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An Application of Strategic Games to a Decision-Making Problem

Uncertainty $\Rightarrow$ *strategic game* $\Rightarrow$ *decision.*

Under the present conditions the analysis of the economic phenomena and processes by means of up-to-date scientific methods and operational research asserts itself as an ever more urgent necessity.

The analysis of a state (situation) of a system, of the phenomena and processes taking place in the system concerned, through the operational research methods and techniques, places the most favourable solutions in view of decision-making at the disposal of the managerial factors. The state of a system, the phenomena and the processes taking place in a system, the evolution of the system, are rendered in mathematical models isomorphic with the state of the system.

The mathematical methods used by the operational research are given by the programming theory (linear, non-linear, parametric, stochastic, dynamic, marginal), the graph theory, the theory of queues, the information theory, the theory of strategic games, the theory of stocks, the wear and equipment replacement theory etc.

The theory of strategic games has for an object, to determine the methods of choosing the best decisions in states of conflict, in which the interests of parties with contrary targets are confronted. States of conflict are met with in all sectors of activity where the notion of contrary interests appears. In general, a state of conflict appears when opposite more assumptions which may be realized with certain probabilities (known or not) a choice is to be made between more solutions for the same assumption, according to a certain performance (or efficiency) criterion. The assumptions can be actions of an opponent who does not disclose his intentions, and the solutions — the possibilities of acting, of that who chooses a solution or takes a decision.

Then, when the one who chooses takes a resolution (decision), he is not aware in front of which action of the opponent finds himself. A concrete example of conflicting state is given by the following problem:

An enterprise, *A* in the field of food industry, which turns out a group of products P (boxes), has a contract concluded with a user, *B* (foreign trade enterprise), in which there is provided for the delivery of 4 000 thousand units from the respective group of products during 10 years. According to the contract, the works are made depending on the yearly demand of user *B*, which demand has not a previously specified lagging. The following data are known:

a) Referring to the demand there are known the following data:

1. By contract there are stipulated:

— The yearly demand should be done in lots of 10 000 units, this ranging between 320 000 and 480 000 units. In case the user *B* is placing annual orders under 320 000 units or over 480 000 units, the enterprise *A* is getting from the user *B* an allowance of 100 Leis for each unit of *P* product unclaimed or additionally delivered. Also, if the overall demand (for 10 years) tops 480 000 units, for each additionally delivered unit the enterprise *A* gets an allowance of 100 Leis.

— The volume of demand for each year is specified by the user B in the third week of the first month of the year concerned.

— The *P* products are delivered as they are manufactured, until the level of the demand is reached.

— If the deliveries of the enterprise *A* do not cover the demand of the user *B* in a year's time, then for each undelivered product in the year concerned the enterprise *A* pays the user *B* a penalty of 100 Leis on the product unit.

— In case of uncovering the user's demand in a year's time, the undelivered remaining has to be covered in the following year.

2. Of the statistical investigation made by enterprise *A* on the demand of the user *B*, it results:

— the annual demands are not influenced by season;

— the annual demands range around a mean value of 400 000;

— the extreme values 320 000 and 480 000 are met only once in 10 years'time, in the time left the demand ranging between 360 000 and 440 000.

b) The state of enterprise *A* is:

1. Production capacity

The enterprise *A* disposes of a production capacity in the space (250 000, 500 000).

Any yearly production level consists of more operation lines, the capacity of each technological line being of 10 000 product units.

The production level may be altered through one technological line or more technological lines every year; any more or less alteration as compared to the year before requires an expense of 400 000 Leis for each technological line put into or out of operation.

At the end of the year before that in which the performance of contract starts, the enterprise *A* has been working with a production level consisting of 40 operation lines, i.e., with a production level of 400 000.

The production level for each year is being established in the last week of the year before and cannot be altered but at the end of the year for the following year.

2. The level of production stock:

The enterprise A disposes of an adequate storing capacity, the storing expenses being of 200 000 Leis for each lot of 10 000 units of P product stored for a year.

When the production level of a year tops the annual demand, the products which exceed the demand are stocked, the remaining stock at the end of each year becoming initial stock in the following year. This initial stock is taken into consideration by establishing the monthly production for sale.

The level of the stock at the end the year before in which the performance of contract starts is of 10 000 units of P product.

It is necessary to determine the annual decision-making policy which has to be observed by enterprise A in order to be able to honour the contract with the user B, and the additional costs borne by enterprise A be minimum.

Remark. It is presupposed that the yearly orders placed by the user in the third week of the first month of each year are of: 37, 35, 48, 32, 43, 44, 38, 43, 39, 36 lots. These are known every year, but only after the establishment of the production level corresponding to the respective year and which cannot be altered.

Before proceeding to the solution of the problem above we shall refer to several notions with regard to the strategic games.

Definition. It is said that a two-person strategic game has been defined if there have been given:

1. A subset X of a n-dimensional Cartesian space X^n, $(X \subset X^n)$, called set of the game states or positions. The member $x \in X$ is called state of the game or game position.

2. A subset X_0 of X, $(X_0 \subset X)$, called initial set (of outset) of the game. Each state $x_0 \in X_0$ is called initial state (of outset) of the game. The game can start from each $x_0 \in X_0$.

3. A subset X_1 of X, $(X_1 \subset X)$, called terminal set (of termination of the game). Each state $x_1 \in X_1$ is called terminal state (of termination) of the game. The game is considered to be over when a certain point $x_1 \in X_1$ is reached.

4. A subset Y of a p-dimensional Cartesian space Y^p, $(Y \subset Y^p)$, called set of the strategies of player I. The member $y \in Y$ is called strategy of player I.

5. A subset Z of a q-dimensional Cartesian space Z^q, $(Z \subset Z^q)$, called set of the strategies of player II. The member $z \in Z$ is called strategy of player II.

6. A single-valued function θ_1 defined on X by values in $\{0, 1\}$, called turn of player I. If $\theta_1(x) = 0$, then the player I cannot choose another strategy to determine himself the alteration of the state of the game; if $\theta_1(x) = 1$, then the player I may choose a new strategy for himself, by taking into account the present state of the game and the result he wishes to obtain.

7. A single-valued function θ_2 defined on X by values in $\{0, 2\}$, called turn of player II. If $\theta_2(x) = 0$, then the player II cannot choose another strategy to determine himself the alteration of the state of the game; if $\theta_2(x) = 2$, than the player II may choose a new strategy for himself, by taking into account the present state of the game and the result he wishes to obtain.

8. A multi-valued function g, defined on $X \times Y \times Z$ by values in X with the properties:

i) $g^{-1}(X_0) = \emptyset$

ii) $g(X_1 \times Y \times Z) = \emptyset$

called game rule. This function describes the development, i.e., the evolution of the game.

9. A real functional f, defined on $X_0 \times X_1 \times X \times Y \times Z$: is called value of the game. This estimates the quality of the game depending on the initial state x_0, the final state x_1, the strategies y and z of the players. Depending on all these, by each state x one of the players is rewarded with the value $f(x_0, x_1, x, y, z)$ at the sacrifice of the other.

There may be asked to determine:

a) initial state $x_0 \in X_0$ of game;

b) terminal state $x_1 \in X_1$ of game;

c) strategy sequences $(y_1^*, y_2^*, ...,)$ for player I and $(z_1^*, z_2^*, ...)$ for player II, which takes the game from the initial state x_0 to the terminal state x_1, by observing the game rule g and the turns θ_1 and θ_2 so that the value

$f(x_0, x_1, x, y_1^*, z_1^*) + f(x_0, x_1, x, y_2^*, z_2^*) + ...$ should be the highest possible for player I and the lowest possible for player II.

The sequences $(y_1^*, y_2^*, ...)$, $(z_1^*, z_2^*, ...)$, the states x_0, x_1 meeting the required condition are the solution of the game. The value corresponding to the solution is called value of the game.

In certain instances there is given the initial state X_0 (it may be even a point x_0), in others the terminal state X_1 (it may be even a point x_1), and in some instances the both states. Sometines there is also known the behaviour of players before the commencement of the game as well as after the game is over. The knowledge of the players' behaviour before the commencement of the game has an effect upon the choice of the initial state and the first strategies.

The game is called discrete, if the game rule g takes discrete values; it is called continuous, if the game rule is a continuous function; it is called differentiated, if the game rule is given by a differentiated system; it is called aleatory, if the game rule is aleatory; it is called with random turn, if θ_1 and θ_2 are aleatory.

Sometimes the functional f is given under integral form to better outline the commencement and the end of the game and dependent of the intermediate states.

There may be met instances when the turns θ_1 and θ_2 are not known by one or both players. Then they play at random. These games shall be named by us casual games. They are also called games without information. When θ_1 and θ_2 are aleatory, the game is called complete information game.

The player, I, who aims at choosing the strategy $(y_1^*, y_2^*, ...)$ so that $f(x_0, x_1 x, y_1^*, z_1^*) + f(x_0, x_1, x, y_2^* z_2^*) + ...$ should be the highest possible, is called maximizing player, and the player II, who aims at choosing the strategy $(z_1^*, z_2^*, ...)$ so that $f(_0, x_1, x, y_1^*, z_1^*) + f(x_0, x_1, y_2^*, z_2^*) + ...$ should be the lowest possible, is called minimizing player. Games may be also met when the two players alternately change their maximizing and minimizing parts.

Any strategy $y \in Y$ is called elementary strategy of player I, while any strategy $z \in Z$ is called elementary strategy of player II. Any sequences of elementary strategies $(y_1, y_2, ...)$, $(z_1, z_2, ...)$ respectively, takingthe game from an initial state $x_0 \in X_0$ to a terminal state $x_1 \in X_1$ are called mixed strategies (addissible) for player I, II respectively.

We note by $Y_{x_0x_1}$ the set of mixed strategies of player I and by $Z_{x_0x_1}$ the set of mixed strategies of player II taking the game from the state x_0 to x_1. Let:

$$Y = \bigcup_{x_0 \in X_0} \bigcup_{x_1 \in X_1} Y_{x_0x_1}$$

and

$$Z = \bigcup_{x_0 \in X_0} \bigcup_{x_1 \in X_1} Z_{x_0x_1}$$

For $y = (y_1, y_2, \ldots) \in Y_{x_0x_1}$ and $z = (z_1, z_2, \ldots,) \in Z_{x_0x_1}$, we put

$$f(x_0, x_1, x, y_1, z_1) + f(x_0, x_1, x, y_2, z_2) + \ldots = f(x_0, x_1, y, z).$$

Because f is a real functional $f(x_0, x_1, y, z)$ shall be also a real functional defined on $X_0 \times X_1 \times Y \times T$.

With these notations, a two-person strategic game may be defined through the problem of the determination of the initial state $x_0 \in X_0$, of the final state $x_1 \in X_1$, of the mixed strategies $y^* \in Y$, $z^* \in Z$ for the players I, *II*, taking the game from x_0 to x_1 by observing the game rule g and the turns θ_1, θ_2 so that:

$$f(x_0, x_1, y^*, z^*) = \max_{y \in Y} \min_{z \in Z} f(x_0, x_1, y, z) = \min_{z \in Z} \max_{y \in Y} f(x_0, x_1, y, z).$$

The value $f(x_0, x_1, y, z)$ is called value of the game. The condition

$$\max_{y \in Y} \min_{z \in Z} f(x_0, x_1, y, z) = \min_{z \in Z} \max_{y \in Y} f(x_0, x_1, y, z)$$

imposes a symmetrical behaviour within the game. Both players, by choosing the strategies y^*, z^*, play cleverly and cautiously. (A player is playing cleverly when by establishing his behaviour he is aware of the conscious and rational behaviour of the other player. A player is playing cautiously when he establishes his behaviour depending on the least unfavourable strategies).

In the case when both players have a rational behaviour (clever and cautious), they play actively. When both players play actively, they choose the strategies y^*, z^*, respectively. The knowledge of y^* and z^* also assures the taking of game decisions by the players.

In certain games, not both players play cleverly and cautiously, but only one. In this case, the player who plays cleverly and cautiously is called active, while the other one passive. This passive player may be called nature, and the game concerned, a game contrary to nature.

In a game contrary to nature, one of the players, for instance, the player I, who is active, aims at choosing that strategy y^* for which:

$$f(x_0, x_1, y^*, z^*) = \max_{y \in Y} \min_{z \in Z} f(x_0, x_1, y, z)$$

or for which:

$$f(x_0, x_1, y^*, z^*) = \min_{y \in Y} \max_{z \in Z} (x_0, x_1, y, z)$$

the other player, that is the player II, who is passive, choosing a casual strategy in relation to the state and value of the game.

In a game contrary to nature, the active player may choose his strategy depending on various targets. Depending on the intended target, he may use one of the criteria:

1. Neumann's criterion. The active player chooses his strategy y^* given by:

$$f(x_0, x_1, y^*, z^*) = \min_{y \in Y} \max_{z \in Z} f(x_0, x_1, y, z)$$

2. Maximin criterion. For each of his possible strategy the active player determines the greatest disadvantage which may appear and then he chooses the least disadvantageous strategy for himself, in other words, he chooses the one whose possibility is less harmful.

3. Maximax criterion. For each of his possible strategy the active player determines the highest gain and chooses that strategy which has the highest gain, whatever be the risks arising from the choice of this strategy.

4. Laplace criterion. The active player regards the strategies of the passive player as being equiprobable, determines the mean value of the gain corresponding to each possible strategy of his and chooses that strategy for himself for which the mean value is the highest one.

5. Wald criterion. The active player chooses the most cautious strategy, specifically a strategy obtained through Neumann's minimax criterion. The choice is made as a consequence of a preliminary estimation of the situation of the game.

6. Hurwicz's criterion. It is at first defined the optimism of the active player, this being a number α ranging between 0 and 1. Then the highest gain M_y and the lowest gain m_y are determined for each strategy and further there is chosen that strategy y^* for which

$$\alpha M_{y^*} + (1 - \alpha) m_{y^*} = \max_{y \in Y} [\alpha M_y + (1 - \alpha) m_y].$$

7. Savage's criterion. A new functional is determined, defined by:

$$g(x_0, x_1, y, z) = \max_{y \in Y} f(x_0, x_1, y, z) - f(x_0, x_1, y, z)$$

and Neumann's minimax criterion is being applied to g.

In a game contrary to nature any of the criteria may be regarded as effective. The choice of one or another is a subjective matter. Each of the criteria is leading to different results. With all the imperfections of criteria, there cannot be conceived the abandoning of the study of phenomena characterized by uncertainty. The ground that at a given moment a decision has to be chosen between more decisions that may be taken for lack of any information, has led to the establishment of some elements which shall allow the comparison of these decisions. Certainly, these elements of comparison cannot but partially solve the problem of decision-making in case of uncer-

tainty. If the comparison between two decisions, on the assumption of complete prevision by probabilities governed by the law of large numbers, is an easy one, the comparison of decisions under conditions of total uncertainty is more complex. In case of a total uncertainty, the choice of certain decisions may be made by using the criterion of the mean value of utility introduced by von Neumann and O. Morgenstern.

If $f(x_0, x_1, y, z)$ represents a possible result, determined by the active player's choice of the strategy y and of the strategy z by the nature (passive player), if the nature chooses z with a probability p_z and if the active player is acting rationally, then a functional $U(f)$ so that the choice of strategy y' is preferred over the choice of strategy y'' if and only that:

$$M_z[U(f(x_0, x_1, y', z))] > M_z[U(f(x_0, x_1, y'', z))]$$

where M_z stands for the mean value in relation to z.

The functional U is called utility functional, $M_z[U(f(x_0, x_1, y, z))]$ the mean value of utility, while p_z is called personal probability of strategy z.

If in the definition of the game given previously we take

1. $X = \{1, 2, ..., i, ..., n\} \times \{1, 2, ..., j, ..., m\} = X_0 = X_1$
2. $Y = \{1, 2, ..., i, ..., n\}$
3. $Z = \{1, 2, ..., j, ..., m\}$
4. $\theta_1(i, j) = 1, \forall (i, j) \in X$
5. $\theta_2(i, j) = 2, \forall (i, j) \in X$
6. $g((i, j), p, q) = (p, q), \forall ((i, j), p, q) \in X \times Y \times Z$
7. $f((i, j), p, q) = a_{pq}; ((i, j), p, q) \in X \times Y \times Z$

then we get a strategic game which according to the form of the values of functional f we call it matrix game. The game deriving from the above problem shall be a contrary to nature matrix game, the nature being the user B who is not interested in the expenses made by the enterprise A.

The demand being made in lots of 10 000 units each and ranging in the space [320 000, 480 000], the user B may annually ask any number of lots of whose value is given by one of the numbers 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48. We regard the choice of any of these as being a strategy. Thus, we may regard the set of the strategies of user B as being $Z = \{32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48\}$.

The enterprise A can annually choose a level of production of the 10 this being formed of a number of operation lines, defined by one of the numbers 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 48, 49, 50. Regarding the choice of a certain number of operation

lines as the strategy of the enterprise A, we may consider that the set of its strategy is $Y = \{25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50\}$.

As functional f we may take the expenses borne by the enterprise A for the performance of the contract with the user B; it has a matrix form, its value, depending on the strategies $y^k \in Y$ and $z^k \in Z$ of the year k, $k = 1, 2, ..., 10$ of players A and B, being given by

$f(y^k, z^k) = 400\,000 \mid y^k - y_*^{k-1} \mid + \max[1\,000\,000\,(z^k - y^k), 200\,000\,(y^k - z^k)]$

where y_*^{k-1} stands for the strategy used by the player A in the year $k-1$ ($y_*^0 = 40$);

$400\,000 \mid y^k - y_*^{k-1} \mid$ stands for the expenses made with a view to changing the production level in the year k as against the year $k-1$.

When $z^k > y^k$, $1\,000\,000\,(z^k - y^k)$ stands for the expenses borne by the enterprise A due to the penalty in the year k.

When $y^k > z^k$, $200\,000\,(y^k - z^k)$ stands for the expenses borne by the enterprise A due to the stockage for the year $k+1$.

There raises the question of establishing the decision-making strategies y_*^k, $k = 1, 2, ..., 10$ of the enterprise A so that the risk of additional expenses should be as small as possible.

By establishing the decisions y_*^k, $k = 1, 2, ..., 10$ we shall take into consideration, instead of function $f(y^k, z^k)$, the function $g(y^k, z^k) = \frac{1}{200\,000} f(y^k, z^k)$; this leads to the same result, but simplifies the computation.

Remark. In establishing the game we did not take into account the possible emergence of demand from the space [320 000, 480 000], for we thought that the user B, playing rationally, would not adopt such disadvantageous strategies for him. In this situation, the enterprise A takes risks.

In the first year the values of function $g(y^1, z^1)$, are given depending on the $(y^1, z^1) \in Y \times Z$, $y_*^0 = 40$ (in table 1). For the user B we consider the strategies 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, since in the stock we have a lot that we may deliver from the outset.

Determining the strategy of the enterprise (player) A for the matrix game defined by the above matrix and using the criteria named it results:

— Neumann's criterion

$\min\limits_{y^1 \in Y} \max\limits_{z^1 \in Z} g(y^1, z^1) = 23 = g(44, 48)$. (There was taken the maximum from each line of the table and then the minimum from the maxims obtained). By using this criterion it results that the enterprise A has to observe the strategy 44 ($y_*^1 = 44$) being sure that through this its expenses, irrespective of the user's choice shall not exceed $23 \times 200\,000 = 4\,600\,000$ Leis. As is noticed from the table, these will however be higher than $8 \times 200\,000 = 1\,600\,00$ Leis, if it takes the decision to proceed during the first year of the performance of the contract with the production level of 44.

Table 1

A \ B	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
25	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140
26	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123	128	133
27	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116	121	126
28	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109	114	119
29	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112
30	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105
31	19	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98
32	17	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91
33	16	15	14	19	24	29	34	39	44	49	54	59	64	69	74	79	84
34	15	14	13	12	17	22	27	32	37	42	47	52	57	62	67	72	77
35	14	13	12	11	10	15	20	25	30	35	40	45	50	55	60	65	70
36	13	12	11	10	9	8	13	18	23	28	33	38	43	48	53	58	63
37	12	11	10	9	8	7	6	11	16	21	26	31	36	41	46	51	56
38	11	10	9	8	7	6	5	4	9	14	19	24	29	34	39	44	49
39	10	9	8	7	6	5	4	3	2	7	12	17	22	27	32	37	42
40	9	8	7	6	5	4	3	2	1	0	5	10	15	20	25	30	35
41	12	11	10	9	8	7	6	5	4	3	2	7	12	17	22	27	32
42	15	14	15	12	11	10	9	8	7	6	5	4	9	14	19	24	29
43	18	17	16	15	14	13	12	11	10	9	8	7	6	11	16	21	26
44	21	20	19	18	17	16	15	14	13	12	11	10	9	8	13	18	23
45	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	15	20
46	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	17
47	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14
48	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17
49	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20
50	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23

The maximin criterion is leading us to the same decision, as the greatest disadvantage for each strategy of A corresponds to the largest expenses. Hence, $y^1_* = 44$.

The maximax criterion. The player A (enterprise) determines the highest gain for each of his possible strategy, i. e., the smallest expense, and chooses that strategy which yields the highest gain, namely the smallest expense. Therefore we take the minimum on each line and then the lowest minimum from this.

There is obtained:

$$\min_{y^1 \in Y} \min_{z^1 \in Z} g(y(y^1, z^1) = \text{p0} = g(40, 40).$$

According to this criterion the enterprise A chooses its production level $y^1_* = 40$. By this choice the largest expenses are of $35 \times 200\,000 = 7\,000\,000$ Leis and the smallest expenses are 0.

Laplace criterion. We regard the strategies of B as being equiprobable (with probability 1/17). The mean values corresponding to the strategies of a are: 100; 93; 86; 79; 72; 65; 58; 51, 35; 45, 06; 39, 12; 33, 53; 28, 29; 23, 41; 20, 35; 14, 71; 10, 88; 11, 12; 12, 30; 13, 53; 15, 88; 17,

05; 19, 35; 22; 25; 28; 31;. The highest mean value is therefore 100; hence, $y_*^1 = 25$. In this case the smallest expenses are of $60 \times 200\,000 = 12\,000\,000$ Leis, while the largest are of $140 \times 200\,000 = 28\,000\,000$ Leis.

Wald's criterion cannot be used since there should be estimated the probability of the demand of B, which is not given in the case examined.

Hurwicz's criterion can neither be applied since there is not known the optimism of player A. This may be estimated depending on his various situations and of his opponent.

It is possible the choice of a decision from the strategies 44, 40, 25. The strategy 25 is eliminated since in this case the smallest expenses top the maximum expenses of strategies 44, 40. The choice is left between 44 and 40. A determination is made to take the decision 44, since the fluctuation of the expenses corresponding to this is smaller than with the case of choosing 40. As a result of the decision taken, the production level of the enterprise A during the first year of the performance of the contract is of 44 operation lines, i.e., 440 000 units.

Remark. The strategies of the enterprise A, which are below the minimum demand or over the maximum demand, are dominant ones. These may be excluded from the outset, since the expenses they presuppose are always higher than those of the others due to the alteration of the production line. We shall make use of this remark below.

We proceed to establishing the decision in the second year. The demand in the first year is of 37. The initial stock for the second year is of $44 + 1 - 37 = 8$.

Since $44 + 8 = 52 > 48$, the maximum demand, the production level is altered from the outset by $52 - 48 = 4$ operation lines; this is possible without altering the expenses, because with the lowering of production level by an operation line there is spent $400\,000 - 200\,000 = 200\,000$ (less the stockage which does not take place any longer), i.e., as much as it might be spent in the following year for a lot of stored products over the upper limit of maximum demand. We may consequently consider that in the second year the level of demand is of $44 - 4 = 40$. Having an initial stock of 8, the strategies of user B are 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40.

Taking into account the previously made remark, the possible strategies, after the elimination of the two dominants of A, are 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40.

We draw up the matrix of $g(y_2, z_2)$ for this case (table 2).

Neumann's criterion gives:

$$\min_{y^2 \in Y} \max_{z^2 \in Z} g(y^2, z^2) = 16 = g(40, 24),$$

hence $y_*^2 = 40$.

The maximin criterion leads us to the same result.

The maximax criterion leads us to

$$\min_{y^2 \in Y} \min_{z^2 \in Z} g(y^2, z^2) = 0 = g(40, 40)$$

hence $y_*^2 = 40$.

Table 2

A \ B	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
25	31	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105
26	30	29	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98
27	29	28	27	26	31	36	41	46	51	56	61	66	71	76	81	86	91
28	28	27	26	25	24	29	34	39	44	49	54	59	64	69	74	79	84
29	27	26	25	24	23	22	27	32	37	42	47	52	57	62	67	72	77
30	26	25	24	23	22	21	20	25	30	35	40	45	50	55	60	65	70
31	25	24	23	22	21	20	19	18	23	28	33	38	43	48	53	58	63
32	24	23	22	21	20	19	18	17	16	21	28	31	36	41	46	51	56
33	23	22	21	20	19	18	17	16	15	14	19	24	29	34	39	44	49
34	22	21	20	19	18	17	16	15	14	13	12	17	22	27	32	37	42
35	21	20	19	18	17	16	15	14	13	12	11	10	15	20	25	30	35
36	20	19	18	17	16	15	14	13	12	11	10	9	8	13	18	23	28
37	19	18	17	16	15	14	13	12	11	10	9	8	7	6	11	16	21
38	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	9	14
39	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	7
40	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

The Laplace criterion leads us to big errors and we do not use it any longer.

The decision taken in the second year is, therefore, to choose $y_*^2 = 40$.

The demand in the second year is of 35. The initial stock in the third year is of $40 + 8 - 35 = 13$. At the beginning we alter the production level of the third year by $40 + 13 - 48 = 5$ operation lines. There is left to us the level $40 - 5 = 35$.

The strategies of player B are 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, and of player A are 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35.

The values of the function $g(y^3, z^3)$ are given in table 3.

Table 3

A \ B	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
25	26	25	24	23	22	21	20	25	30	35	40	45	50	55	60	65	70
26	25	24	23	22	21	20	19	18	23	28	33	38	43	48	53	58	63
27	24	23	22	21	20	19	18	17	16	21	26	31	36	41	46	51	56
28	23	22	21	20	19	18	17	16	15	14	19	24	29	34	39	44	49
29	22	21	20	19	18	17	16	15	14	13	12	17	22	27	32	37	42
30	21	20	19	18	17	16	15	14	13	12	11	10	15	20	25	30	35
31	20	19	18	17	16	15	14	13	12	11	10	9	8	13	18	23	28
32	19	18	17	16	15	14	13	12	11	10	9	8	7	6	11	16	21
33	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	9	14
34	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	7
35	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Neumann's criterion gives:

$$\min_{y^3 \in Y} \max_{z^3 \in Z} g(y^3, z^3) = 16 = g(35, 19),$$

hence $y_*^3 = 35$.

The maximin criterion leads us to the same result.

The maximax criterion leads us to

$$\min_{y^3 \in Y} \min_{z^3 \in Z} g(y^3, z^3) = 0 = g(35, 35)$$

hence $y_*^3 = 35$.

For the production level of the third year there is taken as production level 35 operation lines.

The demand in the third year is of 48. The initial stock in the fourth year is of $40 + 8 - 48 = 0$.

The strategies of player B in the fourth year are 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, (he would not ask over 44, according to the statistical data, but once and he has asked).

The strategies of A are therefore 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44. The values of the function $g(y^4, z^4)$ are given in table 4.

Table 4

A \ B	32	33	34	35	36	37	38	39	40	41	42	43	44
32	6	11	16	21	26	31	36	41	46	51	56	61	66
33	5	4	9	14	19	24	29	34	39	44	49	54	59
34	4	3	2	7	12	17	22	27	32	37	42	47	52
35	3	2	1	0	5	10	15	20	25	30	35	40	45
36	6	5	4	3	2	7	12	17	22	27	32	37	42
37	9	8	7	6	5	4	9	14	19	24	29	34	39
38	12	11	10	9	8	7	6	11	16	21	26	31	36
39	15	14	13	12	11	10	9	8	13	18	23	28	33
40	18	17	16	15	14	13	12	11	10	15	20	25	30
41	21	20	19	18	17	16	15	14	13	12	17	22	27
42	24	23	22	21	20	19	18	17	16	15	14	19	24
43	27	26	25	24	23	22	21	20	19	18	17	16	21
44	30	29	28	27	26	25	24	23	22	21	20	19	18

Neumann's criterion gives:

$$\min_{y^4 \in Y} \max_{z^4 \in Z} g(y^4, z^4) = 2\,4 = g(42, 32)$$

The maximin criterion leads us to the same result.

The maximax criterion leads us to

$$\min_{y^4 \in Y} \min_{z^4 \in Z} g(y^4, z^4) = 0 = g(35, 35),$$

hence $y_*^4 = 35$.

If the player A decides upon $y_*^4 = 42$, the maximum expenses are 24, and the minimum ones 14, if he decides upon $y_*^4 = 35$, then the maximum expenses are 45 and the minimum ones are 0. He makes up his mind to take the decision $y_*^4 = 42$, because the maximum expenses are much less. The production level in the fourth year is therefore 42 operation lines.

The demand in the fourth year is of 32. The initial stock for the fifth year is of $42 - 32 = 10$. From the outset we consider the production level as altered by $42 + 10 - 44 = 8$. Consequently the production level is considered of $42 - 8 = 34$. The strategies of B are 26, 27, 28, 29, 30, 31, 32, 33, 34, because the minimum demand being solicited, it cannot take place according to the statistical data. The strategies of A are 26, 27, 28, 29, 30, 31, 32, 33, 34. The values of $g(y^5, z^5)$ are given in table 5.

Table 5

A \ B	26	27	28	29	30	31	32	33	34
26	16	21	26	31	36	41	46	51	56
27	15	14	19	24	29	34	39	44	49
28	14	13	12	17	22	27	32	37	42
29	13	12	11	10	15	20	25	30	35
30	12	11	10	9	8	13	18	23	28
31	11	10	9	8	7	6	11	16	21
32	10	9	8	7	6	5	4	9	14
33	9	8	7	6	5	4	3	2	7
34	8	7	6	5	4	3	2	1	0

Nemann's criterion gives:

$$\min_{y^5 \in Y} \max_{z^5 \in Z} g(y^5, z^5) = 0 = g(34, 34),$$

hence $y_*^5 = 34$.

The maximin criterion gives $y = 34$.

Alike, the maximax criterion leads us to $y_*^5 = 34$.

The production level in the fifth year is therefore of 34 operation lines. The demand in the fifth year is of 43. The initial stock in the sixth year is of $34 + 10 - 43 = 1$. The strategies of B are 35, 36, 37, 38, 39, 40, 41, 42, 43, while of A 35, 36, 37, 38, 39, 40, 41, 42, 43. The production level being under the maximum level of the demand, there has to be extended the production capacity by a technological line, reaching thus to 35.

The values of $g(y^6, z^6)$ are given in table 6.

Table 6

A \ B	35	36	37	38	39	40	41	42	43
35	0	5	10	15	20	25	30	35	40
36	3	2	7	12	17	22	27	32	37
37	6	5	4	9	14	19	24	29	34
38	9	8	7	6	11	16	21	26	31
39	12	11	10	9	8	13	18	23	28
40	15	14	13	12	11	10	15	20	25
41	18	17	16	15	14	13	12	17	22
42	21	20	19	18	17	16	15	14	19
43	24	23	22	21	20	19	18	17	16

Neumann's criterion leads us to

$$\min_{y^6 \in Y} \max_{z^6 \in Z} g(y^6, h^6) = 21 = g(42, 35)$$

hence $y_*^6 = 42$.

The maximin criterion leads us to the same result.

The maximax criterion leads us to

$$\min_{y^6 \in Y} \min_{z^6 \in Z} g(y^6, z^6) = g(35, 35).$$

hence $y_*^6 = 35$.

We choose $y_*^6 = 42$ as the production level in the sixth year. The demand in the sixth year is of 44. The initial stock for the seventh year is of $42 + 1 - - 44 = -1$. The strategies of B in the seventh year are therefore 37, 38, 39, 40, 41, 42, 43, 44, 45.

The values of $g(y^7, z^7)$ are given in table 7.

Table 7

A \ B	37	38	39	40	41	42	34	44	45
37	10	15	20	25	30	35	40	45	50
38	9	8	13	18	23	28	33	38	43
39	8	7	6	11	16	21	26	31	36
40	7	6	5	4	9	14	19	24	29
41	6	5	4	3	2	7	12	17	22
42	5	4	3	2	1	0	5	10	15
43	8	7	6	5	4	3	2	7	12
44	11	10	9	8	7	6	5	4	9
45	14	13	12	11	10	9	8	7	6

Neumann's criterion leads us to

$$\min_{y^7 \in Y} \max_{z^7 \in Z} g(y^7, z^7) = 11 = g(44, 37),$$

hence $y_*^7 = 44$.

The same answer is given also by the maximin criterion.

The maximax criterion leads us to

$$\min_{y^7 \in Y} \min_{z^7 \in Z} g(y^7, z^7) = 0 = g(42, 42).$$

By strategy $y_*^7 = 44$, the maximum expenses are 11 and the minimum ones are 4. By strategy $y_*^7 = 42$ the maximum expenses are 15 and the minimum ones 0. We decide at risk that the production level should be $y_*^7 = 42$, because $15 - 11 = 4$, that is as much as the minimum expenses would be for 44.

The demand for the seventh year is of 38, to which 1 is added, undelivered in the year before. The initial stock for the eighth year is of $42 - 39 = 3$. The strategies of B are 33, 34, 35, 36, 37, 38, 39, 40, 41. The strategies of A are 33, 34, 35, 36, 37, 38, 39, 40, 41. At first, we alter the production level by $42 + 3 - 44 = 1$. The level is therefore considered $42 - 1 = 41$. The values of $g(y^8, z^8)$ are given in table 8.

Table 8

A \ B	33	34	35	36	37	38	39	40	41
33	16	21	26	31	36	41	46	51	56
34	15	14	19	24	29	34	39	44	49
35	14	13	12	17	22	27	32	37	42
36	13	12	11	10	15	20	25	30	35
37	12	11	10	9	8	13	18	23	28
38	11	10	9	8	7	6	11	16	21
39	10	9	8	7	6	5	4	9	14
40	9	8	7	6	5	4	3	2	7
41	8	7	6	5	4	3	2	1	0

Neumann's criterion leads us to

$$\min_{y^8 \in Y} \max_{z^8 \in Z} g(y^8, z^8) = 8 = g(41, 41),$$

hence $y_*^8 = 41$.

The maximin criterion leads us also to the same result.

The maximax criterion leads us to

$$\min_{y^8 \in Y} \min_{z^8 \in Z} g(y^8, z^8) = 0 = g(41, 41),$$

hence $y_*^8 = 41$. For the eighth year we decide that the level of production be 41 operation lines.

The demand in the eighth year is of 43. The initial stock for the ninth year is $41 + 3 - 43 = 1$. The strategies of B in the ninth year are 35, 36, 37, 38, 39, 40, 41, 42, 43, while of A are 35, 36, 37, 38, 39, 40, 41, 42, 43.

The values of $g(y^9, z^9)$ are given in table 9.

Table 9

A \ B	35	36	37	38	39	40	41	42	43
35	12	17	22	27	32	37	42	47	52
36	11	10	15	20	25	30	35	40	45
37	10	9	8	13	18	23	28	33	38
38	9	8	7	6	11	16	21	26	31
39	8	7	6	5	4	9	14	19	24
40	7	6	5	4	3	2	7	12	17
41	6	5	4	3	2	1	0	5	10
42	9	8	7	6	5	4	3	2	7
43	12	11	10	9	8	7	6	5	4

Neumann's criterion leads us to

$$\min_{y^9 \in Y} \max_{z^9 \in Z} g(y^9, z^9) = 9 = g(42, 35),$$

hence $y_*^9 = 42$.

Alike for the maximin criterion $y^9 = 42$.

The maximax criterion leads us to

$$\min_{y^9 \in Y} \min_{z^9 \in Z} g(y^9, z^9) = 0 = g(41, 41),$$

hence $y_*^9 = 41$.

By taking into account the maximum and the minimum expenses corresponding to the strategies, we decide to take $y_*^9 = 41$.

The demand in the ninth year is 39. The initial stock for the tenth year is $41 + 1 - 40 = 3$. The strategies of B for the tenth year are 33, 34, 35, 36, 37, 38, 39, 40, 41, while of A are 33, 34, 35, 36, 37, 38, 39, 40, 41.

The values of function $g(y^{10}, z^{10})$ are given in table 10.

Table 10

A \ B	33	34	35	36	37	38	39	40	41
33	16	21	26	31	36	41	46	51	56
34	15	14	19	24	29	34	39	44	49
35	14	13	12	17	22	27	32	37	42
36	13	12	11	10	15	20	25	30	35
37	12	11	10	9	8	13	18	23	28
38	11	10	9	8	7	6	11	16	21
39	10	9	8	7	6	5	4	9	14
40	9	8	7	6	5	4	3	2	7
41	8	7	6	5	4	3	2	1	0

Table 11

Year K	The decision taken by enterprise A in the year K(y^k_*)		The initial stock in the year K		The user's demand in the year K (z^k_*)		$g(y^k_*, z^k_*)$	$f(y^k_*, z^k_*)$	Additional expenses in the year K
	number of technological lines	number of P products	numbre of technological lines	number of P products	number of technological lines	number of P products			
1	44	440 000	1	10 000	37	370 000	16	3 200 000	3 200 000
2	40	400 000	8	80 000	35	350 000	13	2 600 000	2 600 000
3	35	350 000	13	130 000	48	480 000	0	0	0
4	42	420 000	0	0	32	320 000	24	4 800 000	4 800 000
5	34	340 000	10	100 000	43	430 000	1	200 000	200 000
6	42	420 000	1	10 000	44	440 000	19	3 800 000	3 800 000
7	42	420 000	−1	−10 000	38	380 000	9	1 800 000	1 800 000
8	41	410 000	3	30 000	43	430 000	1	200 000	200 000
9	41	410 000	1	10 000	39	390 000	3	600 000	600 000
10	41	410 000	3	30 000	36	360 000	8	1 600 000	1 600 000

Neumann's criterion gives

$$\min_{y^{10} \in Y} \max_{z^{10} \in Z} g(y^{10}, z^{10}) = 8 = g(41, 41),$$

hence $y_*^{10} = 41$.

We are led to the same strategy also by the maximin criterion.

The maximax criterion leads us to

$$\min_{y^{10} \in Y} \min_{z^{10} \in Z} g(y^{10}, z^{10}) = 0 = g(41, 41)$$

hence $y_*^{10} = 41$. There is taken as decision in the tenth year $y_*^{10} = 41$. The demand in the tenth year is of 36. The stock the enterprise A is disposing of, after the performance of the contract, is of $41 + 3 - 36 = 8$.

Of the solving of the given problem it results the table 11 which contains the decisions and the yearly additional expenses.

The enterprise A will adopt the decision-making policy for the period of the 10 years given by the most favourable admissible strategy $y_* = (y_*^1, y_*^2, y_*^3, y_*^4, y_*^5, y_*^6, y_*^7, y_*^8, y_*^9, y_*^{10},) = (44, 40, 35, 42, 34, 42, 41, 41, 41, 41)$. The overall additional expenses borne by enterprise A by choosing the decision-making policy y, when the user adopts the policy $z_* = (z_*^1, z_*^2, z_*^3, z_*^4, z_*^5, z_*^6, z_*^7, z_*^8, z_*^9, z_*^{10}) = (37, 35, 48, 32, 43, 44, 38, 43, 39, 36)$, are $3\,200\,000 + 2\,600\,000 + 0 + 4\,800\,000 + 200\,000 + 3\,800\,000 + 1\,800\,000 + 200\,000 + 600\,000 + 1\,600\,000 = 18\,000\,000$.

If the enterprise A makes use of the decision-making policy y_*, irrespective of the policy z_* of user B, the additional expenses borne by the enterprise A are the least unfavourable. These vary in the space [7 200 000, 29 800 000], irrespective of the choice of the policy z_* of user B. Depending on the choice of the policy z_* of user B the additional expenses borne by the enterprise A stand for a multiple number of 200 000 in the space [7 200 000, 29 800 000]. If the enterprise A would not make use of the decision-making policy y_* established above the value of the additional expenses would by far top the established space, irrespective of the policy z choosen by the user B.

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UNE APPLICATIONS DES JEUX STRATEGIQUES A UN PROBLEME DE CHOIX DES DECISIONS

L'auteur présente une façon originelle d'utiliser les jeux stratégiques matriciels pour résoudre un problème de choix de la politique de décision optime. La première partie présente le problème réel, à savoir: une situation de conflit, partiellement contractuelle entre une entreprise *A* et un bénéficiaire extérieur *B E*. On avance ensuite, quelques considérations generales concernant les jeux stratégiques (définitions, classifications, critères) nécessaires à la résolution du problème. Ensuite, on transpose le problème en un modèle de jeux stratégiques à deux personnes contre nature — la situation de conflit entre les entreprises *A* et *B E*. Etablir les stratégies optimes, par rapport á plusieurs critères pour le jeu obtenu entre *A* et *B E*. assure l'établissement de la politique décisionnelle optime.

ПРИМЕНЕНИЕ СТРАТЕГИЧЕСКИХ ИГР В ЗАДАЧЕ ВЫБОРА РЕШЕНИЙ

В настоящей статье указывается оригинальный способ применения матричных стратегических игр в задаче выбора оптимальной политики принятия решений. В первой части статьи формулируется данная конкретная задача, представляющая конфликтное положение, частично договорное между предприятием *A* и внешним заказчиком *B3*. Затем приводятся некоторые общие соображения о стратегических играх (определения, классификации, критерии), необходимые для решения задачи, и затем переносится в модель стратегической игры с двумя лицами против природы конфликтное положение между предприятием *A* и внешним заказчиком *B3*. Установление оптимальных стратегий в соответствии с несколькими критериями для игры между *A* и *B3* обеспечивает установление оптимальной политики принятия решений. Эта оптимальная политика устанавливается в настоящей статье.