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# Labour Efficiency and Optimal Strategies

*Strategic problems can be solved apart from the tactical ones when the best strategy, even under the most unsuccessful tactics is more useful than any other strategy under any tactics.*

1. Consider the following variables:

$x$ = expressed in units of labour efficiency;

$y$ = labour equipment, expressed in productive assets per capita;

$z$ = final social product per capita.

We provide no details with regard to the measuring of labour efficiency since we assume it is a problem of economic analysis; however, we shall bear in mind that the concept of labour efficiency or concepts having a similar meaning appear in the specialty literature. In this respect, we shall remember [1], where the measuring of labour quality is discussed.

Consider the following hypotheses:

$$x, y, z > 0 \tag{1}$$

$$z = f(x, y) \text{ is } 2 - \text{derivable} \tag{2}$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \geqslant 0 \tag{3}$$

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2} \leqslant 0 \tag{4}$$

Hypothesis (1) is natural, since labour efficiency, labour equipment and the final social product are expressed in positive numbers.

Hypothesis (2) is current with production functions; herein, derivability shall be considered also with respect to distributions.

Hypothesis (3) is also natural, illustrating the fact that the final social product is a non-decreasing function in relation to labour efficiency and to labour equipment.

Hypothesis (4) illustrates the idea of marginal non-increasing efficiency, which is also current with production functions.

2. Consider $z$ to be a scalar field, with reference to a Cartesian system of co-ordinates having its domain of definition in the first quadrant.

The equiscalar lines (also termed *isoquanta* in economy) are yielded by equation:

$$f(x, y) = z_0 = \text{constant} \tag{5}$$

On the strength of hypotheses (3) and (4), the equiscalar lines will be shaped as shown in Fig. 1.

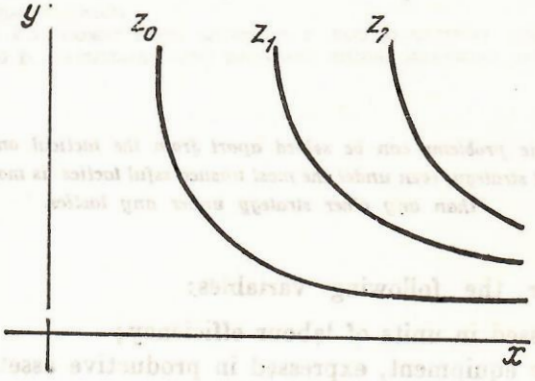


Fig. 1

From (5) we infer:

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \tag{6}$$

which is the differential equation of isoquanta.

The field lines are yielded by the differential equation:

$$\frac{dy}{dx} = \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}} \tag{7}$$

We see in (6) and (7) that the isoquanta and the field lines are non-decreasing and assume the shape illustrated in Fig. 2.

The field lines are trajectories along which the field variation is maximal; as we know, in any point the gradient of the scalar field shows the direc-

tion and sense towards which the scalar field varies with a maximal rate, whereas the gradient vector is tangential to the field line in any point. Writing $\vec{i}$ and $\vec{j}$ for the versors, the gradient is expressed by:

$$\text{grad } z = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \tag{8}$$

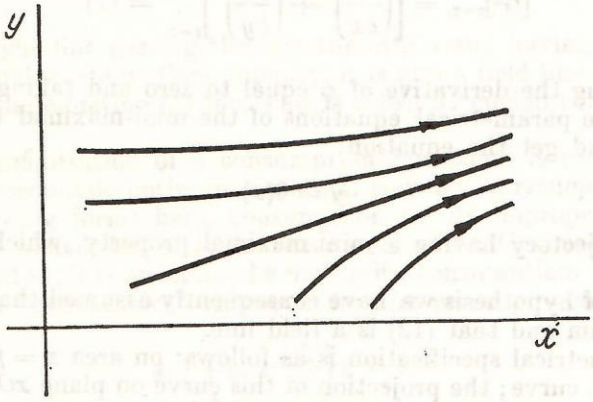


Fig. 2

Following the direction indicated by the arrow, the field lines represent the trajectories of the maximal growth rate (irrespective of the growth velocity).

On the strength of hypothesis (4) we have:

$$\text{div } z = \frac{\partial^2 f}{\partial_x^2} + \frac{\partial^2 f}{\partial_y^2} \leqslant 0 \tag{9}$$

which means that the divergence of the field is non-positive in every point. This illustrates the presence of absorbent sources, which is an expression equivalent to the „marginal non-increasing efficiency", but in terms pertaining to the field theory.

3. There is one field line having the property of representing a mini-maximal growth trajectory, that is to say there is a mini-maximal strategy. Let us make clearer this idea: Point 1 in Fig. 3 marks a circumstance under which labour equipment is high but labour efficiency is lower than the efficiency expected from such equipment. In point 2, labour efficiency is higher than labour equipment. In point 3, labour efficiency coincides with labour equipment. We have one such point on each isoquantum. Joining all the 3-type points we get a trajectory; the hypothesis considered assumes that this trajectory is a field line and, consequently, it is one of the trajectories of the maximal growth rate. Besides, we assume that this trajectory is characterized by a minimum of the maximal rates, that is to say it illustrates a strategy of a mini-maximal type.

In mathematical terms, we have in view to calculate the gradient along isoquantum $z =$ constant and we find the expression of the squared modulus of the gradient as a function of $x$ alone:

$$f(x, y) = z_0 = \text{constant} \tag{10}$$

$$[G^2]_{z=z_0} = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]_{z=z_0} = (x) \tag{11}$$

By making the derivative of $\varphi$ equal to zero and taking into account (11), we get the parametrical equations of the mini-maximal trajectory; we eliminate $z_0$ and get the equation:

$$y = \theta(x) \tag{12}$$

that is the trajectory having a mini-maximal property, which satisfies also equation (7).

By way of hypothesis we have consequently assumed that $\varphi(x)$ shows a unique minimum and that (12) is a field line.

The geometrical specification is as follows: on area $z = f(x, y)$ there is a minimal slope curve; the projection of this curve on plane $xOy$, is the curve $y = \theta(x)$, which is, at the same time, also a field line.

Let us illustrate this property for a Cobb-Douglas function with technical progress:

$$z = xy^{\alpha}$$

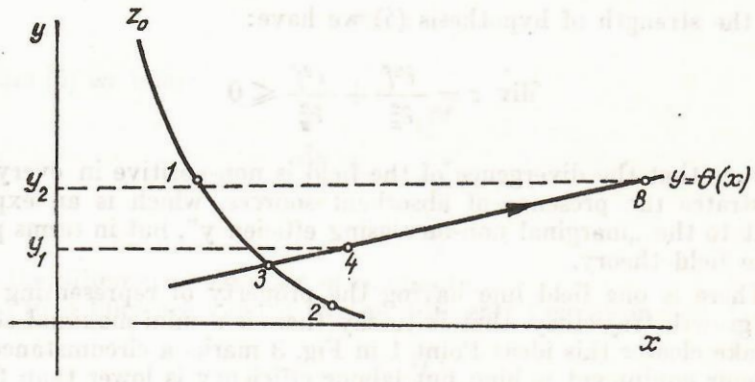


Fig. 3

where $x$ is the technical progress interpreted as labour efficiency, $y$ is labour equipment, and $z$ is the final social product. The gradient is:

$$\text{grad } z = \frac{z}{x}\vec{i} + \alpha\frac{z}{y}\vec{j}$$

and the squared modulus of the gradient, for isoquantum $z = z_0 = \text{constant}$, becomes:

$$(G^2)_{z=z_0} = z_0^2 \left( \frac{1}{x^2} = \frac{\alpha^2}{y^2} \right) = z_0^2 \left[ \frac{1}{x^2} + \alpha^2 \left( \frac{x}{z_0} \right)^{\frac{2}{\alpha}} \right].$$

By making the derivative equal to zero and eliminating $z_0$ we get:

$$y = \sqrt{\alpha} x \tag{12}$$

which is a straight line passing through the origin and having the property of a mini-maximal strategy. Consequently, it is also a field line, that is a solution to differential equation (7) too. This is called a "golden rule in econometrics".

4. The maximization of a consumption functional is usually assumed to be the macroeconomic optimum criterion. Now let us resume this problem under the following form: here, consumption (in its improper meaning) is considered to be what is left after the share assigned for the production means had been subtracted; it is understood a per capita consumption. If we assume a dynamics of an "evaporative" character for the per capita means, we get:

$$y + \left( \mu + \frac{\dot{L}}{L} \right) y = sz \tag{13}$$

where $\mu$ is the wearing out factor of production means (a positive constant), $L$ is the amount of population assumed as labour, and $s$ is the fraction of the final social product assigned for production means. The derivative related to time has been marked by a dot above. We write:

$$\mu + \frac{\dot{L}}{L} = \lambda$$

and thus the expression of per capita consumption becomes:

$$C = z - sz = z - \dot{y} - \lambda y \tag{14}$$

We consider the functional to be maximized under the form:

$$I = \int_{t_0}^{t} (z - \dot{y} - \lambda y) \, \mathrm{d}t \tag{15}$$

The extremals must satisfy Euler's equation:

$$z - \lambda y = K \tag{16}$$

where $K$ is an arbitrary constant.

In order to obtain an optimal strategy, it is required that (16) shall take place at the same time with (12), that is to maximize consumption under an agreement between labour efficiency and labour equipment. Thus we get the system:

$$\begin{cases} f(x, y) - \lambda y = k \\ y = \theta(x) \end{cases} \tag{17}$$

Assuming $\lambda$ = constant, if the system yields a solution $y = y_1$, $x = x_1$, $z = z_1$, then this solution is the stationary solution which is known to be an optimal strategy. The condition $\lambda$ = constant can be explained by $\dot{L} = 0$, that is a constant population (an illustration of simple reproduction); another possibility is $\frac{\dot{L}}{L}$ = constant, that is an exponential increase of population. This is illustrative for simple reproduction when extended to a population varying exponentially. When $\lambda$ is variable, the solution to system (17) illustrates simple reproduction extended to a variable population. In all these instances, functional $I$ varies linearly in relation to $t$.

5. With regard to expanded reproduction, some interesting qualitative aspects can be inferred from the study of jump development of production meanas per capita. It is true that jump development does not occur currently in practice nor is it characteristic of socialist economy; however, the conclusions that have been reached are of interest. Let us take, for instance, the shape of the mini-maximal trajectory in Fig. 3 through a jump from point $A$, where $y = y_1$ = constant, to point $B$, where $y = y_2$ = constant. Before and after the jump there is simple reproduction, with $\lambda$ = constant, that is to say $x$ and $z$ are constant before and after the jump. Hence, we have as an expression of consumption:

$$C = \begin{cases} z_1 - \lambda y_1 & \text{for } t < 0 \\ -(y_2 - y_1)\delta_\tau & \text{(concentrated in point } t = \tau) \\ z_2 - \lambda z y_2 & \text{for } t > \tau \end{cases}$$

where $\tau$ is the moment of the jump and $\delta_\tau$ is the concentrated Dirac's distribution at moment $\tau$. Considering functional $I$ for $t_0 = 0$, we get:

$$I = \begin{cases} (z_1 - \lambda y_1)t & \text{for } t < \tau \\ (z_2 - \lambda y_2)t & \text{for } t > \tau \end{cases}$$

and in point $\tau$ there is a jump of $I$ (that is a decrease) equal to the jump of $y$.

The plotting of $C$ and $I$ is illustrated in Fig. 4, provided that $z_2 - z_1 > \lambda (y_2 - y_1)$. We can see that jump development cannot take place without a decrease in per capita consumption at the moment of the jump, but it can be followed by an increase after the moment of the jump. With regard to cumulated consumption, we can see in the plotting of $I$ that there is an area (the hatched area) illustrating the accumulation "effort", after which cumulated consumption is retrieved at the initial level, in point $\tau'$, but with a sharper increase slope. For a bigger jump, the slope of $I$ increases after the jump, but the effort area increases too (the dotted straight line in the plotting of $I$). The essential problem is the impact of the increase in the effort area as against labour efficiency. We may expect that too sharp an increase of the jump (which means the extension of the hatched area) will result in a decrease of labour efficiency. This problem must be cleared up through economic analysis before carrying out a mathematical study of it.

Similar results are obtained if we assume that expanded reproduction is to take place at the same time with a linear increase in the means of production, starting from a moment $\tau$ until which simple reproduction has taken

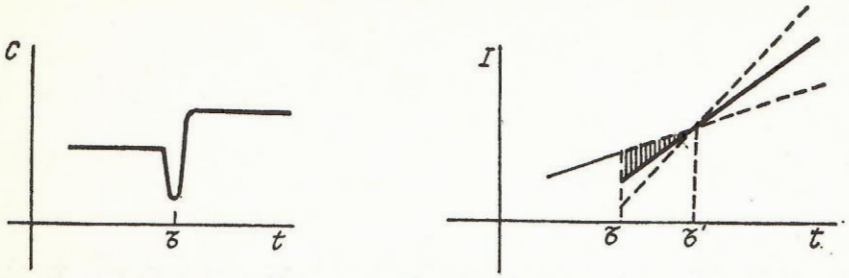


Fig. 4

place. The effort involved by the transition from the dynamics of simple reproduction type to the dynamics of expanded reproduction was already analyzed by K. Marx in his "Capital".

REFERENCES

1. EMILIAN DOBRESCU: *Ritmul creşterii economice.* Editura Politică, Bucureşti 1968.

## L'EFFICIENCE DE LA MAIN-D'OEUVRE ET LES STRATEGIES OPTIMALES

On étudie une propriété des fonctions de production à progrès techniques du genre d'une "loi d'or", mais qui fait ressortir un principe de mini-maxi; on définit une trajectoire optimale de croissance qui est caracterisée par un minimum de rythmes maxima de croissance, ce qui mène à maximiser une fonction de consommation.

Le cas du développement par bond d'une trajectoire optimale à une autre permet une analyse plus profonde du passage de la reproduction simple à la reproduction élargie.

## ЭФФЕКТИВНОСТЬ РАБОЧЕЙ СИЛЫ И ОПТИМАЛЬНЫЕ СТРАТЕГИИ

Изучается свойство функций производства, связанное с техническим прогрессом, в виде «золотого закона», выявляющее принцип мини-максимума; определяется оптимальная траектория роста, характеризующаяся минимумом максимальных темпов роста, что ведет к максимизации функционального уравнения потребления.

Пример скачкообразного развития от одной оптимальной траектории до другой делает возможным более глубокий анализ перехода от простого воспроизводства к расширенному.