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Nature of Shadow Prices and their Utilization in Substantiating Economic Decisions

The utilization of a coresponding mathematical apparatus allows a profound study of the qualitative analysis by means of the quantitative one.

With a view to using value categories as active levers in the planned management of the national economy, it is necessary, besides defining the content and the economic significance of each index, to qualitatively analyse the social-economic relation in which value is produced, with the object of rendering evident the factors which determine its variation.

For correctly solving the problem of labour-value-price ratio, it is necessary to take into account that the working time does not create value only as far as it is socially necessary. Society will acknowledge the utilization value, hence it will also confirm the value as such, in accordance with the social necessity for the given goods. As a matter of fact, practice demonstrates that not any labour, materialized in a product, always finds a confirmation on the market, as social labour. That is why, when analysing the labour-value-price ratio, one should not identify value with the time consumed for producing goods, but should take into consideration to what extent its utilization value satisfies the social necessity.

In connection with the interdependence of categories: labour-value-price, it is essential to pay particular attention to the analysis of the value-price ratio. To this end, due to the complexity of economy, it is necessary to study all connections between economic subjects which contribute to the forming of value. The various enterprises which manufacture the same product are different from one another, from the viewpoint of technical equipping, of

production and labour organization, of production costs level, of the natural working conditions, a.s.o. It is therefore very important to study differences between individual values and, upon this basis, to set up prices which are incentive for the production costs reduction. It is sufficient to recall, e.g., that natural conditions (particularly in farming and extractive industry) are also reflected into the amount of plus-product obtained by enterprises; hence the individual values of obtaining a product will be different. This implies additional incomes for some enterprises. It is necessary, in these conditions, that the plus-product, as a component of price, be differently distributed within the branch, in accordance with the real efforts made by every enterprise in increasing economic efficiency.

These conclusions can be extended up to the level of relations between branches, where the problem of distributing plus-product is much more complex. As far as a product is necessary to the society, the branch must be incited to manufacture it and, concomitantly, to reduce production costs.

Equality between value and its pecuniary expression (price) can exist only when there are ideal conditions for carrying on the reproduction process (i.e. there exist no limitations concerning production and sale factors). Such conditions are usually lacking; it is therefore necessary that the price deviate from the value, in order to be used as an active instrument of planning and management. The working out of a price system must be founded upon social production costs, and further, the value deviations must be established by taking also into account those production factors (production funds, labour forces, land funds), which determine production limitation in certain branches.

In the meantime, an important part in the analysis of the value-price ratio is played by the taking into account of the products quality. Consequently, the price system must reflect the quality difference. Establishing of prices, differentiated according to quality, will contribute to incite enterprises in improving products quality.

As was shown, when working out the price system, it is necessary to analyse resources limits, and to what extent these are influencing the goods value. An answer to this problem can be obtained by the shadow-price analysis, resulting from optimizing production programs.

In connection with the defining of the shadow prices and with the possibility of their utilization for substantiating economic decision, a series of opinions were outlined in the speciality literature. A synthetical presentation of the main points of view can be found in [6].

As far as the defining of shadow-prices is concenrned, it seems that there exists unanimity of opinions, as they are defined as marginal magnitudes, whose amount is determined by the utilization in production of the last in a resource.

În connection with the possibility of using shadow prices in planning, opinions are divided, as a result of the differences concerning the valuation of the labour content, reflected by the shadow prices. Two diametrally oppo-

sed positions arose from the discussions on this theme: between them, all the other positions can be located:

a. Shadow prices can be used as exchange prices *.

b. Shadow prices have no applicability within the framework of the socialist economy.

Shadow prices are indexes resulting as auxiliary values from an optimal production program. As such, shadow prices can be considered as estimates of the availabilities of various resources and of their allotment between various economy sectors. It is essential to thoroughly analyse how these indexes are formed, because of their complexity.

The object of this study is to present and to discuss the nature of shadow prices, as well as the possibilities of using them in estimating the economic efficiency of various production programs. Within the study, different types of programming problems, having as a result shadow prices, are presented and so is the analysis of the various aspects connected with the use of shadow prices in planning.

1. DEFINITION OF THE SHADOW-PRICE CONCEPT AND ECONOMIC INTERPRETATION IN THE CASE OF LINEAR PROGRAMMING

We consider useful to define the shadow-price concept with the help of a linear programming problem, on the one hand, due to the relatively simple mathematical apparatus, and, on the other hand, due to the wider application sphere of this method in planning. The definition of the shadow-price concept, as well as the economic interpretation of this index will be made with the help of an example. For simplicity's sake, we shall consider a linear programming problem which aims at establishing the optimal production program for three products; any of the three available resources can be used, in manufacturing these products. Let

$$f(x) = 3x_1 + 2x_2 + 5x_3 = \max$$

be the linear programming problem, subject to

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &\leqslant 15 \\ 2x_1 + 5x_2 + x_3 &\leqslant 10 \\ 4x_1 + x_2 + 2x_3 &\leqslant 25 \\ x_t &\geqslant 0 \,(j = 1, 2, 3). \end{aligned}$$

where the efficiency function represents the maximum value of production, and the conditions refer to the consumption of theree resources. The optimal solution is given by the final simplex table (Table 1).

*) L.V. Kantorovich (to whom the discovery of these indexes is assigned) and V.V. Novojilov are among the supporters of the first position. They assert that shadow prices, as objectively determined estimates, express socially necessary labour expenses, hence their utilization as exchange prices is possible (as this agrees with the labour-value theory).

Table 1

B	C_i	P_0	3	2	5	0	0	0
			P_1	P_2	P_3	P_4	P_5	P_6
P_3	5	20/7	0	1/7	1	2/7	−1/7	0
P_1	3	25/7	1	17/7	0	−1/7	4/7	0
P_6	0	35/7	0	−63/7	0	0	−14/7	1
Z_j		175/7	3	8	5	1	1	0
$\Delta_j = Z_j - C_j$		*	0	6	0	1	1	0

Accordingly, the problem's optimal solution is $x_1 = \frac{25}{7}$, $x_2 = 0$ and $x_3 = \frac{20}{7}$. In addition to this, we note that, from the third resource, five units $\left(x_6 = \frac{35}{7}\right)$ remained unutilized. Besides the values obtained for the volume of production, in the final simplex table there appear indexes $Z_j = \sum_i c_i a_{ij}$, which have a well-defined interpretation, from the optimal plan viewpoint. It is possible to demonstrate that indexes Z_j, corresponding to the vectors which have built up the initial basis (in the examined case: P_4, P_5 and P_6), represent the optimal solution of the dual problem. If we denote dual variables by $u_i (i = 1, 2, 3)$, we shall have $u_1 = 1$, $u_2 = 1$ and $u_3 = 0$.

It is obvious that, when solving a linear programming problem, there results, besides the optimal solution, some indexes u_i, which correspond to the optimal solution of the dual problem, and consequently the problem of interpreting the economic content of these magnitudes arises. Every magnitude u_i shows which is the increase of the efficiency function, determined by using in production the last unit of the respective resource; hence, u_i represents the estimate of the last unit of this resource, used in production. These estimates, resulting from the optimizing of a production plan, were denominated in the speciality literature "shadow prices". We must point out that shadow prices are estimated in the same units as the efficiency function.

Only resources, which according to the optimal plan are completely used, have shadow prices different from zero. The magnitude of shadow prices u_i depends on the available resource amount (D_i), on the structure of the respective resource (i.e. on coefficients $a_{1i}, a_{2i}..., a_{mi}$), and on the mathematical structure of the model. We remark, at the same time, that the magnitude of shadow prices is valid only in the actual conditions, required by the given problem. If these conditions are changed, shadow prices will change as well.

If resource i is scarce (limited), meaning that the available amount D_i is limited with respect to the necessaries of consumption, shadow price u_i,

will have a positive value. In case a certain resource k is not limited, variable u_k will not be a component of the dual problem optimal solution, hence $u_k = 0$ (e.g., in the examined case, $u_3 = 0$). It results that shadow prices of incompletely utilized resources are always equal to zero. This means, from the economic viewpoint, that the increase of the available amount of resource k will not influence the optimal plan structure.

In general, the shadow price of a resource i can be computed according to relation

$$u_i = \sum_{k=1}^{m} c_k b_{ki} \qquad (i = 1, 2, ..., m).$$

where b_{ki} are the inverse matrix elements of the initial problem. For the analysed case, u_1, e.g., it is computed as follows:

$$u_1 = \sum_{k=1}^{3} c_k b_{ki} = 3\left(\frac{1}{7}\right) + 2 \cdot (0) + 5\left(\frac{2}{7}\right) = 1.$$

The maximum value, which can be obtained for the shadow price of a resource, is determined by the dual problem restrictions system. Each restriction has the form *:

$$\sum_{i=1}^{m} a_{ij} u_i \geqslant p_j \qquad (j = 1, 2, ..., n)$$

and indicates that the sum of costs per unit of product j, expressed in shadow prices, must be at least equal with the price of product j. For one, at least, of the variables u_i $(i = 1, 2, ..., m)$ to have values greater than zero, it is necessary that it appears on the ground of the optimal solution of the dual problem. This means that the auxiliary variable, corresponding to a restriction, should be equal to zero, and consequently:

$$\sum_{i=1}^{m} a_{ij} \cdot u_i = p_j.$$

If u_k is the only shadow price higher than zero, then:

$$u_k = \frac{p_j}{a_{kj}}$$

is its maximum limit, hence:

$$0 \leqslant u_k \leqslant \frac{p_j}{a_{kj}} \qquad (k = 1, 2, ..., m; \quad j = 1, 2, ..., n).$$

The above obtained result evinces that, for any shadow-price combination, the following condition must be observed: production costs, expressed in shadow prices, cannot exceed the price of the respective product.

*) p_j represents the efficiency function coefficients.

A product j will appear in the optimal plan ($x_j > 0$) only when unit production costs, expressed in shadow prices, do not exceed the unit price of product j.

The above stated idea is confirmed by the example presented at the beginning. Product 2 does not appear in the optimal solution ($x_2 = 0$), because its production costs, expressed in shadow prices, exceed its unit price:

$$a_{12}u_1 + a_{22}u_2 + a_{32}u_3 > p_2$$

or, concretely:

$$3 \cdot 1 + 5 \cdot 1 + 1 \cdot 0 = 8 > 2.$$

The other two products appear in the optimal program, since:

$$\sum_{i=1}^{3} a_{ij}u_i = p_j \quad (j = 1,\ 3)$$

As results from above relations, product shadow prices can be defined on the ground of the optimal solution. If we denote by p_k^* the shadow price of product k, the following computation relation will be obtained:

$$p_k^* = u_{m+k} = \sum_{i=1}^{m} u_i a_{ik}$$

or

$$p_k^* = u_{m+k} = \sum_{i=1}^{m} c_i b_{ik}$$

where a_{ik} are the elements of column k in the specific consumption matrix (A), and b_{ik} are the elements of column k in the final simplex table of the initial problem. For illustration, we shall compute the shadow price of product 2:

$$p_2^* = u_5 = 1 \cdot (3) + 1 \cdot (5) + 0 \cdot (1) = 8$$

or

$$p_2^* = u_5 = 3 \cdot \left(\frac{17}{5}\right) + 2 \cdot (0) + 5 \cdot \left(\frac{1}{7}\right) = 8.$$

The manufacturing of a product which does not appear in the optimal plan will be unprofitable from the point of view of resource utilization, as their value, expressed in shadow prices, will exceed the price of the respective product.

Hence the conclusion results that, on the basis of shadow prices, additional information is obtained, for the economic analysis and the efficient allotting of resources. Shadow prices emphasize to what extent the various resources are scarce, and they permit the planner to make decisions concerning their most sensible allotting, the completing of the necessaries from import a.s.o. An increase of the function value, objectively proportional to the shadow

price corresponding to the resource, is achieved by increasing the available amount of the limited resource, since:

$$f(u) = \sum_{i=1}^{m} u_i D_i.$$

Measures to give a stimulus to the rational consumption of scarce resouces can be established on the ground of shadow prices. Such a stimulating means could consist, e.g., in penalizing the enterprises for the immobilized stock of scarce resources, proportionally to shadow prices.

According to the statements, the optimal solution may be considered as a starting point for the economic analysis connected to the efficient allotment of resources. If the available amount of the first resource increases by one unit, the following values for x_i and u_i are obtained (Table 2).

Table 2

	Available amount of first resource (D_1)					
	15	16	26	36	40	41
u_1	1	1	1	1	1	0
x_1	25/7	24/7	14/7	4/7	0	0
u_2	1	1	1	1	1	5
x_2	0	0	0	0	0	0
u_3	0	0	0	0	0	0
x_3	20/7	22/7	42/7	62/7	70/7	70/7
u_4	0	0	0	0	0	7
x_4	0	0	0	0	0	7/7
u_5	6	6	6	6	6	23
x_5	0	0	0	0	0	0
u_6	0	0	0	0	0	0
x_6	5	5	5	5	5	5
$f(x)$	175/7	182/7	252/7	322/7	350/7	350/7

We note that, within certain variation limits of resource 1, the shadow prices have same value. For values exceeding 40 units, the shadow prices have another value, since, in this case, resources 1 and 3 become excess resources. The value of the efficiency function will increase by $\Delta D_1 u_1$, where ΔD_1 repre-

sents the growth of the available amount of resource 1. For $D_1 > 40$ the shadow price of the second resource is $u_2 = 5$, hence the efficiency function value will increase by 5 units, when ΔD_2 grows by a unit.

When interpreting the influence of the variation of available amount D_i upon shadow prices, one must take into account that D_i is a continuous variable. For this reason the shadow price of resource i is defined as follows:

$$u_i = \frac{df}{dD_i}.$$

It is obvious that u_i will have another value when D_i increases (decreases) above a certain limit (e.g., $D_1 > 40$).

Hence, another limit of shadow prices results, i.e. they keep their validity and consequently their role, in case of little increases of the resources volume.

A particular part is played, in planning economic development, by the determination of optimal magnitude of accumulations and investments, respectively. The main problems to which the working out of an optimal investment plan must answer, can be synthesized as follows:

a) Which is the volume of investments required for a certain period?

b) How must investments be distributed between the various sectors and branches of the national economy?

c) Which is the most appropriate method of stimulating the rational use of resources, allotted to each sector?

The solving of these problems belongs to the application field of dynamic programming. We shall present a linear model of dynamic programming, with the help of which the optimal investment program will be established, and thereafter we shall analyse the possibility of using the resulting shadow prices in establishing priorities concerning the distribution of the investment fund between branches.

It is necessary to decompose the production of each branch in its components, in order to construct the dynamic model. If we denote by $x_i(t)$ the production level of branch i in the year t, we can write:

$$x_i(t) = \sum_{j=1}^{n} x_{ij}(t) + y_i^{(1)}(t+1) + y_i^{(2)}(t) \qquad (i = 1, 2, ..., u) \qquad (1)$$

It results, from above relation, that the production of branch i is intended for the current production consumptions $\sum_{j=1}^{n} x_{ij}(t)$, for the unproductive consumption in the next period $[y_i^{(1)}(t+1)]$ and for increasing the production stock $[y_i^{(2)}(t) = S_i(t+1) - S_i(t)]$ After introducing direct costs coefficients a_{ij} and d_{ij} (fixed funds consumption coefficients), following dynamic model is obtained:

$$\begin{aligned} x_i(t) &= \sum_{j=1}^{n} a_{ij} x_j(t) + y_i^{(1)}(t+1) + y_i^{(2)}(t) \\ S_i(t) &= \sum_{j=1}^{n} d_{ij} x_j(t). \end{aligned} \qquad (2)$$

Equation system (2) consists of $2n$ equations and contains $4n$ unknowns: $x_i(t)$, $y_i^{(1)}(t+1)$, $S_i(t)$, and $S_i(t+1)$. Magnitude $S_i(t)$ represents the fixed funds stock existing in branch i during the year t. For $t=0$, $S_i(0)$ represents the fixed funds existing in the reference year, hence these elements are given.

Solving the system requires to previously settle down $2n$ values. The great number of degrees of freedom of the system allows to compute a great number of program versions, even in the case in which the solution has resulted from an optimizing process.

In what follows we shall take into account a linear model, referring to an optimizing process on a one-year period, for avoiding involved notations. For the same reason, the model takes into consideration an economy having only two branches. The pre-set values are: $\bar{S}_1(t)$ and $\bar{S}_2(t)$.

We consider the maximizing of the national income * as an object of optimization in the year $t+1$, and we denote by c_1 and c_2 the prices of products intended to final consumption, we then obtain the model:

$$F = c_1[y_1^{(2)}(t) + y_1^{(1)}(t+1)] + c_2[y_2^{(2)}(t) + y_2^{(1)}(t+1)] = max \tag{3}$$

in conditions:

$$\begin{aligned}
&[y_1^{(2)}(t) + y_1^{(1)}(t+1)] - (1-a_{11})x_1(t) + a_{12}x_2(t) \leqslant 0\\
&[y_2^{(2)}(t) + y_2^{(1)}(t+1)] + a_{21}x_1(t) - (1-a_{22})x_2(t) \leqslant 0\\
&d_{11}x_1(t) + d_{12}x_2(t) \leqslant \bar{S}_1(t)\\
&d_{21}x_1(t) + d_{22}x_2(t) \leqslant \bar{S}_2(t)\\
&x_1(t) \geqslant 0; \quad x_2(t) \geqslant 0\\
&[y_1^{(2)}(t) + y_1^{(1)}(t+1)] \geqslant 0; \quad [y_2^{(2)}(t) + y_2^{(1)}(t+1)] \geqslant 0
\end{aligned} \tag{4}$$

By solving this model we obtain the values of indexes

$$x_1(t),\ x_2(t),\ [y_1^{(2)}(t) + y_1^{(1)}(t) + 1)] \text{ and } [y_2^{(2)}(t) + y_2^{(1)}(t+1)]$$

The dual of the problem is formulated as follows:

$$F^* = u_1\bar{S}_1(t) + u_2\bar{S}_2(t) = min$$

subject to:

$$\begin{aligned}
&p_1 \geqslant c_1\\
&p_2 \geqslant c_2\\
&-(1-a_{11})p_1 + a_{21}p_2 + d_{11}u_1 + d_{21}u_2 \geqslant 0\\
&a_{12}p_1 - (1-a_{22})p_2 + d_{12}u_1 + d_{22}u_2 \geqslant 0\\
&p_1 \geqslant 0; \quad p_2 \geqslant 0; \quad u_1 \geqslant 0 \text{ and } u_2 \geqslant 0.
\end{aligned}$$

*) Other optimization criteria may also be taken into account.

The economic interpretation of variables p_1 and p_2, as well as of variables u_1 and u_2, is similar to that in the previous paragraphs, e.g. positive shadow prices correspond to the production funds with a capacity below the necessity limit, while shadow prices equal to zero correspond to the excess production funds. In case of production funds, a shadow price equal to zero means that, in the respective branch, during year t, higher investments were done, than was necessary for the optimal structure of the national economy, while a positive shadow price reflects the reverse situation. Hence, by solving the above linear programming problem, there results information concerning shadow prices, on whose basis decision can be made, as regards the mproving of the distribution of the investment fund, between the branches of the national economy.

Other mathematical programming models can be used in the modelling of economic efficiency increase. It is obvious that, when adopting other models, different shadow prices result, as was already pointed out.

The utilization of shadow prices in planning must be done with much discrimination, on the one hand, due to the fact that the mathematical models, they are resulting from, reflect more or less properly the studied economic phenomenon or the inter-conditioning with other economic phenomena, and, on the other hand, due to the deficiencies of the prices system used in estimating indexes included into the model.

2. SHADOW PRICES IN THE CASE OF PROGRAMMING WITH INTEGERS

The modelling of economic processes requires the use of discrete variables. Linear programming problems with discrete variables imply a series of peculiarities, as concerns the methodology of obtaining optimal solution. The shadow price analysis which results will permit an easier understanding of their economic significance.

We showed in the previous paragraph that the magnitude of shadow prices of resources depends on the specific consumption coefficients and on the available amount of the respective resource. When re-optimizing, additional conditions are used, which influence, either specific consumption vectors, or the available amounts vector. Hence, shadow prices of the optimal solution, obtained after re-optimizing, will be different from shadow prices of the ancient optimum. We must take into account, when economically interpreting shadow prices, the restrictions added to the initial problem.

A particular case of re-optimization is the programming with integers [5]. We can easily establish, when graphically representing the region of possible solutions of an usual linear programming problem *, and of a programming problem with integers, that the re-optimizing process consists in adding some additional restrictions, which eliminate part of the region of possible solutions of the original problem (see Fig. 1).

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*) Consider the case: $f(x) = \sum_{j=1}^{n} c_j x_j = max$ subject to $\sum_{j=1}^{n} a_{ij} x_j \leqslant D_i \quad (i = 1, 2, \ldots, m)$.

Where c_i and a_{ij} are integers.

The possible solutions region is OABCD, for the usual programming problem, and OEFGH, for the problem whose solution must be expressed in integers. This solution includes any possible solution, expressed in integers, of the original problem. Any basis solution of the new problem is a solution in integers. It results that, by adding a restriction, e.g. KL, the discrete optimal solution of the linear programming problem will be obtained. In Fig. 1, the maximum value of a corresponding efficiency function will be obtained in point F.

We can use the following procedure, in order to construct the new restrictions. Let us consider any equation met with during the solving of the problem; this equation will be written under the form *:

$$\bar{x}_i = D_i + \sum_{j=1}^{n} a_{ij}(-x_j) \quad (i = 0, 1, ..., m). \tag{2.1}$$

where $\bar{x}_i$ are the basic variable. For mathematical simplification, we suppose that $a_{ij} \geqslant 0$. Components of vector P_0 can be decomposed as follows: $D_i = k + f_i$ where k is an integer, and $0 \leqslant f_i \leqslant 1$. By replacing in relation (2.1), we obtain:

$$\sum_{j=1}^{n} a_{ij}x_j = f_i + (k - \bar{x}_i). \tag{2.2}$$

The left-hand member of equality (2.2) is a positive number (as $a_{ij} \geqslant 0$) and differs from the right-hand member by the integer $(k - \bar{x}_i)$. If we eliminate this number, relation (2.2) becomes:

$$\sum_{j=1}^{n} a_{ij}x_j \geqslant f_i \tag{2.3}$$

which is an additional restriction, satisfied by any non-negative discrete solution of the original problem. Any change of coefficients a_{ij} by a non-negative integer (a^*_{ij}) determines another inequality (2.3), in which $\sum_j a_{ij}x_j$, was changed, i.e.:

$$\sum_j a^*_{ij}x_j \geqslant f_i. \tag{2.4}$$

In the case when a coefficient $a_{ik} < 0$, we can add an integer, in order to obtain $a^*_{ik} > 0$. By using the same reasoning we can write a more powerful condition than (2.3) i. e.

$$\sum_j f_{ij}x_j \geqslant f_i \tag{2.5}$$

where f_{ij} is the fractional part pf a_{ij}. We obtain, from inequalities (2.5), additional restrictions, which are added to the original problem, with a view to obtaining a discrete solution

$$s_i = -f_i - \sum_j f_{ij}(-x_j) \; [s_i \geqslant 0; \; f_{ij} \geqslant 0] \tag{2.6}$$

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*) Relation (2.1) can be an eficiency function

$$z = z_0 + \sum_{j=1}^{n} z_j(-x_j)$$

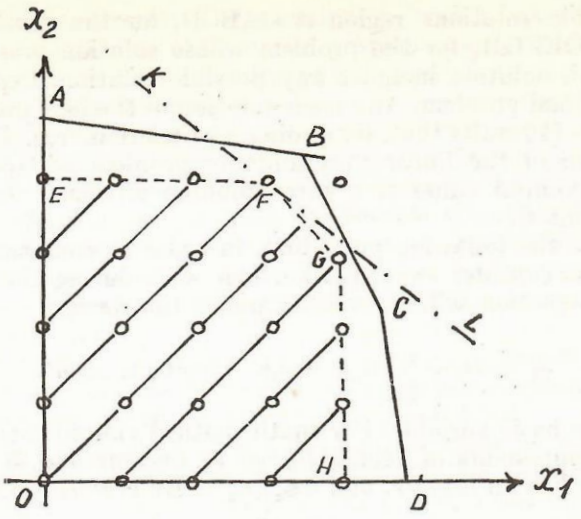


Fig. 1.

where s_i are auxiliary variables. These variables must be discrete, as they represent term $(k - \bar{x}_i)$ in relation (2.2).

Hence, the obtaining of the discrete optimal solution can be achieved, by this procedure, in two stages:

1. Compute optimal solution of the original problem.

2. If the obtained solution is not discrete, add additional restriction of the type of equation (2.6), and re-optimize the original problem solution. From a practical point of view, it is important that this successive-optimizing process be achieved with a number of iterations as small as possible. This can be carried out by selecting that restriction (2.6) for which f_i has the greatest possible value. *

As with any linear programming problem, there corresponds, to this optimal solution, a dual optimal solution, which consists in shadow prices of resources and products. These shadow prices have the same properties as the shadow prices of a usual linear programming problem, but they also have particular properties:

a) They vary according to the selection of additional restrictions.

b) They are prices expressed in integers (if the re-optimization is carried on until all elements in the final simplex table are integers).

c) They will not correspond to the scarce resources in the original problem, as they are influenced by the shadow prices of the added resources (corresponding to the additional restrictions).

In the optimal point in Fig. 1, not one of the three resources — graphically represented by straight lines AB, BC, CD, — is wholly used, hence the

*) In point B, in which the maximum value of the efficiency function was obtained, in the original problem, variables x_j in equation (2.6) had zero value (as they were non-basic). Ratio f_i/f_{ij} represents the non-zero coordinate of hyperplane in x_j on axis x_j; hence by increasing these coordinates, we shall obtain a restriction, which will eliminate a part as large as possible from the unhatched area of possible solutions region of the original problem.

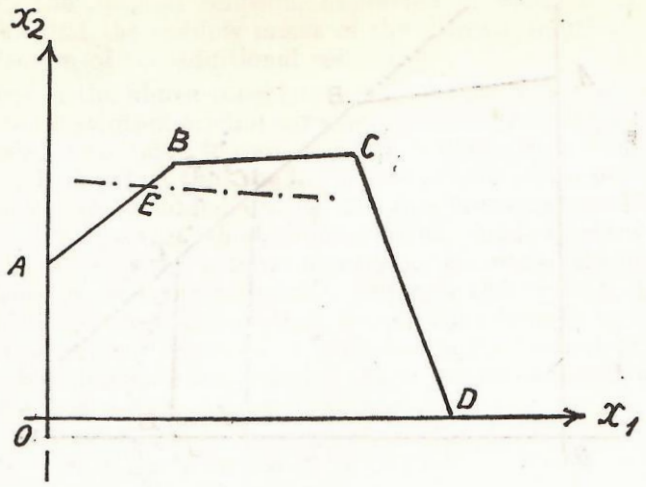


Fig. 2.

corresponding shadow prices will be equal to zero; in the optimal solution of the original problem, only the resource represented by straight line CD was not wholly used. It is possible that, in the optimal solution of the original problem, the shadow price of a resource be equal to zero. *E. g.*, if we suppose, in Fig. 2, that C is the optimum point of the original problem, the resource represented by straight line AB is not fully used, hence the corresponding shadow price is zero, while in the discrete optimal solution (point E), this resource is fully used, hence the corresponding shadow price is higher than zero.

The mathematical mean of shadow prices u_i of original resources is higher than or at least equal with the mathematical mean of the shadow prices in the discrete optimal solution, i.e.:

$$\frac{\sum_{i=1}^{m} u_i D_i}{\sum_{i=1}^{m} D_i} \geqslant \frac{\sum_{i=1}^{m+n} u_1^* D_i}{\sum_{i=1}^{m+n} D_i}$$

where u_i^* represents the shadow prices of the discrete solution (additional restrictions are included).

When interpreting the influence exerted by the modification of the available amount (D_i) of a certain resource upon the total value of the efficiency function, one must take into account the fact that, in the discrete programming, the variation of D_i is expressed in undivisible units ΔD_i. In order to illustrate the difference between $\frac{\Delta f}{\Delta D_i}$ (influence of ΔD_i upon the efficiency function), and $\frac{df}{dD_i}$, we shall consider the graph in Fig. 3.

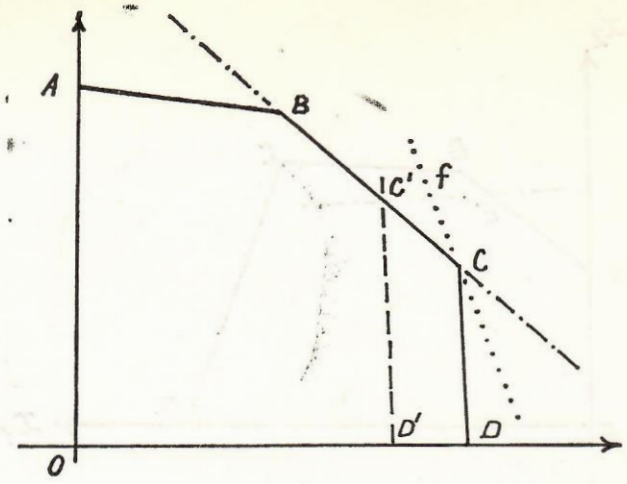


Fig. 3.

We suppose that the optimal solution of the problem was obtained in point C, that restriction CD refers to resource i, and that additional restriction LK is superposed upon the straight line BC. If the decreasing by a unit of the available amount $D_i(\Delta D_i^-)$ corresponds to the straight line $C'D'$, then $\frac{\Delta f}{\Delta D_i^-} = \frac{df}{dD_i}$ (as the new optimum point C' occurs at the intersection with straight line BC, and shadow price $\frac{df}{dD_i}$ is constant in the interval BC).

But in the case in which decreasing of D_i by a unit shifts straight line CD beyond point B, shadow price $\frac{df}{dD_i}$ will change (its absolute value increases), and in this case $\frac{\Delta f}{\Delta D_i^-}$ will not be equal to the shadow price obtained at the optimal point C. Hence, the absolute value of $\frac{\Delta f}{\Delta D_i^{-1}} \geqslant \frac{df}{dD_i}$, which means that decreasing of D_i by a unit will reduce the value of the efficiency function by at least $\frac{df}{dD_i}$ (shadow price of resource i in point C). In the case of increasing D_i by a unit, the value of $\frac{\Delta f}{\Delta D_i^+}$ can be higher or smaller than $\frac{df}{dD_i}$. We can also consider that, if straight line LK passes through point C, but has a smaller negative slope than that of straight line f (which represents the efficiency function), shadow price $\frac{df}{dD_i}$ will have another

value, although the optimal solution, expressed in integers, is the same. Hence, it results that the shadow prices of the discrete solution vary according to the selection of the additional restrictions.

On the basis of the above considerations, we find that shadow prices of the discrete optimal solution are not satisfactory for estimating scarce resources, as they assign zero value to resources which normally are not in excess. This is explained by the fact that shadow prices of artificial resources (corresponding to the resources added during the re-optimizing process) tend to having possible values, a fact which diminishes the shadow prices of the original problem. This is why, for estimating original resources, the discrete solution shadow prices must be re-estimated. Through this process, the shadow prices of artificial resources are allotted, proportionally with s_{ij}, to the shadow prices of the original resources (s_{ij} represents the weight of the initial inequalities in the additional restrictions). Thus the recomputed price of the original resource i is obtained from relations:

$$u_i' = u_i + \sum_k s_{ki} u_k, \; [s_{ki} \geqslant 0] \quad (i = 1, 2, \ldots, m) \tag{2.7}$$

where u_k represents the shadow prices of the artificial resources. In this way, any resource having the recomputed price equal to zero will be in excess, in the true economic sense, i.e. any increase in the stock of this resource will not modify the optimal production structure.

3. INTERPRETATION OF SHADOW PRICES IN THE CASE OF THE OPTIMIZATION OF A NONLINEAR FUNCTION

Essentially, an optimizing process consists in computing those variable values, for which the object-function has a maximum (minimum) value. This principle can also be applied in the case of nonlinear function, but the method of finding the optimal solution is different. If it is necessary to find the optimal value of a function

$$f = f(x_1, \; x_2, \ldots, x_n)$$

with the following given restrictions * (for simplifying, we shall consider only two):

$$\begin{gathered} g(x) = g(x_1, \; x_2, \ldots, x_n) = D_1 \\ h(x) = h(x_1, \; x_2, \ldots, x_n) = D_2 \end{gathered} \tag{3.1}$$

we can use Lagrange's multiplications method. The idea which underlies this method consists in constructing a function $L(X, \lambda)$, where the problem res-

*) These functions must be continuous and continuously differentiable. For a fuller tratment see [11].

trictions are also included. To this end, we use multiplicators λ_i, and function $L(X, \lambda)$ will have the form:

$$L = L(X, \lambda) = f + \lambda_1[D_1 - g(x)] + \lambda_2[D_2 - h(x)]. \tag{3.2}$$

Multiplicators λ_i are considered as additional variables, and, in this case, the partial derivatives with respect to $x_1, x_2, ..., x_n, \lambda_1$ and λ_2 will be:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= f_i + \lambda_1 g_i + \lambda_2 h_i = 0 \quad (i = 1, 2, ..., n) \\ \frac{\partial L}{\partial \lambda_1} &= D_1 - g(x) = 0 \\ \frac{\partial L}{\partial \lambda_2} &= D_2 - h(x) = 0 \end{aligned} \tag{3.3}$$

By solving this equation system, we obtain values $x_j (j = 1, 2, ..., n)$ and $\lambda_i (i = 1, 2)$. The last two equations guarantee the observing of restrictions (3.1).

Multiplicators λ_1 and λ_2 have a precise economic interpretation. They represent the magnitude by which the optimal value of the efficiency function is modified when D_1 and D_2 are modified by a unit; hence, they can be interpreted as shadow prices of the two resources. Mathematically, shadow prices λ_1 and λ_2 represent the partial derivatives of function $L(x, \lambda)$ with respect to D_1 and D_2:

$$\frac{\partial L}{\partial D_1} = \lambda_1 \text{ and } \frac{\partial L}{\partial D_2} = \lambda_2.$$

As an exemple, we shall consider a consumer who can make a choice between two objects, whose prices are p_1 and p_2, respectively. If the consumer's budget is C, the problem to be solved consists in maximizing a consumption function of the form:

$$f = f(x_1, x_2) = max, \left[f_1 = \frac{\partial f}{\partial x_1} > 0, \; f_2 = \frac{\partial f}{\partial x_2} > 0 \right]$$

subject to:

$$x_1 p_1 + x_2 p_2 = C$$

or

$$C - x_1 p_1 - x_2 p_2 = 0.$$

The Lagrange function corresponding to this problem will have the form:

$$L = f(x_1, x_2) + \lambda[C - x_1 p_1 - x_2 p_2].$$

For determining the critical points of this function, we annul the first derivatives:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= f_1 - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= f_2 - \lambda p_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= C - x_1 p_1 - x_2 p_2 = 0. \end{aligned}$$

We can easily demonstrate that the solution of the above system represents a maximum point. The first two equations are equivalent to:

$$\frac{f_1}{p_1} = \frac{f_2}{p_2} = \lambda.$$

This relation shows that, for maximizing function f, the consumer must establish the consumption structure in such a way, as to let above relation be verified for every product. The common value of these fractions is precisely the shadow price λ of the given resource.

The same way as in the linear programming, a dual problem corresponds to a nonlinear programming problem. A nonlinear programming problem, in a general form, can be formulated this way *:

$$f(x) = f(x_1, x_2, ..., x_n) = max$$

subject to:

$$\begin{aligned} h_1(x) &= h_1(x_1, x_2, ..., x_n) \leqslant D_1 \\ h_2(x) &= h_2(x_1, x_2, ..., x_n) \leqslant D_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ h_m(x) &= h_m(x_1, x_2, ..., x_n) \leqslant D_m \end{aligned} \tag{3.4}$$

where the objective-function is supposed as being concave, and functions h_i convex. If the first-order partial derivatives of these functions, with respect to x_j, are continuous, then the following nonlinear programming function can be formulated:

$$g(x, \lambda) = f(x) + \sum_i \lambda_i [D_i - h_i(x)] - \sum_j x_j \left[\frac{\partial f}{\partial x_j} - \sum_i \lambda_i \frac{\partial h_i}{\partial x_j} \right] = min.$$

subject to:

$$\begin{aligned} \lambda_1 \frac{\partial h_1}{\partial x_1} + ... + \lambda_m \frac{\partial h_m}{\partial x_1} &\geqslant \frac{\partial f}{\partial x_1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1 \frac{\partial h_1}{\partial x_n} + ... + \lambda_m \frac{\partial h_m}{\partial x_n} &\geqslant \frac{\partial f}{\partial x_n} \\ \lambda_i \geqslant 0 \ (i = 1, 2, ..., m) & \end{aligned} \tag{3.5}$$

which is considered the dual of problem (3.4) [1] and [13]. A series of the usual properties of the dual problem are also valid in the case of nonlinear programming. We must point out that, in the case of nonlinear programming, the initial problem is not the dual of its own dual. The explanation consists in the fact that, by passing to dual, the number of variables increases.

*) We suppose that $f(x)$ represents the maximum profit.

Dual variables $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the shadow prices of resources, and $\partial h_i / \partial x_j$ represents the additional amount of resource i, required to produce an additional unit of product j. The right-hand side of the inequalities constituting the restrictions of the dual problem, $\frac{\partial f}{\partial x_j}$, represent the profit obtained from an additionally manufactured unit of product j.

It is obvious that the economic interpretation of dual variables and of restrictions in nonlinear programming is similar to their interpretation in the linear case. In exchange, the economic interpreation of the objective-function is more complex. We notice that the objective-function of the dual problem is a Lagrange-type function, including all restrictions of the initial problem. To simplify the interpretation of function $g(x, \lambda)$, we shall consider that $f(x)$ reached its maximum value; hence, it can be treated as a constant, in discussing the minimum of function $g(x, \lambda)$.

The term $[D_i - h_i(x)]$ represents the unutilised amount of resource i; consequently, minimizing term $\lambda_i[D_i - h_i(x)]$ means minimizing the unutilized value of resource i. In interpreting the last term of function $g(x, \lambda)$ we must take into account the economic content of the elements in brackets:

$$\left[\frac{\partial f}{\partial x_i} - \sum_i \lambda_i \frac{\partial h_i}{\partial x_j}\right].$$

As was shown, $\frac{\partial f}{\partial x_j}$ represents the profit obtained from an additionally manufactured unit of product j; $\sum_i \lambda_i \frac{\partial h_i}{\partial x_j}$ represents the production costs, expressed in shadow prices, included in product j. The term in brackets is nonpositive (see restrictions of dual problem) and represents the losses which would result by the increase of production x_j over the optimal value. As a synthesis, minimizing function $g(x, \lambda)$ means minimizing the unutilised value of resources, and of losses, which would be obtained by not observing the optimal plan.

CONCLUSIONS

Mathematically, shadow prices represent Lagrange multiplicators, corresponding to an optimum problem with restrictions, or the variables of a dual programming problem. The maximum value of the efficiency function $f(x)$ of the initial problem is obtained when resources are used in such a way that their value, expressed in shadow prices, be equal to efficiency function.

$$f(x), \text{ i.e. } \sum_i p_j x_j = \sum_i u_i D_i.$$

The mathematical connection, existing between the initial and the dual programming problem, has determined the setting of a problem, similar from the economic viewpoint. Essentially, this problem consists in defining the

economic content of shadow prices and in establishing the possibilities of using them in planning.

Shadow prices result from the mathematical modelling process of the national economy, hence their magnitude depends on certain actual conditions (e.g., certain temporary resource limitations, objectives of economic policy expressed by the model, the mathematical structure of the model wherefrom they resulted, the limited caracter of some resources).

Obviously, the utilization of a corresponding mathematical apparatus allows to go thoroughly into the qualitative analysis, through the agency of the quantitative analysis; we must point out that it is very difficult to work out a macroeconomic model, detailed to such an extent, as to include all elements determining the variation of economic phenomena. In general, mathematical models include a series of simplifying hypotheses, influencing the solution. In addition to this, working out an optimal plan of the national economy implies the estimate of resources in the existing exchange prices. Consequently, prices will result, which reflect the deficiencies of the existing price system. Taking into account these considerations, we can conclude that shadow prices cannot be used as exchange prices, because they would not be able to perform the economic and political functions, required from a price system in a socialist economy.

Another conclusion which results, is that the use of mathematical methods in planning must be accompanied by the analysis of connections between the indexes, which make up the object of planning, by the analysis of the obtained solutions, and by the interpreting of the new indexes, appearing in these solutions. On the basis of the obtained information, the correction of the mathematical optimal solutions is carried out. Thus, the correction of possible imperfection of a solution obtained from an optimal model of the national economy can be achieved by market studies and by studying the using possibilities of value levers in planning.

Nevertheless, shadow prices can be used in planning, as they supply information concerning bottlenecks, from the optimal plan viewpoint, as well as their estimate.

As is known, current prices system must be brought up-to-date at certain time intervals, in order to reflect modifications taking place in economy as a consequence of the technical progress. When bringing prices up-to-date, shadow prices of resources and products can be used, as they render evident the sectors of economy where prices highly deviate from their value.

Shadow prices reflect the relative deficit of the various resources, as against the economic efficiency. Hence, on the basis of the shadow-price analysis, it is possible to select a suitable solution, when the problem of extending production and of increasing the plan efficiency by mobilizing additional resources, is set.

Shadow prices are relevant in making a decision with regard to the economic increasing. Due to the fact that actual prices do not reflect, hence do not transmit all direct and indirect influences taking place into the costs, as well as into the final demand, their utilization in working out an optimal investment plan could indicate the allotment of investments only towards activities which use some few fixed funds and would unfavourably affect the other branches.

Along the same line, the problem of additional investments finds its place. It is well known that, in certain conditions, increasing the investments beyond a certain limit does not lead to the increasing in the same proportion of the economic efficiency: moreover, losses can take place. A characteristic example for illustrating the above idea can be found in agriculture. Thus, it was demonstrated that — under certain conditions of techniques — production does not rise at the same rate, when achieving additional investments above a certain limit (see 10). The solving of this problem forms the object of an investment optimizing problem. In this case, shadow prices corresponding to the resource "investments" will furnish indications on the economic efficiency of the increase by a unit of investments, beyond a certain limit.

Shadow prices result from the optimal solution of a programming problem, where the essential interdependences of production factors are included, and, consequently, they can be used as an instrument of allotting resources in the framework of the planned development of economy. Besides establishing priorities in working out investment plan, shadow prices can be used in establishing the measures to be taken with a view to stimulating rational consumption of resources. Thus, on the ground of the ratio between shadow prices of various resources, norms of replacing resources can be computed; differentiated taxes on production funds, or differentiated financing of stock can be established, in terms of shadow-prices magnitude.

REFERENCES

1. BALINSKI, M.L., BAUMOL, W.J.: *The Dual in Nonlinear Programming and its Economic Interpretation.* The Review of Economic Studies, vol. XXXV(3), Nr. 103, July 1968.
2. BIRMAN, I.: *Optimalnoe planirovanie.* Moskva, 1968.
3. DORFMAN , R., SAMUELSON, P., SOLOW, R.: *Linear Programming and Economic Analysis.* McGraw-Hill Book Company, New York, Toronto, London, 1958.
4. DREZE, J.: *Some Postwar Contributions of French Economists to Theory and Public Policy.* American Economic Review, vol. LIV, No. 4, June 1964.
5. GOMORY, R., BAUMOL, W.: *Integer Programming and Pricing,* Econometrica, vol. 28, No. 3, July 1960.
6. HOROVITZ, M.: *Semnificaţia economică a preţurilor umbră.* Revista de statistică, nr. 6, 1968.
7. IANCU, A.: *Pentru un sistem de preţuri care să se bazeze pe concepţia mărimilor extremale.* Probleme economice, nr. 8, 1968.
8. KANTOROVICI, L.V.: *Calculul economic al folosirii optime a resurselor.* Editura ştiinţifică, Bucureşti, 1966.
9. NOVOJILOV, V.V.: *Măsurarea cheltuielilor şi a rezultatelor în planificarea optimă.* Editura ştiinţifică, Bucureşti, 1969.
10. PARPALA, O.: *Eficienţa economică a investiţiilor suplimentare în agricultură.* Lupta de clasă, nr. 2, 1969.
11. TEREHOV, L.L.: *Economico-matematicheskie metody.* Moskva, 1968.
12. WILLIAMSON, O.: *Peak-Load Pricing and Optimal Capacity under Indivisibility Constraints.* The American Economic Review, vol. LVI, No. 4, Sept. 1966.
13. WOLFE, PH.: *A. Duality Theorem for Nonlinear Programming.* Quarterly of Applied Mathematics, 19, 1961.

LA NATURE DES PRIX-OMBRE ET LEUR UTILISATION DANS L'ARGUMENTATION DES DECISIONS ÉCONOMIQUES

Le but de cette étude est celui de présenter et de mettre en discussion la nature des prix ombre et les possibilités de leur utilisation dans l'évaluation de l'efficacité économique de différents programmes de production. On présente différents types de problèmes de programmation mathématique et on fait l'analyse de toute une série d'aspects issus de l'utilisation des prix ombre dans la planification. On démontre que les prix-ombre procurent des informations supplémentaires pour l'affectation efficace des ressources, car ils mettent en évidence les lieux étroits au point de vue du plan optime.

L'analyse des prix-ombre permet à celui qui planifie, de prendre des décisions concernant l'assurance du nécessaire des ressources déficitaires, en les allouant de façon rationnelle, l'importation, etc., de consolider les mesures destinées à stimuler la consommation rationnelle des ressources, d'établir les priorités concernant la répartition du fonds d'investissement par branches. Aussi on relève une série de limites de l'utilisation des prix-ombre dans l'économie.

ПРИРОДА ТЕНЕВЫХ ЦЕН И ИХ ПРИМЕНЕНИЕ ПРИ ОБОСНОВАНИИ ЭКОНОМИЧЕСКИХ РЕШЕНИЙ

Предметом этой статьи являются изложение и обсуждение природы теневых цен и возможностей их применения при оценке экономической эффективности различных производственных программ. В статье указываются различные типы вопросов математического программирования и на их основе рассматривается ряд аспектов в связи с применением теневых цен в планировании. Указывается, что теневые цены обеспечивают дополнительные сведения для эффективного выделения ресурсов, так как они выявляют узкие места в отношении оптимального плана. Анализ теневых цен позволяет плановику принимать решения об обеспечении потребности из дефицитных ресурсов путем их правильного выделения, импорта и т.д., обосновывать меры по стимулированию рационального расхода ресурсов, устанавливать очередность в распределении фондов капитальных вложений по отраслям. Подчеркивается также ряд ограничений в применении теневых цен в народном хозяйстве.