e^{-\mu_{cs}\, t} \ldots \ a.s.o$$

where μ_{AB}, μ_{BC}, are the crushing intensities from a class to another;

$$S_{ji}(\tau,\ 0) = \sum_{r=1}^{3} k_{ji}^{(3)}(\tau,\ 0) = k_{ji}^{(1)}(\tau,\ 0) + k_{ji}^{(2)}(\tau,\ 0) + k_{ji}^{(3)}(\tau,\ 0)$$

Developing the terms pertaining to $S_{ji}(\tau,\ 0)$ we obtain:

$$S_{ji}(\tau,\ 0) = k_{ji}(\tau,\ 0) + \sum_{l=A}^{D} \int_0^{\tau} k_{jl}(\sigma,\ 0)\, k_{li}(\tau,\ \sigma)\, d\sigma +$$

$$+ \sum_{l=A}^{D} \int_0^{\tau} k_{jl}(\tau,\ \theta)\, d\sigma \sum_{m=A}^{D} d\sigma \int_0^{\tau} k_{lm}(\beta,\ \sigma)\, k_{mi}(\tau,\ \beta)\, d\beta$$

Because in case of simple crushing it is possible only to pass from a higher to a lower class, $k_{j1} = 0$ and $k_{1m} = 0$, it follows that:

$$S_{ji}(\tau,\ 0) = k_{ji}(\tau,\ 0) = f_{ji}(\tau,\ 0)\ e^{-\int_0^{\tau} \Sigma f_{ji}(\tau,\ 0)\, dt}.$$

Considering the crushing intensity to be constant:

$$f_{ji}(\ \sigma,\ 0) = \alpha_{ji} = \text{constant}$$

$$\Sigma f_{ji}(\sigma,\ 0) = \mu_{jl} = \text{constant},$$

it follows:

$$S_{ji}(\tau,\ 0) = \alpha e^{-\mu_{ij}\ \tau}.$$

Introducing this value into relationship (55), we obtain:

$$N_i(t) = N_i(0)\, p_{ii}(\theta,\ t) + \sum_{\substack{j=A \\ i=A}}^{D} N_j \int_0^{\tau} p_{ij}(\tau,\ t)\, \alpha e^{-\mu_{ji}\tau}\ dt. \qquad (56)$$

Since

$$p_{ii}(0, t) = e^{-\mu t}$$

$$\alpha_{ji} p_{ij}(\tau, t)\, e^{-\mu_{ji}\tau} = \alpha_{ji} e^{-\mu_{ji}(t-\tau)}\, e^{-\mu_{ji}\tau} = \alpha_{ji} e^{-\mu_{ji} t}$$

it follows:

$$N_i(t) = N_i(0)\, e^{-\mu t} + \sum_{\substack{j=A \\ i=A}}^{D} N_j(0) \int_0^t \alpha e^{-\mu_i \tau}\, d\tau,$$

which, by integration, yields:

$$N_i(t) = N_i(0)\, e^{-\mu t} + \sum_{\substack{j=A \\ i=A}}^{D} N_j(0)\, \frac{\alpha_{ji}}{\mu_{ji}}\, (1 - e^{-\mu_{ji}\tau}) \qquad (57)$$

This relationship is simiar to the one indicated by other research workers [11], [12], and has experimentally been proved.

Relationship (55) enables the calculation of some more complex crushing circuits as well, but in such cases the volume of calculuses becomes too great, which requires the use of a computer.

REFERENCES

1. HUBER-PANU I., GEORGESCU B.: *Variația indicilor tehnologici ai flotației în funcție de durata procesului în cazul unui singur concentrat.* Studii și cercetări de metalurgie I, Nr. 1, 1956.
2. ARBITER N. HARRIS C. C.: *Flotation Kinetics.* Froth Flotation: 50-th Anniversary (pag. 215—246). New York, AIME, 1962.
3. DOBRESCU L.: *Unele aplicații ale calculului statistic în flotație.* Studii și cercetări de metalurgie, 9, nr. 2, 1964.
4. BALL B., FUERSTENAU D. W.: *A two-phase distributed parameter model of the flotation process.* Compte rendu — 9-ème Congrès International de la préparation des minerais, Prage, 1970.
5. ONICESCU O., MIHOC G., IONESCU-TULCEA C.T.: *Calculul probabilităților și aplicații.* Editura Academiei R.P.R., 1956.
6. GALBRUN H.: *Traité de calcul des probabilités*, Paris, 1947.
7. TAUCER R.: *Sulla teoria dei gruppi* Atti. del Congr. Inter. dei Mat. Bologna, 1928.
8. IONESCU H.: *Studiul distribuțiilor statistice cu caracteristici variabile.* Buletinul științific al Academiei R.P.R., secțiunea de științe matematice și fizice. III nr. 1, 1951.
9. GAUDIN M. A.: *Principles of mineral dressing.* Mc. Graw — Hill, New York, 1939.
10. TAGGART A.: *Handbook of mineral dressing.* J. Wiley. New York, 1948.
11. PANU-HUBER: *Cinetica procesului de sfărîmare.*
12. AUSTIN L. G., LIMPEL R.P.: *Zür Theorie der Zerkleinerung.* Aufbereitungstechnik 7 (1966), pag. 10—20.
13. FUERSTENAU D. W. and collab.: *Dechema Monographie 57*, nr. 993—1026, I, Weinheim (1967) Trans AIME nr. 241 (1968).
14. SZANTHO E. V., FUHRMAN N., KORN M, RALF L.: *Zerkleinerungsverhalten von Körnerkollektiven bei der troken-und nassmahlung in Kugelmühlen.* Compte rendu 9-ème Congres Intern. de la Preparation des Minerais—Prage 1970.
15. IVANIER L., BOUCRAUT M.: *Concentration à sec des minerais de fer à limonite et à hematite par séparation magnétique à haute intensité.* Compte rendu. 9-éme Congrés interne de la préparation des minerais — Prage 1970.
16. POLKIN S. I.: *Prepararea minereurilor.* Editura Tehnică, 1956.

DES MODELES MATEMATIQUAS POUR CARACTERISER LES PROCESSUS DE PREPARATION DES SUBSTANCES MINERALES UTILES

Prenant comme point de départ l'idée que tous les processus de préparation des minerais, quel que soient les différences qui existent, ont une caractéristique commune, à savoir: la composition statistique du système — constitué par un nombre énorme de granules minérales — qui avec le temps passe d'un état à un autre (de minera brut en produit) on a aboutit à la conclusion qu'on peut établir un modèle mathématique général, qui peut être utilisé pour représenter tous les processus.

Dans la première partie de l'article on établit deux modèles simples, qui peuvent être appliqués dans le cas des processus de préparation des minerais, ces modèles pouvant être représentés, au point du vue mathématique, par un processus statistique soumis aux lois des chaînes Markov. Dans la deuxième partie de l'article on présente un modèle généralisée pour déterminer l'évolution qui suit les lois d'une chaîne à relations complètes et qui peuvent être intégrées au problèmes pour les statistiques variables. On démontre la possibilité de passer du modèle généralisé aux modèles simples du genre Markov.

La partie finale contient quelques exemples d'application des modèles mathématiques aux processus de préparation des minerais.

МАТЕМАТИЧЕСКИЕ МОДЕЛИ ОХАРАКТЕРИЗОВАНИЯ ПРОЦЕССОВ ПРИГОТОВЛЕНИЯ ПОЛЕЗНЫХ ИСКОПАЕМЫХ

Исходя из того обстоятельства, что все процессы приготовления руд, независимо от существующих между ними различий, имеют общую характеристику, а именностатист ический состав системы (образуемый огромным числом минеральных зерен), переходящий во времени от одного состояния к другому (от сырой руды к видам продукции), пришли к заключению, что можно установить общую математическую модель, применимую для представления всех процессов.

В первой части статьи устанавливаются две более простые модели, применимые для таких процессов приготовления руд, которые могут быть математически представлены статистическим процессом, подчиняющимся законам цепей Маркова. Во второй части статьи устанавливается обобщенная модель для определения хода процессов приготовления руд, которые подчиняются законам цепи с полными связями и могут быть включены в общую задачу переменных статистик. Указывается возможность перехода от обобщенной модели к простым моделям типа Маркова.

В последней части статьи приводится несколько примеров применения математических моделей при различных процессах приготовления руд.