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An Algorithm for Scheduling *n* Works on *m* Machines

The use of scientific methods for decision making within the productive systems leads to a higher economic efficiency.

The ever growing complexity of the economic activity in Romania inevitably requires the thorough replacement of the methods based on experience and intuition for decision making within the productive systems by scientific methods in order to avail to the full of the advantages yielded by the laws governing the socialist economy. A necessary condition for carrying out such a replacement as quickly as possible lies in the development at a rapid rate of the methods employed for solving mathematical models of productive systems. These methods are to secure optimal solutions to the mathematical models, which turn into optimal decisions within the productive system.

One of the problems inherently arising in many productive systems is the scheduling of n works on m machines. It stands to reason that, when stating the problem, the machine may be substituted by any other work spot where a certain operation is to be performed. This main problem originates many other collateral problems, depending on the criteria underlying the scheduling and some other requirements of the production process.

The scheduling problem for which an algorithm is given further resides in finding the order (succession) of performing n works on m machines so that the time during which the machines do not run —between the operations required by the works — is as short as possible.

The conditions under which scheduling is carried out are as follows:

1. Each work is achieved through n operations. All the works have the same technological flow on the m machines and no relation of the order (succession) of operations is imposed.

2. One machine cannot be employed for performing two operations at the same time and does not start performing a new operation before the one under way has not been completed.

3. The operations for a certain work are scheduled in agreement with the productive process.

We write:

L^j = work j, $(j = 1, 2, ..., n)$;

m_i = machine i, $(i = 1, 2, ..., m)$;

O_i^j = operation i of work j;

t_i^j = working time for operation O_i^j.

For instance, t_i^j is computed according to formula

$$t_i^j = t_{i,p}^j + t_{i,op}^j + t_{i,a}^j + t_{i,o}^j + t_{i,t}^j \tag{1}$$

where

$t_{i,p}^j \geqslant 0$ is the preparatory and closing time;

$t_{i,op}^j \geqslant 0$ is the operating time;

$t_{i,a}^j \geqslant 0$ is the work spot attendance time;

$t_{i,o}^j \geqslant 0$ is the repose time;

$t_{i,t}^j \geqslant 0$ is the break time contingent upon the technology.

It follows that there can be $t_i^j = 0$, which means that there is a fictitious operation O_i^j.

We also write:

$x_i^j \geqslant 0$ is the time unemployed by machines for work L^j, between the operations O_{i-1}^j and O_i^j. In other words, x_i^j is the storing time of work L^j between operations O_{i-1}^j and O_i^j;

$y_{i-1}^j \geqslant 0$ is the time unemployed by machine m_i between operations O_i^{j-1} and O_i^j.

Subject to the conditions stated above, the problem of finding the succession of n works on m machines has to be solved, so that

$$M = min \sum_{i=1}^{m-1} \sum_{j=2}^{n} y_i^j \tag{2}$$

Fig. 1 gives the Gantt graph of the problem.

This type of problem can currently be met with in operative programming of production in the field of machine building and textile industries, in road construction, a.s.o.

An analytical solution to this problem is known if there are solely two machines and an unlimited number of works, or three machines and an unlimited number of works subject to certain restraints [3].

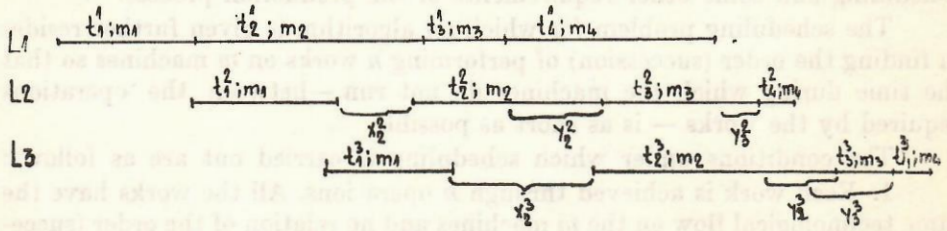

Fig. 1

The general problem of scheduling can be solved theoretically, particularly through mixed linear programming. Such methods of approach are not, however, practical at least on account of the long time they require for computation.

We further give here an algorithm through which a solution to (2) is reached, satisfying many practical problems regarding the number of the necessary arithmetical operations.

Let $\mathcal{P}$ be the set of permutation works L^j, $(j = 1, 2, ..., n)$. We notice in Fig. 1 that, for a $p \in \mathcal{P}$, there are relationships

$$\left.\begin{aligned} x_2^2 &= max\ (t_2^1 - t_1^2;\ 0) \\ x_3^2 &= max\ (t_2^1 + t_3^1 - t_1^2 - t_2^2 - x_2^2;\ 0) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_k^2 &= max\left(\sum_{i=2}^{k} t_i^1 - \sum_{i=1}^{k-1} t_i^2 - \sum_{i=2}^{k-1} x_i^2;\ 0\right) \end{aligned}\right\} \tag{3}$$

From relationship (3) we easily get relationship

$$\sum_{i=2}^{k} x_i^2 = \max_{2 \leqslant u \leqslant k} \left(\sum_{i=2}^{u} t_i^1 - \sum_{i=1}^{u-1} t_i^2;\ 0\right) \tag{4}$$

Let

$$\bar{t}_i^j = t_i^j + x_i^j. \tag{5}$$

Then, we can write

$$\sum_{i=2}^{k} x_i^l = \max_{2 \leqslant u \leqslant k} \left(\sum_{i=2}^{u} \bar{t}_i^{l-1} - \sum_{i=1}^{u-1} t_i^l;\ 0\right) \tag{6}$$

Or, considering (5), relationship (6) can also be written

$$\left.\begin{aligned} \sum_{i=2}^{k} x_i^l = \max_{2 \leqslant u \leqslant k} \left(\sum_{i=2}^{u} x_i^{l-1} + \sum_{i=2}^{u} t_i^{l-1} - \sum_{i=1}^{u-1} t_i^l;\ 0\right) \\ (2 \leqslant k \leqslant m;\ 3 \leqslant l \leqslant n) \end{aligned}\right\} \tag{7}$$

Similarly, we infer relationships

$$\left.\begin{aligned} y_1^2 &= t_1^2 + x_2^2 - t_2^1 \\ y_2^2 &= \sum_{i=1}^{2} t_i^2 + \sum_{i=2}^{3} x_i^2 - \sum_{i=2}^{3} t_i^1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_k^2 &= \sum_{i=1}^{k} t_i^2 + \sum_{i=2}^{k+1} x_i^2 - \sum_{i=2}^{k+1} t_i^1 \end{aligned}\right\} \tag{8}$$

With the aid of (8) we get

$$\left.\begin{aligned} y_1^2 + y_2^2 &= 2t_1^2 + t_2^2 + 2x_2^2 + x_3^2 - (2t_2^1 + t_3^1) \\ \sum_{i=1}^{3} y_i^2 &= 3t_1^2 + 2t_2^2 + t_3^2 + 3x_2^2 + 2x_3^2 + x_4^2 - (3t_2^1 + 2t_3^1 + t_4^1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{k} y_i^2 &= \sum_{i=0}^{k-1} (k-i)\, t_{i+1}^2 + \sum_{i=0}^{k-1} (k-i)\, x_{i+2}^2 - \sum_{i=0}^{k-1} (k-i)\, t_{i+2}^1 \end{aligned}\right\} \quad (9)$$

And

$$\left.\begin{aligned} y_1^3 &= t_1^3 + x_2^3 - t_2^2 - x_2^2 \\ y_2^3 &= t_1^3 + t_2^3 + x_2^3 + x_3^3 - (t_2^2 + t_3^2) - (x_2^2 + x_3^2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_k^3 &= \sum_{i=1}^{k} t_i^3 + \sum_{i=1}^{k} x_{i+1}^3 - \sum_{i=1}^{k} t_{i+1}^2 - \sum_{i=1}^{k} x_{i+1}^2 \end{aligned}\right\} \quad (10)$$

It follows from (10)

$$\left.\begin{aligned} \sum_{i=1}^{k} y_i^3 &= \sum_{i=0}^{k-1} (k-i)\, t_{i+1}^3 + \sum_{i=0}^{k-1} (k-i)\, x_{i+2}^3 - \\ &- \sum_{i=0}^{k-1} (k-i)\, t_{i+2}^2 - \sum_{i=0}^{k-1} (k-i)\, x_{i+2}^2;\ 2 \leqslant k \leqslant m-1 \end{aligned}\right\} \quad (11)$$

Generally, relationship

$$\left.\begin{aligned} \sum_{i=1}^{k} y_i^l &= \sum_{i=0}^{k-1} (k-i)\, t_{i+1}^l + \sum_{i=0}^{k-1} (k-i)\, x_{i+2}^l - \sum_{i=0}^{k-1} (k-i)\, t_{i+2}^{l-1} - \\ &- \sum_{i=0}^{k-1} (k-i)\, x_{i+2}^{l-1};\ 2 \leqslant l \leqslant n;\ 2 \leqslant k \leqslant m-1 \end{aligned}\right\} \quad (12)$$

follows.

From (4) and (9) we get

$$\left.\begin{aligned} \sum_{i=1}^{k} y_i^2 &= \max_{2 \leqslant u \leqslant k+1} \left(\sum_{i=2}^{u} t_i^1 - \sum_{i=1}^{u-1} t_i^2;\ 0 \right) + \sum_{i=1}^{k} t_i^2 - \sum_{i=2}^{k+1} t_i^1 + \\ &+ \max_{2 \leqslant u \leqslant k} \left(\sum_{i=2}^{u} t_i^1 - \sum_{i=1}^{u-1} t_i^2;\ 0 \right) + \sum_{i=1}^{k-1} t_i^2 - \sum_{i=2}^{k} t_i^1 + \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \max\ (t_2^1 - t_1^2;\ 0) + t_1^2 - t_2^1 \end{aligned}\right\} \quad (13)$$

Or

$$\sum_{i=1}^{k} y_i^2 = \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u-1}^{k} t_i^2 - \sum_{i=u}^{k+1} t_i^1;\ 0\right)+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u-1}^{k-1} t_i^2 - \sum_{i=u}^{k} t_i^1;\ 0\right)+ \qquad (14)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^2 - t_2^1;\ 0)$$

Similarly, from (7) and (12) we infer

$$\sum_{i=1}^{k} y_i^l = \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k+1} x_i^{l-1};\ 0\right)+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k} x_i^{l-1};\ 0\right)+ \qquad (15)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^l - t_2^{l-1} - x_2^{l-1};\ 0)$$

We see that

$$\sum_{i=1}^{k} y_i^l \leqslant \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=1}^{k+1} x_i^{l-1} + x_{k+1}^{l-1};\ 0\right)+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k} x_i^{l-1};\ 0\right)+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^l - t_2^{l-1} - x_2^{l-1};\ 0) \leqslant$$

$$\leqslant \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k+1} x_i^{l-1} + \sum_{u\leqslant k} x_k^{l-1} + x_{k+1}^{l-1};\ 0\right)+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k} x_i^{l-1} + x_k^{l-1};\ 0\right)+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^l - t_2^{l-1} - x_2^{l-1};\ 0) \leqslant$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^l - t_i^{l-1}) - \sum_{u=2} x_2^{l-1};\ 0\right)+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^l - t_i^{l-1}) - \sum_{u=2} x_2^{l-1};\ 0\right)+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \max\,(t_1^l - t_2^{l-1} - x_2^{l-1};\ 0) \leqslant$$

$$\leqslant \max_{2 \leqslant u \leqslant k+1} \left(\sum_{i=u}^{k+1} (t_{i-1}^l - t_i^{l-1});\ 0 \right) +$$

$$+ \max_{2 \leqslant u \leqslant k} \left(\sum_{i=u}^{k} (t_{i-1}^l - t_i^{l-1});\ 0 \right) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \max\,(t_1^l - t_2^{l-1};\ 0) \tag{16}$$

since $x_i^j \geqslant 0;\ 2 \leqslant i \leqslant m;\ 2 \leqslant j \leqslant n.$

Let

$$\left.\begin{aligned}
&\sum_{j=2}^{k+1} \max_{2 \leqslant u \leqslant j} \left(\sum_{i=u}^{j} (t_{i-1}^{l_1} - t_i^{l_2});\ 0 \right) = Y_1^{l_2,\, l_1} \\
&\sum_{j=2}^{k+1} \max_{2 \leqslant u \leqslant j} \left(\sum_{i=u}^{k+1} (t_{i-1}^{l_1} - t_i^{l_2}) - \sum_{u=2}^{l_2} x_2^{l_2};\ 0 \right) = Y_2^{l_2,\, l_1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\sum_{j=2}^{k+1} \max_{2 \leqslant u \leqslant j} \left(\sum_{i=u}^{j} (t_{i-1}^{l_1} - t_i^{l_2}) - \sum_{i=u}^{j} x_i^{l_2} + \sum_{\substack{j \geqslant k \\ u \leqslant k}} x_k^{l-1} + \sum_{j=k+1} x_{k+1}^{l-1};\ 0 \right) = Y_{k-1}^{l_2,\, l_1} \\
&\sum_{j=2}^{k+1} \max_{2 \leqslant u \leqslant j} \left(\sum_{i=u}^{j} (t_{i-1}^{l_1} - t_i^{l_2}) - \sum_{i=u}^{j} x_i^{l_2} + \sum_{j=k+1} x_j^{l-1};\ 0 \right) = Y_k^{l_2,\, l_1} \\
&\sum_{j=2}^{k+1} \max_{2 \leqslant u \leqslant j} \left(\sum_{i=u}^{j} (t_{i-1}^{l_1} - t_i^{l_2}) - \sum_{i=u}^{j} x_i^{l_2};\ 0 \right) = Y_{k+1}^{l_2,\, l_1}
\end{aligned}\right\} \tag{17}$$

Making use of the notations in (17) we can write relationship (16) under the form

$$\sum_{i=1}^{k} y_i^l = Y_{k+1}^{l-1,\, l} \leqslant Y_k^{l-1,\, l} \leqslant \ldots \leqslant Y_1^{l-1,\, l}. \tag{18}$$

From relationship (15) we can infer that

$$\sum_{i=1}^{k} y_i^l \geqslant \max_{2 \leqslant u \leqslant k+1} \left(\sum_{i=u}^{k+1} (t_{i-1}^l - t_i^{l-1}) - \sum_{\substack{i=u \\ 2 \leqslant u \leqslant k}}^{k} x_i^{l-1};\ 0 \right) - x_{k+1}^{l-1} +$$

$$+ \max_{2 \leqslant u \leqslant k} \left(\sum_{i=u}^{k} (t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{k} x_i^{l-1};\ 0 \right) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \max_{2 \leqslant u \leqslant 3} \left(\sum_{i=u}^{3} (t_{i-1}^l - t_i^{l-1}) - \sum_{i=u}^{3} x_i^{l-1};\ 0 \right) +$$

$$+ \max\,(t_1^l - t_2^{l-1} - x_2^{l-1};\ 0) \geqslant$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\geqslant \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^{l}-t_{i}^{l-1})-x_{u-2}^{l-1};\ 0\right)-\sum_{i=3}^{k+1}x_{i}^{l-1}+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^{l}-t_{i}^{l-1})-x_{u-2}^{l-1};\ 0\right)-\sum_{i=3}^{k}x_{i}^{l-1}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^l-t_2^{l-1}-x_2^{l-1};\ 0)\geqslant$$

$$\geqslant \max_{2\leqslant u\leqslant k+1}\left(\sum_{i=u}^{k+1}(t_{i-1}^{l}-t_{i}^{l-1});\ 0\right)-\sum_{i=2}^{k+1}x_{i}^{l-1}+$$

$$+\max_{2\leqslant u\leqslant k}\left(\sum_{i=u}^{k}(t_{i-1}^{l}-t_{i}^{l-1});\ 0\right)-\sum_{i=2}^{k}x_{i}^{l-1}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\max\ (t_1^l-t_2^{l-1};\ 0)-x_2^{l-1} \tag{19}$$

Let

$$X_{k_1,k}^{l-1,\,l}=\sum_{i=k_1}^{k}x_i^l \tag{20}$$

$$\overline{X}_{k}^{l_1,\,l_2}=\max_{2\leqslant u\leqslant k}\left(\sum_{i=2}^{u}t_i^{l_1}-\sum_{i=1}^{u-1}t_i^{l_2};\ 0\right),\quad l_1\neq l_2 \tag{21}$$

It follows from (4) and (7) that

$$\left.\begin{aligned} X_{2,k}^{1,2}&=\overline{X}_{k}^{1,2}\\ X_{2,k}^{2,3}&\leqslant\overline{X}_{k}^{2,3}+X_{2,k}^{1,2}=\sum_{i=1}^{2}\overline{X}_{k}^{i,i+1}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ X_{2,k}^{l-1,\,l}&\leqslant X_{2,k}^{l-2,\,l-1}+\overline{X}_{k}^{l-1,\,l}=\sum_{i=1}^{l-1}\overline{X}_{k}^{i,i+1}\end{aligned}\right\} \tag{22}$$

With the aid of (20), relationships (18) and (19) can be written

$$Y_1^{l-1,\,l}-\sum_{i=2}^{k+1}X_{2,i}^{l-2,\,l-1}\leqslant Y_2^{l-1,\,l}-\sum_{i=3}^{k+1}X_{3,i}^{l-2,\,l-1}\leqslant\ldots\leqslant Y_k^{l-1,\,l}-$$

$$-X_{k+1,\,k+1}^{l-2,\,l-1}\leqslant\sum_{i=1}^{k}y_i^l\leqslant Y_k^{l-1,\,l}\leqslant\ldots\leqslant Y_1^{l-1,\,l} \tag{23}$$

Or, by making use of (22), from (23), we get

$$Y_1^{l-1,\,l} - \sum_{i=2}^{k+1}\sum_{h=1}^{l-2} \overline{X}_i^{h,h+1} \leqslant Y_2^{l-1,\,l} - \sum_{i=2}^{k+1}\sum_{h=1}^{l-2} \overline{X}_i^{h,h+1} + kx_2^{l-1} \leqslant \ldots \leqslant Y_k^{l-1,\,l} -$$

$$- \sum_{i=2}^{k+1}\sum_{h=1}^{l-2} \overline{X}_i^{h,h+1} + \sum_{i=2}^{k} (k-i+2)\, x_i^{l-1} \leqslant \sum_{i=1}^{k} y_i^l \leqslant Y_k^{l-1,\,l} \leqslant$$

$$\leqslant Y_{k-1}^{l-1,\,l} \leqslant \ldots \leqslant Y_1^{l-1,\,l} \tag{24}$$

And

$$\sum_{j=2}^{l}\left(Y_1^{j-1,\,j} - \sum_{i=2}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{j-2} \overline{X}_i^{h,h+1}\right) \leqslant \sum_{j=2}^{l}\left(Y_2^{j-1,\,j} - \sum_{i=3}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{j-2} \overline{X}_i^{h,h+1} + \underset{j\geqslant 3}{kx_2^{j-1}}\right) \leqslant \ldots$$

$$\leqslant \sum_{j=2}^{l}\left(Y_k^{j-1,\,j} - \sum_{i=2}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{j-2} \overline{X}_i^{h,h+1} + \sum_{i=2}^{k} (k-i+2)\, x_i^{j-1}\right) \leqslant \sum_{j=2}^{l}\sum_{i=1}^{k} y_i^j \leqslant \sum_{j=2}^{l} Y_k^{j-1,\,j} \leqslant$$

$$\leqslant \sum_{j=2}^{l} Y_{k-1}^{j-1,\,j} \leqslant \ldots \leqslant \sum_{j=2}^{l} Y_1^{j-1,\,j}. \tag{25}$$

Making $p \in \mathfrak{Q}$ conspicuous, it follows

$$\underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l}\left(Y_1^{j-1,\,j} - \sum_{i=2}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{j-2} \overline{X}_i^{h,h+1}\right)_p \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l}\left(Y_2^{j-1,\,j} + \underset{j\geqslant 3}{kx_2^{j-1}} - \sum_{i=2}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{i-2} \overline{X}_i^{h,h+1}\right)_p \leqslant$$

$$\ldots \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l}\left(Y_k^{j-1,\,1} + \sum_{i=2}^{k} (k-i+2)\, x_i^{j-1} - \sum_{i=2}^{k+1}\sum_{\substack{h=1\\ j\geqslant 3}}^{j-2} \overline{X}_i^{h,h+1}\right)_p \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l}\sum_{i=1}^{k} y_{i,p}^j \leqslant$$

$$\leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l} Y_{k,p}^{j-1,\,j} \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l} Y_{k-1,\,p}^{j-1,\,j} \leqslant \ldots \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{l} Y_{1,\,p}^{j-1,\,j}. \tag{26}$$

From (26) we infer

$$\underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{n} Y_{1,p}^{j-1,\,j} - \underset{p\in\mathfrak{Q}}{max} \sum_{j=3}^{n}\sum_{i=2}^{m}\sum_{h=1}^{j-2} \overline{X}_{i,p}^{h,h+1} \leqslant$$

$$\leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{n}\sum_{i=1}^{m-1} y_{i,p}^j \leqslant \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{n} Y_{1,p}^{j-1,\,j}, \tag{27}$$

In order to find

$$M = \underset{p\in\mathfrak{Q}}{min} \sum_{j=2}^{n}\sum_{i=1}^{m-1} y_{i,p}^j$$

we first get

$$Y_1^1 = \min_{p \in \mathfrak{P}} \sum_{j=2}^{n} Y_{1,p}^{j-1,j}$$

and

$$X_1^1 = \max_{p \in \mathfrak{P}} \sum_{j=3}^{n} \sum_{i=2}^{m} \sum_{h=1}^{j-2} \bar{X}_{i,p}^{h,h+1},$$

Let graph $G = (L, M)$ where (see Fig. 2) the set of nodes L is in one-to-one correspondence with the set of works L_i, $1 \leqslant i \leqslant n$. Each node L_i is connected by an arc with each node L_j, $1 \leqslant j \leqslant n$, $j \neq i$. If we associate to each arc $u(i, j)$ the value $l(u) = Y_1^{i,j}$, which we term as the length of $u(i, j)$, then

$$\sum_{j=2}^{n} Y_1^{j-1,j} = \sum_{u \in \mu} l(u) = l(\mu)$$

where μ is a Hamiltonian path in G. It follows that

$$\min_{p \in \mathfrak{P}} \sum_{j=2}^{n} Y_{1,p}^{j-1,j} = \min_{\mu \in \mathcal{H}} \sum_{u \in \mu} l(u) \tag{28}$$

where $\mathcal{H}$ is the set of Hamiltonian paths in G.

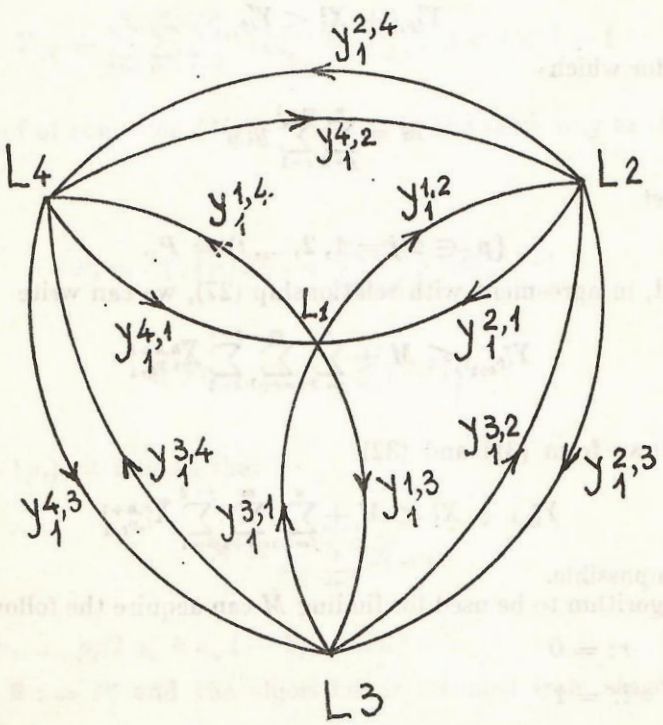

Fig. 2

We notice that we get $n(n-1)$ different values of $l(u)$ at most, and $n!$ different values of $l(\mu)$. It follows that, in order to find $\min_{\mu \in \mathfrak{M}} l(\mu)$, it is necessary to compute the $n(n-1)$ values $l(u)$ and to find the Hamiltonian path of minimal length out of the set $n!$ of Hamiltonian paths.

Similarly, we can find X_1^1 as well, if we consider $l(u(i, j)) = \sum_{t=2}^{m} \bar{X}_t^{i,j}$ which is inferred with the aid of formula (21)

Let

$$Y_r^{j-1,j} := Y_r^{j-1,j} + \sum_{i=2}^{r} x_i^{j-1}(m-i+1);\ r = 2, 3, \ldots, m-1$$

and

$$Y_r = \left\{ \sum_{j=2}^{n} Y_{r,p_k}^{j-1,} = Y_{(p_k)}^r / Y_{(p_1)}^r \leqslant Y_{(p_2)}^r \leqslant \ldots \leqslant Y_{(p_{n!})}^r, p_k \in \mathfrak{L} \right\}. \tag{29}$$

If

$$Y_{(p_1)}^r + X_1^1 \geqslant Y_{(p_t)}^r,\ 1 \leqslant t \leqslant n!,\ p_t \in \mathfrak{L} \tag{30}$$

and

$$Y_{(p_1)}^r + X_1^1 < Y_{(p_{t+1})}^r \tag{31}$$

then $\bar{p} \in \mathfrak{L}$, for which

$$M = \sum_{j=2}^{n} \sum_{i=1}^{m-1} y_{i,\bar{p}}^j$$

belongs to set

$$\{p_j \in \mathfrak{L} / j = 1, 2, \ldots, t\} = P_1.$$

Indeed, in agreement with relationship (27), we can write

$$Y_{(p_{t+1})}^r \leqslant M + \sum_{j=3}^{n} \sum_{i=r+1}^{m} \sum_{h=1}^{j-2} \bar{X}_{i,p_{t+1}}^{h,h+1} \tag{32}$$

if $\bar{p} = p_{t+1}$.

It follows from (31) and (32)

$$Y_{(p_1)}^r + X_1^1 \leqslant M + \sum_{j=3}^{n} \sum_{i=r+1}^{m} \sum_{h=1}^{j-2} \bar{X}_{i,p_{t+1}}^{h,h+1} \tag{33}$$

which is impossible.

The algorithm to be used for finding M can acquire the following stages:

1.1 $r := 0$

1.2 $l := 1$

2.1 $r := r + 1$

2.2 $Y^r_{(p_l)}$ and X^1_1 are found.

3.1 $l := l + 1$

3.2 $Y^r_{(p_l)}$ is found.

3.3 If $Y^r_{(p_1)} + X^1_1 \geqslant Y^r_{(p_l)}$ the algorithm is resumed from step 3.1; if $Y^r_{(p_1)} + X^1_1 < Y^r_{(p_l)}$ it follows—according to (31) — that M is ascertained with the aid of

$$p \in P^r_1 \subset \mathfrak{Q}, \quad P^r_1 = \{p_1, p_2, \ldots, p_{l-1}\}.$$

3.4 If $P^r_1 = \{p_1\} \Rightarrow M = \sum_{j=2}^{n} \sum_{i=1}^{m-1} y^j_{i,p_1}$.

If $P^r_1 = \{p_1; p_2 \ldots, p_k / 2 \leqslant k \leqslant l-1\}$, we remove from P^r_1 those permutations p_t of works L_i, $(i = 1, 2, \ldots, n)$ for which

$$Y^r_{(p_1)} + \sum_{j=3}^{n} \sum_{i=2}^{m} \sum_{h=1}^{j-2} \overline{X}^{h,h+1}_{i,p_t} < Y^r_{(p_t)}, \quad 2 < t < l-1. \tag{34}$$

The proof of condition (34) is achieved in the same way as the proof of condition (31).

3.5 Let

$$P^r_2 = P^r_1 \setminus \left\{ p_t / Y^r_{(p_1)} + \sum_{j=3}^{n} \sum_{i=2}^{m} \sum_{h=1}^{j-2} \overline{X}^{h,h+1}_{i,p_t} < \right.$$

$$\left. < Y(p_t), \quad 2 \leqslant t \leqslant l-1 \right\}. \tag{35}$$

If $P^r_2 = \{p_1\}$, it follows that

$$M = \sum_{j=2}^{n} \sum_{i=1}^{m-1} y^j_{i,p_1};$$

if $P^r_2 = \{p_1, p_2, \ldots, p_k / 2 \leqslant k \leqslant l-1\}$, then

3.6 $\mathfrak{Q} := P^r_2$ and the algorithm is resumed from stage 1.2.

We notice that M is ascertained after m iterations, at most (see (26)).

CONCLUSIONS

When thinking out the algorithm for scheduling n works on m machines, we have had in view to reach M with as few mathematical operations as possible.

By working out the algorithm we have seen that the increase in the number of operations for each work does not lead to a serious increase in the number of arithmetical operations, because with the aid of values t_i^j we have calculated $Y_1^{i,j}$ and $\sum_{i=2}^{m} \bar{X}_i^{i,j}$ which are $2\,n(n-1)$ in number. In turn, the increase in the number of works L_j leads to a rapid increase in the number of arithmetical operations, because, for instance, in order to compute $Y^1(p_t)$ we have had to calculate the length of $n!$ Hamiltonian paths in graph G, as well as to schedule them in an increasing order.

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ALGORITHME POUR L'ORDONNANCEMENT DE n TRAVAUX SUR m MACHINES

L'algorithme de calcul présenté se rapporte à la détermination de la solution optimale du problème d'ordonnancement de n travaux sur m machines, de telle manière que le temps de non-utilisation des machines (entre les opérations des travaux) soit minimal.

En mettant en évidence certaines relations entre x_i^j (le temps non-utilisé par les grandeurs m_i pour le travail j entre les opérations O_{i-1}^j et O_i^j) et y_{i-1}^j (le temps non-utilisé par la machine m_i entre les opérations O_i^{j-1} et O_i^j) on obtient un algorithme pour la détermination de

$$M = min \sum_{i=1}^{m-1} \sum_{j=2}^{n} y_i^j \ .$$

АЛГОРИТМ ДЛЯ СОСТАВЛЕНИЯ ДОКУМЕНТОВ НА ОПЛАТУ n РАБОТ НА m МАШИН

Представляемый алгори м расчета касается определения оптимального решения задачи составления документов на оплату n работ на m машин таким образом, чтобы время неиспользования машин (между операциями работ) было наименьшим.

Принимая в расчет некоторые отношения между x_i^j (время, неиспользуемое машиной m_i для работы j между операциями O_{i-1}^j и O_i^j) и y_{i-1}^j (время, неиспользуемое машиной m_i между операциями O_i^{j-1} и O_i^j), получаем алгоритм для определения величины:

$$M = min \sum_{i=1}^{m-1} \sum_{j=2}^{n} y_i^j \ .$$