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Mathematical Modelling of Transport Demand Prediction

(Matrix Starting Point—Place of Destination)

The prediction of the transport demand becomes an absolutely necessary stage for designing a transport network under the demographic development of big towns.

1. INTRODUCTION

The provision of good intraurban conveyance at large and of public transport services in particular is one of the ever more difficult problems arising from the growth of population, especially in big towns. Difficulties proceed from the necessity of coping with an ever higher demand for conveyance from one district of the town to another under the existing transport network and the growing motoring.

Hence the outstanding importance of the traffic prediction for designing an optimal transport network to positively meet the conveyance demand in agreement with the town systematization guiding projects.

The problem can be solved along two stages:

I. To find the present and prospective transport assignment.

II. To ascertain the transport network (structure and characteristics) optimally satisfying the conveyance demand.

The "preferential balance" is one of several mathematical models thought out in order to find a solution to the first stage. Theoretical and practical research [1], [2], [3] carried out by the Belgian Society of Applied Mathematics and Economics with regard to the transport network organization in Brussels and Charleroi, as well as the study for organizing the underground transport network in Milan, have pointed out that the mathematical model of preferential balance enables a fair long-range prediction of the transport demand.

2. THE MODEL OF PREFERENTIAL BALANCE

2.1. *Basic elements.*

Under the research carried out by the Bucharest Transport Company, the Romanian capital has been divided into zones taking into account:

— geographical and physical characteristic features of the city;

— socio-economic characteristic features of the population.

These criteria had to be correlated in such a manner that they should ascertain a division into zones approximately homogeneous as to the number of dwellings and jobs. Thirty three zones have thus resulted.

For each zone the following data have been found:

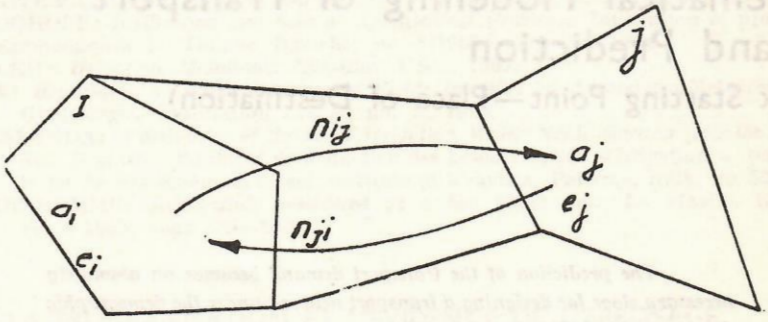

Fig. 1

a_i, a_j = active population residing in zone I and J, respectively;
e_i, e_j = population working in zone I and J, respectively;
n_{ij} = number of inhabitants residing in zone I and working in zone J.

These data check the following relationships:

$$\sum_{i=1}^{m}\sum_{i=1}^{m} n_{ij} = N; \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{m} e_j = N \tag{1}$$

where N stands for the total number of passengers in the city, and m for the number of zones.

Since the system achieved is a closed one, restrictions (2) are limit conditions:

$$\sum_{j=1}^{m} n_{ij} = a_i; \quad \sum_{i=1}^{m} n_{ij} = e_j \tag{2}$$

Their plotting is given in Fig. 2.

The geographical position of the zones and the possibilities of travelling from one zone to another play an important part in the phenomenon under investigation.

The calculus has included, as data for the model, a group of coefficients "C_{ij}" representing the relative possibilities for travelling from zone I to zone J. Distance, mean time required, cost, etc., have been calculated in order to find these coefficients. Thus, we have obtained matrix "C_{ij}" characterizing the transport network ($C_{ij} \neq C_{ij}$).

With regard to the active population, in agreement with their preferences for choosing the job and residence, three categories of passengers stand out:

Passengers of category [α] subordinate their choice for job to their residence.

Passengers of category [β] subordinate their choice for residence to their job.

Passengers of category [γ] make their choice for job irrespectively of their choice for residence, or choose their residence irrespectively of the place of their job.

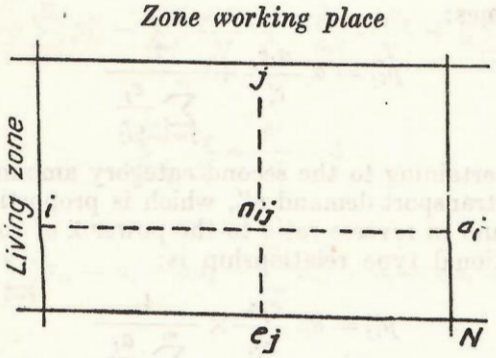

Fig. 2

Thus, passengers of categories [α] and [β] take into account coefficients C_{ij}, whereas passengers of category [γ] disregard these coefficients.

Yet, everybody's desire is generally not compatible with the working and living possibilities. This is mirrored by the appearance of divergences which, in order to ensure a balance, must be solved up.

2.2. *Formulating the model.*

We assume that the N passengers are grouped in the three afore said categories in proportions α, β, γ, so that we have relationship

$$\alpha + \beta + \gamma = 1 \tag{3}$$

Passengers pertaining to the first category amount to αa_i in zone I and bring about a transport demand p'_{ij}, which is proportional to the number of jobs in zone J and in reverse ratio to power λ of coefficient C .

The gravitational type relationship is:

$$p'_{ij} = k\alpha \frac{a_i e_j}{c_{ij}^{\lambda}} \tag{4}$$

The sum total of all the passengers pertaining to category [α], living in zone I and working in all the zones J is equal to $\alpha\, a_i$:

$$\sum_{j=1}^{m} p'_{ij} = \alpha a_i \tag{5}$$

Coefficient k in relationship (4) is found by means of relationship (5):

$$\sum_{j=1}^{m} p'_{ij} = k\alpha a_i \sum_{j=1}^{m} \frac{e_j}{c_{\hat{ij}}}$$

hence:

$$k = \frac{1}{\sum_{j=1}^{m} \frac{e_i}{c_{\hat{ij}}}}$$

and thus (4) becomes:

$$p'_{ij} = \alpha \frac{a_i e_j}{c'_{ij}} + \frac{1}{\sum_{j=1}^{m} \frac{e_j}{c_{\hat{ij}}}} \tag{4'}$$

Passengers pertaining to the second category amount to $\beta\, e_j$ in zone J and bring about a transport demand p''_{ij}, which is proportional to the number of jobs in zone I and in reverse ratio to the power λ of coefficient C_{ij}.

The gravitational type relationship is:

$$p''_{ij} = k\beta \frac{e_j a_i}{c'_{ij}} \times \frac{1}{\sum_{i=1}^{m} \frac{a_i}{c_{\hat{ij}}}} \tag{6}$$

Coefficient k is found like in the previous instance using relationship (7).

The sum total of all the passengers pertaining to category [β] is:

$$\sum_{i=1}^{m} p''_{ij} = \beta e_j \tag{7}$$

Passengers pertaining to the third category amount to γN and bring about a transport demand

$$q_{ij} = \gamma \frac{a_i e_j}{N} \tag{8}$$

knowing

— the total transport demand of persons residing in zone I and working in zone J, yielded by relationship

$$\tau_{ij} = p'_{ij} + p''_{ij} + q_{ij} \tag{9}$$

— the total demand for dwellings in zone I, yielded by relationship

$$\sum_{j=1}^{m} \tau_{ij}$$

— the total demand for jobs in zone J, yielded by relationship

$$\sum_{i=1}^{m} \tau_{ij}$$

The differences between demands $\sum_{j=1}^{m} \tau_{ij}$ and supplies a_i are yielded by relationship

$$a_i - \sum_{j=1}^{m} \tau_{ij} = a_i - \sum_{j=1}^{m} p'_{ij} - \sum_{j=1}^{m} p''_{ij} - \sum_{j=1}^{m} q_{ij} =$$

$$= a_i - \alpha a_i - \sum_{j=1}^{m} p''_{ij} - (1 - \alpha - \beta) \sum_{j=1}^{m} \frac{a_i e_j}{N} =$$

$$a_i - \alpha a_i - \sum_{j=1}^{m} p''_{ij} - \sum_{j=1}^{m} \frac{a_i e_j}{N} + \alpha \sum_{j=1}^{m} \frac{a_i e_j}{N} + \beta \sum_{j=1}^{m} \frac{a_i e_j}{N}; \qquad (10)$$

Because:

$$\sum_{j=1}^{m} \frac{e_j}{N} = \frac{N}{N} = 1$$

(10) becomes

$$a_i - \sum_{j=1}^{m} \tau_{ii} = a_i - \alpha a_i - \sum_{j=1}^{m} p''_{ij} - a_i + \alpha a_i + \beta a_i \qquad (10')$$

Writing

$$a''_i = \frac{1}{\beta} \sum_{j=1}^{m} p''_{ij} \qquad (11)$$

We obtain

$$a_i - \sum_{j=1}^{m} \tau_{ij} = \beta(a_i - a''_i) \qquad (10'')$$

Similarly, for the differences between the demands $\sum_{i=1}^{m} \tau_{ij}$ for jobs and supplies we obtain:

$$e_j - \sum_{i=1}^{m} \tau_{ij} = \alpha(e_j - e'_j) \qquad (12)$$

where

$$e'_j = \frac{1}{\alpha} \sum_{j=j}^{m} p'_{ij} \qquad (13)$$

It would be ideal that both demand for jobs and demand for dwellings were met and, consequently, the equations were called off. However, in practice there is always a play of demand and supply which makes equations (10″) and (12) not to yield nought as a result.

The variation of the demand for jobs and of the demand for dwellings brings about a transport variation in the sense that an increase in the number of jobs or of dwellings in a zone induces an increase in the number of passengers participating in the interzone traffic.

The traffic between zone I and zone J has a constant component and a variable component at one time, which is similar to the voltage variation at one time in an electric circuit with elements R, L, C:

$$n_{ij} = n_{ij}^{constant} + n_{ij}^{variable} \tag{14}$$

The constant component $n_{ij}^{constant}$ represents the original situation of traffic n_{ij}, that is

$$n_{ij}^{constant} = p'_{ij} + p''_{ij} + q_{ij} \tag{15}$$

The variable component $n_{ij}^{variable}$ represents the variations of traffic n_{ij} as a consequence of the play between demand and supply.

$$n_{ij}^{variable} = \alpha(e_j - e'_j)\frac{a_i}{N} + \beta(a_i - a''_i)\frac{e_j}{N}. \tag{16}$$

Relationship (16) stands for the sum of variations (see relationships 10″ and 12) that have occurred as a result of the increase in the number of jobs in zone J referred to the inhabited zone I, and of the increase in the number of dwellings in zone I referred to the working zone J.

By replacing (15) and (16) in (14) we get:

$$n_{ij} = p'_{ij} + p''_{ij} + q_{ij} + \alpha(e_j - e'_j)\frac{a_i}{N} + \beta(a_i - a''_i)\frac{e_j}{N} \tag{17}$$

This relationship is the equation of traffic between zone I and zone J at one time.

In agreement with relationships (4′), (16) and (18), equation (17) becomes:

$$n_{ij} = \alpha \frac{a_i e_j}{c_{\hat{ij}} \sum_{i=1}^{m} \frac{e_j}{c_{\hat{ij}}}} + \beta \frac{a_i e_j}{c_{\hat{ij}} \sum_{i=1}^{m} \frac{a_i}{c_{\hat{ij}}}} + \tag{17′}$$

$$+ (1 - \alpha - \beta)\frac{a_i e_j}{N} + \alpha(e_j - e'_j)\frac{a_i}{N} + \beta(a_i - a''_i)\frac{e_j}{N}$$

we write:

$$\theta_{ij} = k_{ij} \cdot 1 \frac{1}{c_{ij}} \tag{18}$$

where k_{ij} are proportionality coefficients having value 1 or 0. The k_{ij} for which they meet conditions:

$$\left.\begin{aligned} n_{ij} &< s \\ n_{ij} &< 0{,}1\, a_i \\ n_{ij} &< 0{,}1\, e_i \end{aligned}\right\} \quad \text{such as} \quad \left[\sum_{i=1}^{m}\sum_{j=1}^{m}(n_{ij} > s)\right] \geqslant 0{,}8N \tag{19}$$

will have value 0.

All the other k_{ij} have value 1.

This means that, practically, the equations for which traffic n_{ij} has got small values shall not be represented.

By replacing in (17′) we get:

$$n_{ij} = \alpha \frac{a_i e_j \theta_{ij}}{k_{ij} \sum_{j=1}^{m} \frac{\theta_{ij} e_j}{k_{ij}}} + \beta \frac{a_i e_j \theta_{ij}}{k_{ij} \sum_{i=1}^{m} \frac{\theta_{ij} a_i}{k_{ij}}} + \frac{a_i e_j}{N} - \alpha \frac{a_i e_j}{N} - \beta \frac{a_i e_j}{N} +$$

$$+ \alpha \frac{a_i e_j}{N} - \alpha \frac{a_i e'_j}{N} + \beta \frac{a_i e_j}{N} - \beta \frac{a''_i e_i}{N} \qquad (17'')$$

Write:

$$n'_{ij} = \frac{\theta_{ij} a_i e_j}{\sum_{k=1}^{m} \theta_{ik} e_k}; \quad n''_{ij} = \frac{\theta_{ij} a_i e_j}{\sum_{n=1}^{m} \theta_{nj} a_n} \qquad (20)$$

and replacing in (17″) we get:

$$n_{ij} = \alpha n'_{ij} + \beta n''_{ij} + \frac{a_i e_j}{N} - \alpha \frac{a_i e'_j}{N} - \beta \frac{a''_i e_j}{N} \qquad (17'')$$

Equation

$$n_{ij} = \alpha \left(n'_{ij} - \frac{a_i e'_j}{N} \right) + \beta \left(n''_{ij} - \frac{a''_i e_j}{N} \right) + \frac{a_i e_j}{N} \qquad (17'')$$

is the equation of the passengers' traffic between zones I and J, where n'_i, n''_{ij} e'_j and a''_i are yielded by relationships (20), (13) and (11).

Experience (with Paris, Brussels, Baltimore, etc.) has proved that the parameters α, β, and λ, have very small variations in time. Thus, for Baltimore, β had a value of 0.705 in 1926, and 0.642 in 1946.

The characteristic coefficients of a town α, β, λ, are found by imposing condition

$$n_{ij} = n^*_{ij} \qquad (21$$

where n^*_{ii} are data representing the number of passengers travelling from zone I to zone J of the city. These data are obtained through a survey sample of the city inhabitants.

The number of passengers involved by the model is yielded by relationship

$$\sum_{i,j} (n^*_{ij} - n_{ij}) > 0 \qquad (21')$$

For λ once settled, equations (17^{IV}) are linear in α and β.

If there is a value of α for which the family of straight lines representing the above equations in the plane of co-ordinates (α, β) make up a pencil whose vertex pertains to field:

$$0 \leqslant \alpha + \beta \leqslant 1, \tag{22}$$

the model is variable. Otherwise, a system of divergent straight lines is obtained for any value of λ.

The family of straight lines (17^{IV}) does not represent a pencil of straight lines with an intersection point for all the straight lines of the family. For this reason we shall find a point tending to the vertex of the pencil. This point is found from the condition that the sum of the squared distances is minimal, which means that the point thus found is the closest to all the straight lines of the family.

The intersections of equations (17^{IV}) with the axes of co-ordinates (α, β) are yielded by relationships:

$$\alpha_{ij} = \frac{n^{\circ}_{ij} - \frac{a_i e_j}{N}}{n'_{ij} - \frac{a_i e_j}{N}} \tag{23}$$

$$\beta_{ij} = \frac{n^{\circ}_{ij} - \frac{a_i e_j}{N}}{n''_{ij} - \frac{a''_i e_j}{N}}$$

Hence, equations (17^{IV}) can be written under the form

$$\frac{\alpha}{\alpha_{ij}} + \frac{\beta}{\beta_{ij}} = 1 \text{ for } i, j = 1, \ldots N \tag{24}$$

The distances from one point (α, β) to the straight lines (24) are yielded by relationship

$$d_{ij} = \frac{\beta_{ij}\alpha + \alpha_{ij}\beta - \alpha_{ij}\beta_{ij}}{\pm \sqrt{\beta^2_{ij} + \alpha^2_{ij}}} \tag{25}$$

The point required (α_0, β_0) is the minimal point of function:

$$D(\alpha, \beta) = \sum_{i=1}^{m} \sum_{j=1}^{m} d^2_{ij} \tag{26}$$

which is obtained by solving the system

$$\begin{cases} \dfrac{\partial D(\alpha,\beta)}{\partial \alpha} = 0 \\ \dfrac{\partial D(\alpha,\beta)}{\partial \beta} = 0 \end{cases} \quad \text{namely} \quad \begin{cases} \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} 2d_{ij}\dfrac{\partial d_{ij}}{\partial \alpha} = 0 \\ \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} 2d_{ji}\dfrac{\partial d_{ij}}{\partial \beta} = 0 \end{cases} \tag{27}$$

In this way the city structure parameters, for an arbitrarily chosen value of λ have been found.

Some other values for λ have also been calculated.

Out of all the values of α, β, λ thus found, the pair minimizing relationship (21′) is chosen.

In order to find the passengers flow between diverse zones of the city, the Bucharest Transport Company carried out a sample survey in 1969. To that end, about one million questionnaires were circulated, of which approximately 350,000 were retrieved, that is 45%.

It follows, from similar works [1], [2], [3], that for a sample of 30 to 40% of the surveyed collectivity, more than 80% of the travels are correctly described with a probability of 95%. Consequently, it was thought that the number of questionnaires retrieved and found valid was adequate for proceeding to data ordering.

By applying the model of preferential balance to the structure of Bucharest, divided into 33 zones, for the gross city coeficients the following values were obtained:

$$\alpha = 0.12; \quad \beta = 0.19; \quad \gamma = 0.69 \qquad \lambda = 3.9$$

These data are of service to the prediction of the prospective transport demand. Applying relationships (14), (15), (16) where a_i, e_j and N are data included in the city development and systematization plans and α, β, γ, λ, are ascertained from the "model of preferential balance", n_{ij} for the required period is calculated.

This work has been programmed on an IBM 360/40 computer. Its flow chart for automatic computation is given in Fig. 3, where:

1. Data checking unit: checks the data of matrix N_{ij} and C_{ij} punched on the cards:
   - — cards ordering;
   - — numerical checking of cards;
   - — maximal value checking;
   - — supplementary checking of matrix N_{ij}, if the line and column sum totals are closed.
2. Finding k_{ij} in agreement with relationship (19).
3. Values of λ shall be surveyed beginning with $\lambda = \lambda$ initial.
4. Finding intersections α_{ij}, β_{ij} and errors proceeds in accordance with relationships (23) and (21′) for the value λ settled.

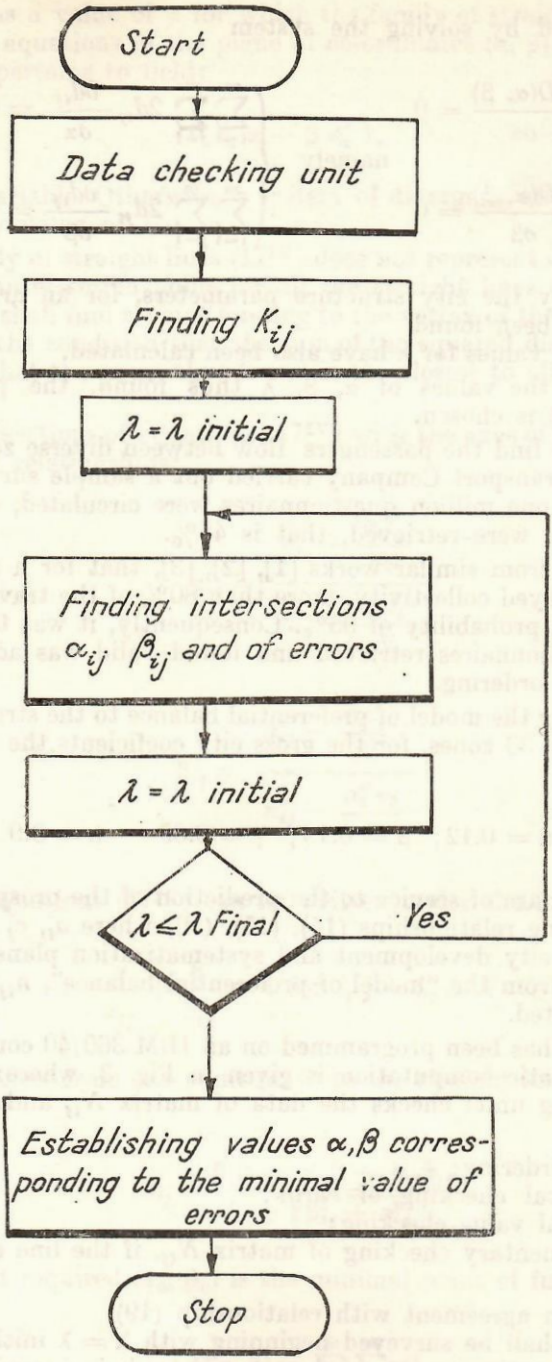
Start
Data checking unit
Finding K_{ij}
$\lambda = \lambda$ initial
Finding intersections α_{ij} β_{ij} and of errors
$\lambda = \lambda$ initial
$\lambda \leq \lambda$ Final
Yes
Establishing values α, β corresponding to the minimal value of errors
Stop

Fig. 3

5. A new value $\lambda = \lambda_{initial} + \Delta\lambda$ is put forward.
6. If λ is smaller or equal to λ_{final}, the computation of parameters for the new value of λ is resumed. Otherwise, one passes to unit 7.
7. The minimal value of the error is found, and to this end α, β are established.

RÉFERENCES

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LE MODELAGE MATHEMATIQUE DE LA PREVISION DU NECESSAIRE DE TRANSPORT (LA MATRICE: POINT DE DEPART—DESTINATION)

Dans cet article on présente „le modèle de l'équilibre de préférence" à l'aide duquel on réalise le modelage mathématique de la prévision de la demande de transport.

Pour une zone de la ville, le modèle définit les grandeurs:

a_i — voyageurs qui habitent I;

e_j — voyageurs qui travaillent à J;

n_{ij} — voyageurs qui se deplacent entre la zone I et la zone J;

N — le nombre total des voyageurs de la ville ainsi qu'une répartition des voyageurs par rapport à la priorité accordée, dans le choix, soit au domicile, soit au lieu de travail pour les catégories:

α — domicile fixe, β — lieu de travail fixe, γ — choix indépendant

Dans la première phase on détermine les coefficients caractéristiques de la ville pour la période donnée α et β et une grandeur dépendant du réseau de transport λ; en exprimant la sollicitation de transport de tous les voyageurs:

$$\tau_{ij} = f_{ij}(\alpha, \beta, \lambda, a_i, e_j, N)$$

et mettant la condition du présent $\tau_i = n_{ij}$ (d'un sondage) a_i, e_j, N connus et tenant compte du fait que pour λ fixe l'équation est linéaire, on détermine ce λ pour lequel la famille des droites forme un fascicule dans le plan, (α, β,) α_o, β_o étant les coordonnées de l'angle.

Dans la deuxième phase α_o β_o, λ, a_i, e_j, N étant connus, on détermine pour l'avenir n_{ij}.

L'auteur présente les résultats de l'application du modèle à l'étude de la ville de Bucarest.

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ПРОГНОЗА ПОТРЕБНОСТИ В ПЕРЕВОЗКАХ (МАТРИЦА «МЕСТО ОТПРАВКИ—МЕСТО НАЗНАЧЕНИЯ»)

В настоящей статье приводится «модель предпочтительного равновесия», которая осуществляет математическое моделирование прогноза спроса на перевозки. По зональной структуре города модель определяет величины:

a_i — пассажиры, проживающие в зоне I;

e_j — пассажиры, работающие в зоне J;

n_{ij} — пассажиры, совершающие поездки между зоной I и зоной J;
N — общее число пассажиров в городе с разбивкой пассажиров — в зависимости от предпочтения при выборе либо местожительства, либо места работы — на категории:
α — постоянное местожительство,
β — постоянное место работы,
γ — независимый выбор.

В первой стадии определяются характерные коэффициенты города для данного периода α и β и зависящая от транспортной сети величина λ; обозначив спрос на перевозки всех пассажиров:

$$\tau_{ij} = f_{ij}\ (\alpha,\ \beta,\ \lambda,\ a_i,\ e_j,\ N)$$

и поставив условие, что в настоящее время $\tau_{ij}=n_{ij}$(на основе выборочного обследования), где a_i, e_j, N известны, а также учитывая то обстоятельство, что для постоянной величины λ уравнение является линейным, автор определяет ту величину λ, при которой семейство прямых образует пучок в плоскости (α, β), причем α_0, β_0 являются координатами вершины.

Во второй стадии, когда известны α_0, β_0, λ, a_i, e_j, N, определяется на будущее n_{ij}.
В статье приводятся результаты применения модели при изучении г. Бухареста.