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Shortest Route Subject to Improvements Through Dynamic Programming

The programming theory, to which dynamic programming pertains, is to be understood as part of the general knowledge of rational activity.

INTRODUCTION

A network is defined by a set of points called nodes and a set of arcs, each of which joins two nodes. For a shortest route problem, each arc is associated, a positive number representing its length. In the given set of nodes, one node is termed as source and the other as sink. The classical routing problem is then to determine the shortest route and its length from source to sink. Many papers discussing the variants of the original shortest route problem have appeared (1 to 7). Many a times it is physically possible to provide short cuts between any two nodes by expending money in various ways, for example, we may install one more machine to speed up the work. Therefore a more general situation of practical interest could be that where it is possible to improve the existing link and thus reducing the effective length. All improvements cause expenditure, therefore, there is an upper limit to the number of improvements.

Let the original length l_{ij}^0 (joining nodes i and j) after giving $1, 2, 3, \ldots, k$ improvements becomes $l_{ij}^1, l_{ij}^2, l_{ij}^3, \ldots, l_{ij}^k$ such that $l_{ij}^0 \geqslant l_{ij}^1 \geqslant l_{ij}^2 \geqslant \ldots \geqslant l_{ij}^k \geqslant 0$. The problem is to distribute optimally a given number k of improvements in the entire path such that the length of the path joining source to sink is minimised.

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The following two different cases will be considered:

(1) determination of optimal route and its length with improvements in a *directed network.*

(2) determination of optimal route and its length with improvements in a *non-directed network.*

In order to solve the above general problems, in this paper, functional equations have been developed by using Bellman's principle of optimality. Numerical examples have been presented.

CASE I: The shortest route through a Directed network.

Define f_i^k = the minimal length of the path from node i to N where 'inbetween links' are subject to a total k number of improvements which are distributed optimally.

Using Bellman's Principle of optimality, we write

$$f_i^k = \min_j \, | l_{ij}^r + f_j^{k-r} | \qquad (1)$$

$$j > i \qquad \text{with } 0 \leqslant r \leqslant k$$

where we keep the understanding that all pair of nodes are connected such that $l_{ij} = \infty$ for an arc which is in fact missing. Equation (1) is a recurrence relation and holds for $i = N - 1, N - 2, ..., 2, 1$.

The boundary condition corresponding to equation (1) is evidently

$$f_N^k = 0 \; \forall \; k \; \varepsilon \, I^+ \text{ (set of positive integers)} \qquad (2)$$

because when the node N is reached, no further movement is then required.

Numerical Example

To illustrate the computation, the network in Figure I is considered for finding out the shortest path between nodes 1 to 4 by providing at the most, two improvements in the entire path.

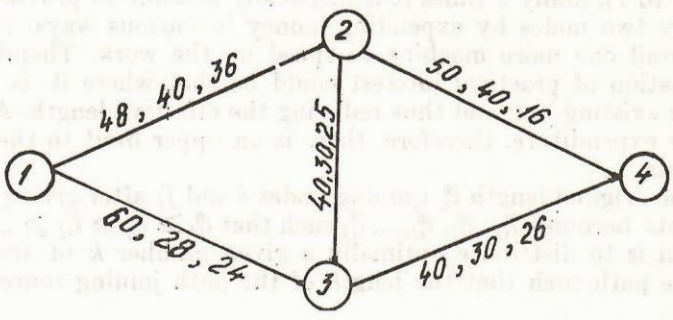


Fig. 1

The numerical figures on each link denote the values of l_{ij}^k where $i, j = 1, 2, 3, 4$, and $k = 0, 1, 2$. In this example every improvement reduces the length of the path but there shall be no loss of generality in computation if we come

across cases where $l_{ij}^{r-1} = l_{ij}^{r}$ meaning thereby that an improvement is not possible over the arc which may reduce its length. The computations are made as follows:

Step I At node $N = 4$, by considering equations (2);

$$f_4^k = 0;\ k = 0,\ 1,\ 2.$$

Step 2 At node 3 i.e. taking $i = 3 = N - 1$, using equation (1),

we write $f_3^k = min\ \{1_4^{k-r}\}$ with $R = 0,\ 1,\ 2.$

$$f_3^0 = 40\ (3 \overset{0}{\rightarrow} 4),$$

$$f_3^1 = 30\ (3 \overset{1}{\rightarrow} 4),$$

$$f_3^2 = 26\ (3 \overset{2}{\rightarrow} 4).$$

The figures in parenthesis indicate the path. Numerical figures over arrow indicated the number of improvements on that link.

Again using equation (1) for node 2, we get

$$f_2^k = min\,|\,(l_{24}^k + f_4^{k-r}),\ (l_{23}^r + f_3^{k-r})\,|$$
$$R = 0,\ 1,\ 2$$

This gives

$$f_2^0 = 50\ (2 \overset{0}{\rightarrow} 4)$$

$$f_2^1 = 40\ (2 \overset{1}{\rightarrow} 4)$$

$$f_2^2 = 16\ (2 \overset{2}{\rightarrow} 4)$$

finally from node 1 we have

$$f_1^0 = min\ \{(1_{12}^0 + f_2^0);\ (1_{13}^0 + f_3^0)\}$$
$$= min\ \{48 + 50;\ 60 + 40\}$$
$$= 98 \text{ having path } (1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$$

$$f_1^1 = min\ \{(1_{12}^0 + f_2^1),\ (1_{12}^1 + f_2^0);\ (1_{13}^0 + f_3^1),\ (1_{13}^1 + f_3^0)\}$$
$$= min\ \{48 + 40,\ 40 + 50;\ 60 + 30,\ 28 + 40\}$$
$$= min\ \{88,\ 90;\ 90,\ 68\}$$
$$= 68 \text{ having path } (1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$$

$$f_1^2 = min\ \{(1_{12}^0 + f_2^2),\ (1_{12}^1 + f_2^1),\ (1_{12}^2 + f_2^0);\ (1_{13}^0 + f_3^2),\ (1_{13}^1 + f_3^1),\ (1_{13}^2 + f_3^0)$$
$$= min\ \{48 + 16,\ 40 + 40,\ 36 + 50,\ 60 + 26,\ 28 + 30,\ 24 + 40\}$$
$$= 58 \text{ with paths } (1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$$

The above values of f_i^k are given in table I.

Table I

$\downarrow k$ \ $\rightarrow i$	1	2	3
0	$98(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$	$50(2 \overset{0}{\rightarrow} 4)$	$40(3 \overset{0}{\rightarrow} 4)$
1	$68(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$	$40(2 \overset{1}{\rightarrow} 4)$	$30(3 \overset{1}{\rightarrow} 4)$
2	$58(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$	$16(2 \overset{2}{\rightarrow} 4)$	$26(3 \overset{2}{\rightarrow} 4)$

From Column 1 in table I, we infer that the shortest route from node 1 to 4

1. without improvement is 98 $(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
2. with *one* improvement is 68 $(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
3. with *two* improvements is 58 $(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$.

CASE 2: The Shortest Path through a non-directed network.

The network of fig. 1 is again considered for determining the shortest path such that the movement in either direction along any arc is allowed. As before let l_{ij}^k denote the length of the arc from node i to j with k improvements. Further we assume that $l_{ij} = l_{ji}$ for all $i, j, \varepsilon N$.

Define $g_i^{k,n}$ = length of the shortest route from node i to node N such that the path consists of n distinct arcs with a total of k improvements distributed optimally.

The functional equation

$$g_i^{k,(n+1)} = \min_{j \neq i} \{l_{ij}^r + g_i^{(k-r),n}\} \tag{3}$$

$$i = N-1,\ N-2, \ldots 2,\ 1 \text{ and } i \neq N;\ n < N$$

defines a new sequence $\{g_i^{k,n}\}$.

The boundary conditions are given by

$$g_i^{k,1} = l_{iN}^k \tag{4}$$

$$i = 1,\ 2,\ \ldots,\ N-1$$

$$g_N^{k,n} = 0 \tag{5}$$

Let us now illustrate the calculations in the case of *non-directed* network problem of fig. 1. Here $l_{ij} = l_{ji}$ such that $i, j = 1, 2, 3,4$ and $N = 4$. We

now form a table II for distance matrix $L = (l_{ij}^k)$ where each row has k subrows. Here $k = 0, 1, 2$

Table II

$L = l_{ij}^k =$

$i\downarrow$	$j\rightarrow$ / $k\downarrow$	1	2	3	4
1	0	∞	48	60	∞
	1	∞	40	28	∞
	2	∞	36	24	∞
2	0	48	∞	40	50
	1	40	∞	30	40
	2	36	∞	25	16
3	0	60	40	∞	40
	1	28	30	∞	30
	2	24	25	∞	26
4	0				
	1	∞	∞	∞	∞
	2				

It may be noted that all elements in the last row of matrix L have been put equal to inifinity because it is not desirable to move further from node 4 after once we have reached there.

We now design another table III showing the values of $g_j^{k,1}$ from table II such that j denotes the column index and k the row index taking values 0, 1, 2. The number 1 in the superscript signifies the number of steps or arcs travelled in reaching node $N = 4$ from node j.

Table III

$n = 1$ (Route from j to N via only one arc)

$g_j^{k,1}$

k \ j	1	2	3
0	∞	50_4	40_4
1	∞	40_4	30_4
2	∞	16_4	26_4

The subscript, inside the square of the table points out the optimal destination to be reached from the current node j. There is no need of attaching a subscript on unbounded values.

In order to find the values of $g_j^{k,2}$ $(j = 1, 2, 3, 4; k = 0, 1, 2)$ the following procedure is adopted:

Determination of $g_1^{k,2}$

Step 1 The values of the row $(k = 0)$ in Table III are written on a card separately and moved over each subrow of 1 st row of Table II. The values obtained after addition are written in the first subrows of each main row in Table IV.

Step 2 Similarly the elements of row $(k = 1)$ in Table III are written on a card are moved over the first two subrows of main 1 st row in Table II. The resulting values are then written in the 2nd subrow positions with a tick mark (ν) in table IV.

Step 3 Exactly in a similar way the elements of row $(k = 2)$ in Table III are added with the elements of 1st subrow with 1st main row of Table II and written in the last line of Table IV.

Table IV

		k node i	1	2	3	minimum length of path with improvements
$g_{j=1}^{k,2}$	$g_1^{0,2}$	$g_j^{0,2} = l_{ji}^0 + g_i^{0,1}$	∞	$48 + 50 = 98$	$60 + 40 = 100$	$98(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
	$g_1^{1,2}$	$g_j^{1,2} = l_{ji}^1 + g_i^{0,1}$	∞	$40 + 50 = 90$	$28 + 40 = 68$	$68(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
		$= l_{ji}^0 + g_i^{1,1}$	∞	$48 + 40 = 88$	$68 + 30 = 90$	
	$g_1^{2,2}$	$g_j^{2,2} = l_{ji}^2 + g_i^{0,1}$	∞	$36 + 50 = 86$	$24 + 40 = 64$	
		$= l_{ji}^1 + g_i^{1,1}$	∞	$40 + 40 = 80$	$28 + 30 = 58$	$58(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$
		$= l_{ji}^0 + g_i^{2,1}$	∞	$48 + 16 = 64$	$60 + 26 = 86$	

Determination of $g_2^{k,2}$ In Table III elements of row $(k = 0)$ are written on a card and moved over all the subrows of main 2nd row of Table II. The values are obtained as in Step 1. Similarly Steps 2 and 3 are repeated. The value obtained in Steps 1, 2 and 3 are tabulated in Table V.

Table V

		node k / i	1	2	3	minimum length of path with improvements
$g_{j=2}^{k,2}$	$g_2^{0,2}$	$g_j^{0,2} = l_{ji}^0 + g_i^{0,1}$	∞	∞	80	$80(2 \overset{0}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
	$g_2^{1,2}$	$g_j^{1,2} = l_{ji}^1 + g_i^{0,1}$	∞	∞	70	$70(2 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
		$= l_{ji}^0 + g_i^{1,1}$	∞	∞	70	$\text{or}(2 \overset{0}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$
	$g_2^{2,2}$	$g_j^{2,2} = l_{ji}^2 + g_i^{0,1}$	∞	∞	65	
		$= l_{ji}^1 + g_i^{1,1}$	∞	∞	60	$60(2 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$
		$= l_{ji}^0 + g_i^{2,1}$	∞	∞	66	

Determination of $g_3^{k,2}$ These values are obtained as in Steps 1, 2 and 3 after considering main 3rd row in Table II and Table III as usual. These values are tabulated in Table VI.

Table VI

		node k / i	1	2	3	Minimum length of path with improvements
$g_{j=3}^{k,2}$	$g_3^{0,2}$	$g_j^{0,2} = l_{ji}^0 + g_i^{0,1}$	∞	90	∞	$90(3 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
	$g_3^{1,2}$	$g_j^{1,2} = l_{ji}^1 + g_i^{0,1}$	∞	80	∞	$80(3 \overset{1}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
		$= l_{ji}^0 + g_i^{1,1}$	∞	80	∞	$\text{or}(3 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 4)$
	$g_3^{2,2}$	$g_j^{2,2} = l_{ji}^2 + g_i^{0,1}$	∞	75	∞	
		$= l_{ji}^1 + g_i^{1,1}$	∞	70	∞	
		$= l_{ji}^0 + g_i^{2,1}$	∞	56	∞	$56(3 \overset{0}{\rightarrow} 2 \overset{2}{\rightarrow} 4)$

Determination of $g_4^{k,2}$ Elements of 4th row in Table II are combined with the row elements of Table III according to Steps 1, 2 and 3 mentioned beforehand. Values of $g_4^{k,2}$ are ∞.

The values in last columns of Tables IV, V and VI are written in a consolidated way in the following Table VII.

Table VII

$n = 2$ (Route from j to N via two arcs)

k \ j	1	2	3
0	$98(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$	$80(2 \overset{0}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$	$90(3 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
1	$68(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$	$70(2 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$ or $(2 \overset{0}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$	$80(3 \overset{1}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$ or $(3 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 4)$
2	$58(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$	$60(2 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$	$56(3 \overset{0}{\rightarrow} 2 \overset{2}{\rightarrow} 4)$

In a similar way, a Table showing the minimum length of path obtained in starting from node 1 and reaching node 4 for $n = 3$ is given below:

Table VIII

k \ j	1
0	$128(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
1	$118(1 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$ or $(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
2	$108(1 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 4)$

In a N node network problem the route connecting two given nodes can have at the most $(N - 1)$ links. Therefore, in the present problem of four nodes we need only the values corresponding to $n = 3$ and $j = 1$.

The shortest path between nodes 1 and 4 is obtained by taking the minimum value from Column 1 of Tables III, VII and VIII. This is given below in Table IX.

Table IX

The shortest path for $k = 0, 1$ and 2.

k \ n	1	2	3	*Min.*
0 2	∞	$98(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$	$128(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$	$98(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$
1 2	∞	$68(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$	$118(1 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$ or $(1 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$	$68(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 4)$
2 2	∞	$58(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$	$108(1 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 2 \overset{0}{\rightarrow} 4)$ or $(1 \overset{1}{\rightarrow} 3 \overset{0}{\rightarrow} 2 \overset{1}{\rightarrow} 4)$	$58(1 \overset{1}{\rightarrow} 3 \overset{1}{\rightarrow} 4)$

Though this approach is quite general in nature but the calculations involved are evidently lengthy and tedious. Therefore, the problem involving more nodes with more number of improvements might be looked into in subsequent publication.

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LE CHEMIN MINIMAL DANS UN GRAPHE SOUMIS A L'AMELIORATION SUCCESSIVE PAR PROGRAMMATION DYNAMIQUE

Le problème de l'obtention d'un chemin minimal d'une station de départ jusqu'à une autre station de destination, dans un réseau de stations, est généralement un problème fondamental dans la recherche opérationnelle et notamment des chemins dans un réseau. Dans ce travail on considère la possibilité d'améliorer les chemins existants entre deux stations quelconques. Le problème considéré est de décider quelle liaison, ou liaisons, devraient être améliorées pour obtenir le chemin nominal entre deux stations pour un certain nombre d'améliorations dans le chemin entier.

Les équations fonctionnelles utilisant le principe de l'optimalité de Bellman ont été développées et résolues pour deux cas différents. Pour illustrer les calculs, on a fourni deux exemples numériques.

НАИМЕНЬШИЙ ПУТЬ В ГРАФЕ, ПОДВЕРГАЕМОМ ПОСЛЕДОВАТЕЛЬНЫМ УЛУЧШЕНИЯМ ПУТЕМ ДИНАМИЧЕСКОГО ПРОГРАММИРОВАНИЯ

Задача получения наименьшего пути от отправной станции до станции назначения в сети станций является, в общем, основной задачей при операционном исследовании и, в частности, задачей путей в сети. В настоящей статье рассматривается возможность улучшения путей, существующих между любыми двумя станциями. Рассматриваемая задача заключается в том, чтобы решить, какую связь — или связи — необходимо было бы улучшить для получения наименьшего пути между двумя станциями для данного числа улучшений всего пути.

Функциональные уравнения, использующие принцип оптимальности Беллмана, были развиты и решены для двух различных случаев. Для иллюстрирования расчетов даются численные примеры.