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The Industrial Process Control by Means of Cumulative Sum Charts Drawn Up on the Basis of the Dispersion of the Selection Amplitude

To investigate production and industrial systems means, in a way to investigate human life itself.

I. CUMULATIVE SUM CHARTS METHOD

A method meant for the preventive quality control in respect to the precision of a certain industrial process is the cumulative sum charts method, where the parameter under investigation is the dispersion of the measured quality characteristic.

By means of this type of charts we can detect the sharp variations of the precision parameter and, consequently, have a possibility to interfere so that the precision may acquire acceptable values in proportion to the length of the tolerance interval.

The charts are drawn up in agreement with certain volume selections; the volume will be fixed or variable from a selection to another. As statistics (selection functions) we shall consider

a) the selection dispersion s^2;

b) the selection amplitude w.

We shall deal further with the cumulative sum charts, making use of the results of some selections of a different volume and of some selections of the same volume.

II. CONTROL CHARTS BASED ON THE SELECTION DISPERSION

Consider an industrial process where X is a quality characteristic subject to control. This characteristic is a random variable defined by the population C of products, which we assume to be normal. The distribution of variable X will then be:

$$n(x;\ m,\ \sigma) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{-(x-m)^2}{2\sigma^2}},\quad x \in R \tag{1}$$

depending on parameters m and σ.

We assume that the process is subject to control and that the dispersion of the quality characteristic X for the whole batch is σ_0^2 (hypothesis H_0: $\sigma^2 = \sigma_0^2$).

We assume our target to be the detection by control of an increase in the dispersion from value σ_0^2 to value σ_1^2, where $\sigma_1 > \sigma_0$ (hypothesis H_1: $\sigma^2 = \sigma_1^2 > \sigma_0^2$).

In order to get the sequential ratio of probability underlying the drawing up of the cumulative sum control charts we make use of distinct volume selections, $n_1, n_2, \ldots, n_r$, the values obtained in each selection being:

$$\begin{aligned} &x_{11},\ x_{12},\ \ldots,\ x_{1n_1}\\ &x_{21},\ x_{22},\ \ldots,\ x_{2n_2}\\ &\vdots\\ &x_{r1},\ x_{r2},\ \ldots,\ x_{rn_r}. \end{aligned} \tag{2}$$

Consider $s_1^2, s_2^2, \ldots s_r^2$ to be the unshifted selection dispersions yielded by the relationship we know:

$$s_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 /_{n_i - 1}\quad i = 1,\ 2,\ \ldots,\ r \tag{3}$$

where $\bar{x}_i$ are the selection means assumed to be estimations of mean m.

The sequential ratio of probability comparing hypotheses H_0: $\sigma^2 = \sigma_0^2$ and H_1: $\sigma^2 = \sigma_1^2 > \sigma_0^2$ is obtained, following the r selections, by calculating the ratio of probabilities p_{1r} and p_{0r}, where

$$\begin{aligned} p_{1r} &= \prod_{i=1}^{r} \prod_{j=1}^{n_i} \left[\frac{1}{\sigma_1 \sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_{ij} - \bar{x}_i}{\sigma_1}\right)^2} \right]\\ p_{0r} &= \prod_{i=1}^{r} \prod_{j=1}^{n_i} \left[\frac{1}{\sigma_0 \sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_{ij} - \bar{x}_i}{\sigma_0}\right)^2} \right] \end{aligned} \tag{4}$$

Taking into account values p_{1r} and p_{0r}, the sequential ratio of probability is:

$$\rho = \frac{p_{1r}}{p_{0r}} = \left(\frac{\sigma_0}{\sigma_1}\right)^{\sum_{i=1}^{r}(n_i-1)} \cdot e^{-\frac{1}{2}\left[\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right]\sum_{i=1}^{r} s_i^2 (n_i-1)} \tag{5}$$

Wald's inequality yield:

$$\frac{\alpha_1}{1-\alpha_0} < \rho < \frac{1-\alpha_1}{\alpha_0} \tag{6}$$

where α_0, respectively α_1, are the risks of type I and II, that is to say α_0 is the probability of detecting a dispersion variation from σ_0^2 to σ_1^2 without its existing in reality, and α_1 is the probability of not detecting such a variation although it actually exists.

By making use of the inequality in the right side of relationship (6), that is the inequality corresponding to the hypothesis assumed, we have:

$$\left(\frac{\sigma_0}{\sigma_1}\right)^{\sum_{i=1}^{r}(n_i-1)} e^{-\frac{1}{2}\left[\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right]\sum_{i=1}^{r} s_i^2 (n_i-1)} < \frac{1-\alpha_1}{\alpha_0} \tag{7}$$

Writing $k_i = n_i - 1$ and calculating the logarithm, we get:

$$\sum_{i=1}^{r} k_i \ln \frac{\sigma_0}{\sigma_1} - \frac{1}{2}\left[\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right]\sum_{i=1}^{r} k_i s_i^2 < \ln \frac{1-\alpha_1}{\alpha_0} \tag{8}$$

or

$$\sum_{i=1}^{r} k_i s_i^2 > \frac{2}{\sigma_0^{-2}-\sigma_1^{-2}} \ln \frac{1-\alpha_1}{\alpha_0} + \frac{2\sum_{i=1}^{r} k_i \ln \frac{\sigma_1}{\sigma_0}}{\sigma_0^{-2}-\sigma_1^{-2}}. \tag{9}$$

If instead of inequality (9) we consider equality:

$$\sum_{i=1}^{r} k_i s_i^2 = \frac{2}{\sigma_0^{-2}-\sigma_1^{-2}} \ln \frac{1-\alpha_1}{\alpha_0} + \frac{2\sum_{i=1}^{r} k_i \ln \frac{\sigma_1}{\sigma_0}}{\sigma_0^{-2}-\sigma_1^{-2}} \tag{10}$$

then equality (10) is the equation of a straight line in a plane where the ordinate and the abscissa of a point are, respectively:

$$\sum_{i=1}^{r} k_i s_i^2 \quad \text{and} \quad \sum_{i=1}^{r} k.$$

It follows that on a cumulative sum chart, where we have marked the points of co-ordinates

$$\left(k_1, \frac{k_1 s_1^2}{\sigma_0^2}\right), \left(k_1 + k_2, \frac{k_1 s_1^2 + k_2 s_2^2}{\sigma_0^2}\right), \left(k_1 + k_2 + k_3, \frac{k_1 s_1^2 + k_2 s_2^2 + k_3 s_3^3}{\sigma_0^2}\right), \ldots$$

the straight line (10) is a control straight line whose parameters are approximately yielded by relationships:

$$d = -\frac{\ln \alpha_0}{\ln \frac{\sigma_1}{\sigma_0}} \tag{11}$$

and:

$$\theta = \operatorname{arctg} \frac{2 \ln \left(\frac{\sigma_1}{\sigma_0}\right)}{1 - \left(\frac{\sigma_0}{\sigma_1}\right)^2}. \tag{12}$$

Indeed, if in relationship (10) we assume $\alpha_1 \approx 0$ and multiply both members by factor $\frac{1}{\sigma_0^2}$, we get:

$$\frac{1}{\sigma_0^2} \sum_{i=1}^{r} k_i s_i^2 = -\frac{2 \ln \alpha_0}{1 - (\sigma_0/\sigma_1)^2} + \frac{2 \ln (\sigma_1/\sigma_0)}{1 - (\sigma_0/\sigma_1)^2} \sum_{i=1}^{r} k_i \tag{13}$$

It follows that

$$d = \frac{\frac{-2 \ln \alpha_0}{1 - \left(\frac{\sigma_0}{\sigma_1}\right)^2}}{\frac{2 \ln \sigma_1/\sigma_0}{1 - (\sigma_0/\sigma_1)^2}} = \frac{-\ln \alpha_0}{\ln \sigma_1/\sigma_0},$$

$$\theta = \operatorname{arctg} \frac{2 \ln \frac{\sigma_1}{\sigma_0}}{1 - \left(\frac{\sigma_0}{\sigma_1}\right)^2}.$$

By means of values d and θ yielded by formulas (11) and (12) we can draw up the cumulative sum chart with a control limit for increases of dispersion σ^2, which we plot in Fig. 1, where on the abscissa-axis stand values $\sum k_i$ and on the ordinate-axis stand values $\sum k_i s_i^2/\sigma_0^2$

If we want to detect through control a decrease in dispersion from value σ_0^2 to value $\sigma_1^2 (\sigma_1' < \sigma_0)$, then, by making use of the inequality in the left-side

of relationaship (6), through an analogous reasoning we find the control limit straight line $P'Q'$ (Fig. 2), for which

$$d' = -\frac{\ln \alpha_0}{\ln \frac{\sigma_0}{\sigma_1'}} \tag{14}$$

and

$$\theta' = \operatorname{arctcg} \frac{2 \ln\left(\frac{\sigma_0}{\sigma_1'}\right)}{(\sigma_0/\sigma_1')^2 - 1}. \tag{15}$$

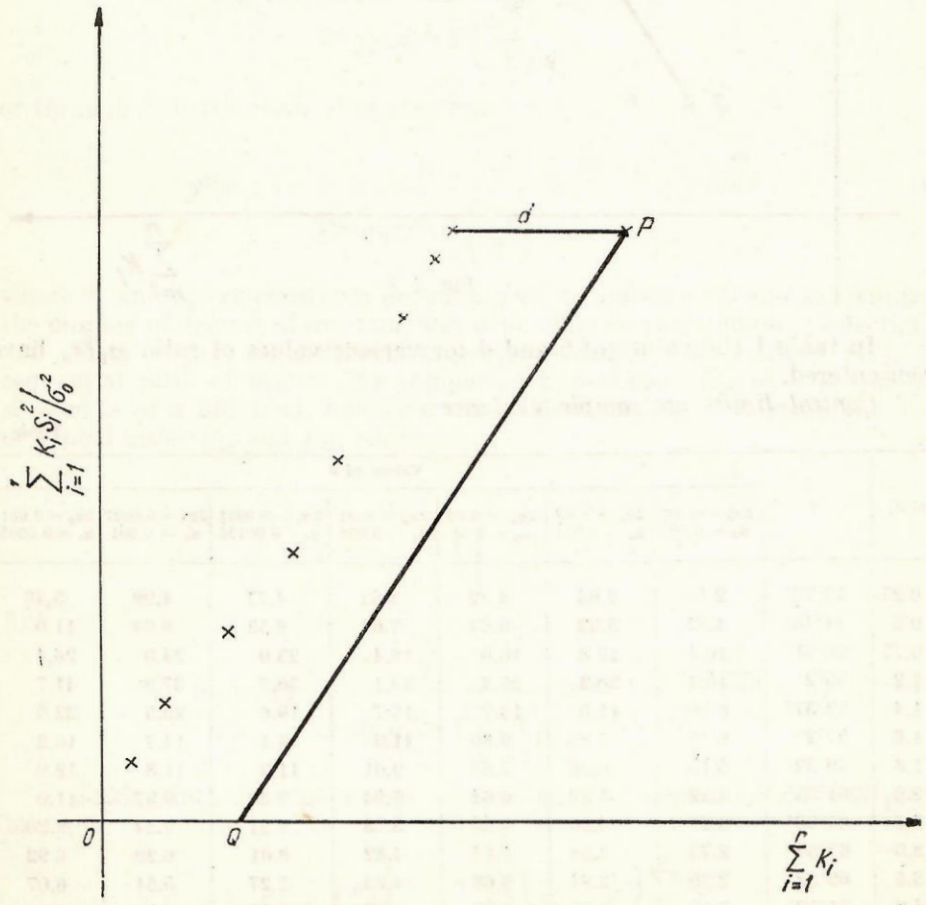


Fig. 1

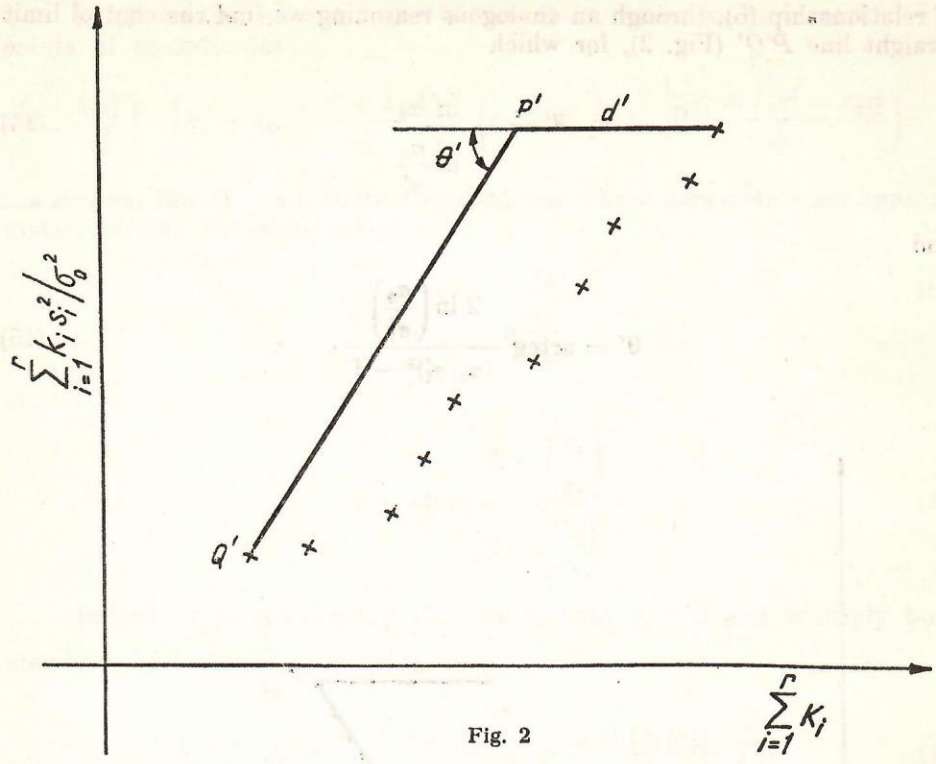


Fig. 2

In table I the values of θ and d for various values of ratio σ_1/σ_0 have been entered.

Control limits on sample variance

Table I

σ_1/σ_0	θ	Values of d						
		$2\alpha_0 = 0.10$ $\alpha_0 = 0.05$	$2\alpha_0 = 0.05$ $\alpha_0 = 0.025$	$2\alpha_0 = 0.02$ $\alpha_0 = 0.01$	$2\alpha_0 = 0.01$ $\alpha_0 = 0.005$	$2\alpha_0 = 0.0027$ $\alpha_0 = 0.00135$	$2\alpha_0 = 0.0027$ $\alpha_0 = 0.001$	$2\alpha_0 = 0.001$ $\alpha_0 = 0.0005$
0.25	10°20′	2.16	2.66	3.32	3.82	4.77	4.98	5.48
0.5	24°48′	4.32	5.32	6.64	7.64	9.53	9.97	11.0
0.75	36°31	10.4	12.8	16.0	18.4	23.0	24.0	26.4
1.2	50°2′	16.4	20.2	25.3	29.1	36.2	37.9	41.7
1.4	52°37′	8.90	11.0	13.7	15.7	19.6	20.5	22.6
1.6	57°2′	6.37	7.85	9.80	11.3	14.1	14.7	16.2
1.8	59°32′	5.10	6.28	7.83	9.01	11.2	11.8	12.9
2.0	61°35′	4.32	5.32	6.64	7.64	9.53	9.97	11.0
2.5	65°23′	3.27	4.03	5.03	5.78	7.21	7.54	8.30
3.0	67°57′	2.73	3.36	4.19	4.82	6.01	6.29	6.92
3.5	69°52′	2.39	2.94	3.68	4.23	5.27	5.51	6.07
4.0	71°19′	2.16	2.66	3.32	3.82	4.77	4.98	5.48

III. CONTROL CHARTS BASED ON THE SELECTION AMPLITUDE

Consider now a statistics using not the selection dispersions s_i^2 but the selection amplitudes

$$w_i = x_{i\,max} - x_{i\,min}, \qquad i = 1, 2, ..., r \tag{16}$$

The use of the selection amplitudes is quite right, since the calculus in agreement with relationship (16) is obviously simpler than the calculus corresponding to the selection dispersion yielded by relationship (3).

We assume again the population C to be normal and all the selections to have the same volume n.

With respect to the distribution of the selection amplitude, E. S. Pearson pointed out that it can be approximated either through a distribution $\varkappa$ of the form:

$$\chi(m_1) = \frac{2}{2^{m_1/2}(c_1\sigma)^{m_1}\,\Gamma\left(\frac{m_1}{2}\right)}\, w_i^{m_1-1}\, e^{-\frac{w_i^2}{2(c_1\sigma)^2}} \tag{17}$$

or through a distribution χ^2 of the form:

$$\chi^2(m_2) = \frac{1}{2^{m_2/2}(c_2\sigma)^{m_2}\,\Gamma\left(\frac{m_2}{2}\right)}\, w_i^{\frac{m_2}{2}-1}\, e^{-\frac{w_i}{2(c_2\sigma)^2}} \tag{18}$$

where c_1 and c_2 are constants depending on n, whereas m_1 and m_2 represent the number of degrees of freedom, also depending on the volume of selection n.

With respect to distribution $\chi(m_1)$ yielded by relationship (17), the sequential ratio of probability comparing hypotheses H_0: $\sigma^2 = \sigma_0^2$ and H_1: $\sigma^2 = \sigma_1^2 > \sigma_0^2$ is obtained, following the r selections, by calculating the ratio of probabilities p_{1r} and p_{0r}, where

$$p_{1r} = \prod_i \frac{2}{2^{m_1/2}(c_1\sigma_1)^{m_1}\,\Gamma\left(\frac{m_1}{2}\right)}\, w_i^{m_1-1}\, e^{-\frac{w_i^2}{2(c_1\sigma_1)^2}}$$

and

$$p_{0r} = \prod_i \frac{2}{2^{m_1/2}(c_1\sigma_0)^{m_1}\,\Gamma\left(\frac{m_1}{2}\right)}\, w_i^{m_1-1}\, e^{-\frac{w_1^2}{2(c_1\sigma_0)^2}} \tag{19}$$

Taking into account values p_{1r} and p_{0r}, the sequential ratio of probability is:

$$\rho = \frac{p_{1r}}{p_{0r}} = \left(\frac{\sigma_0}{\sigma_1}\right)^{rm_1} e^{-\frac{1}{2c_1^2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^{r} w_i^2}. \tag{20}$$

On imposing Wald's restriction $\rho < \frac{1-\alpha_1}{\alpha_0}$, corresponding to the example under consideration, by applying logarithms we obtain:

$$(rm_1)\ \ln\frac{\sigma_0}{\sigma_1} - \frac{1}{2c_1^2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum_{i=1}^{r} w_i \leqslant \ln\frac{1-\alpha_1}{\alpha_0}. \qquad (21)$$

Instead of inequality (21), consider equality:

$$\frac{1}{c_1^2}\sum_{i=1}^{r} w_i^2 = \frac{2rm_1 \ln\frac{\sigma_0}{\sigma_1}}{\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}} - \frac{2\ln\frac{1-\alpha_1}{\alpha_0}}{\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}} \qquad (22)$$

or, simplifying (22) by $1/\sigma_0^2$, equality:

$$\frac{1}{c_1^2\sigma_0^2}\sum_{i=1}^{r} w_i^2 = \frac{2rm_1 \ln\frac{\sigma_0}{\sigma_1}}{\left(\frac{\sigma_0}{\sigma_1}\right)^2 - 1} - \frac{2\ln\frac{1-\alpha_1}{\alpha_0}}{\left(\frac{\sigma_0}{\sigma_1}\right)^2 - 1}. \qquad (22')$$

Equality (22′) is the equation of a straight line on a plane where the current co-ordinates are:

$$rm_1 \text{ and } \frac{1}{c_1^2\sigma_0^2}\sum_{i=1}^{r} w_i^2.$$

The straight line (22′) is a control line.

If in (22′) we assume $\alpha_1 \approx 0$, then values d and θ corresponding to this control limit are:

$$d = \frac{-2\ln\alpha_0}{2\ln\frac{\sigma_1}{\sigma_0}} = -\frac{\ln\alpha_0}{\ln\frac{\sigma_1}{\sigma_0}} \qquad (23)$$

and

$$\theta = \operatorname{arctg}\frac{2\ln\frac{\sigma_1}{\sigma_0}}{1 - \left(\frac{\sigma_0}{\sigma_1}\right)^2}. \qquad (24)$$

By means of values d and θ yielded by formulas (23) and (24) we can draw up the cumulative sum chart with the control limit for dispersion increases. It is plotted in Fig. 3, where on the abscissa-axis stand values rm_1, and on the ordinate-axis stand values $(c_1\sigma_0)^{-2}\sum_i w_i^2$.

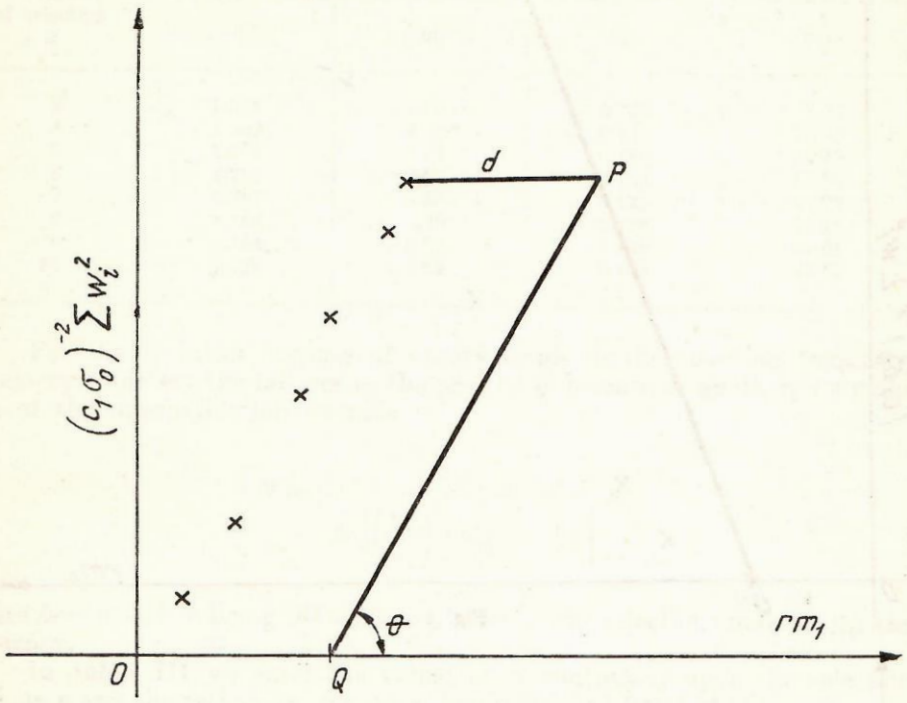


Fig. 3

If, on the chart, we have points under the control line PQ, this is a proof of the increase in dispersion of the quality characteristic under investigation as against the expected level σ_0^2.

This entails either a downright interference, or a more careful supervision of the industrial process.

If we want to detect through control a decrease in dispersion from value σ_0^2 to value $\sigma_1'^2$ ($\sigma_1' < \sigma_0$), then, by making use of Wald's inequality $\frac{\alpha_1}{1-\alpha_0} < \rho$, we find in a similar way the control limit for decreases in dispersion. Fig. 4.

If we approximate the distribution of the selection amplitude through a distribution $\varkappa^2(m_2)$, the reasoning and calculus are similar to those in distribution $\varkappa(m_1)$, the values d and θ being yielded, also in this example, by formulas (23) and (24).

The cumulative sum control chart has, in this instance, the values rm_2 on the abscissa, and values $(\sigma_0 c_2)^{-1}\sum w_i$ on the ordinate. Fig. 5.

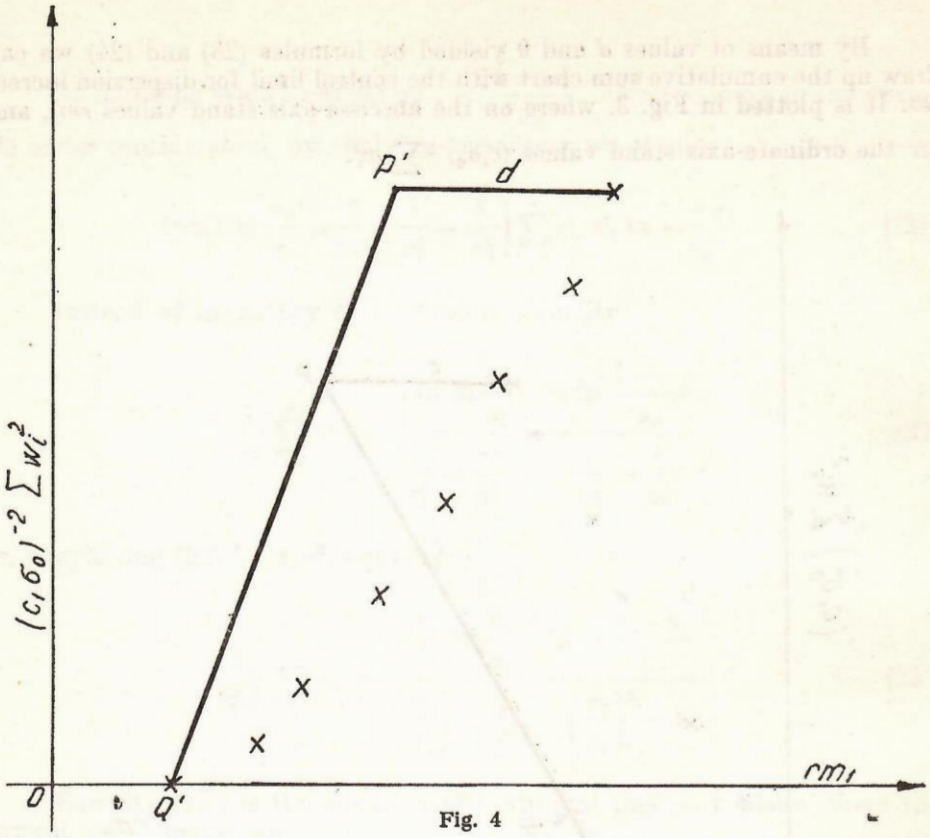


Fig. 4

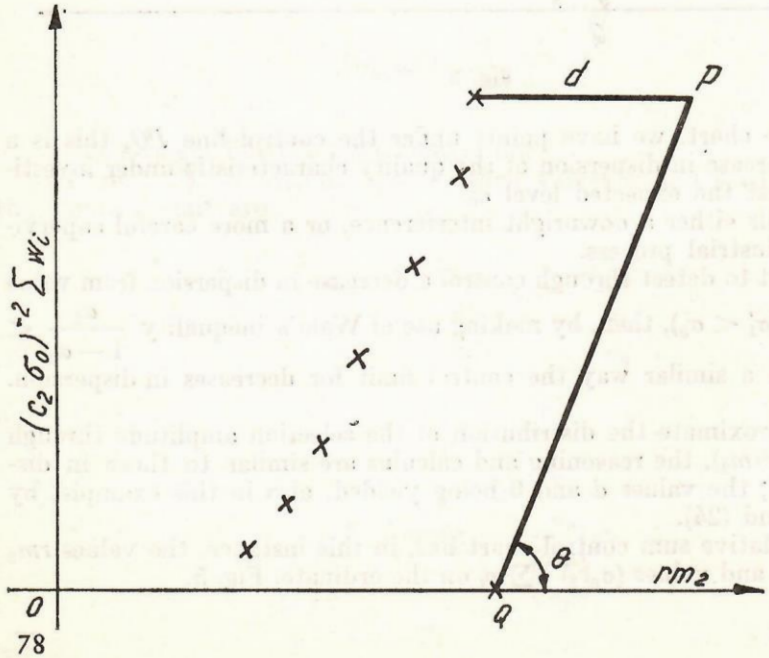


Fig. 5

The values c and m for different selection volumes in both distributions [$\chi(m_1)$ and $\chi^2(m_2)$] are entered in table II.

Table II

Volume of selection n	Distribution $\chi(m_1)$		Distribution $\chi^2(m_2)$	
	c_1	m_1	c_2	m_2
3	1.378	1.93	0.233	7.27
4	1.302	2.95	0.188	10.95
5	1.268	3.83	0.160	14.49
6	1.237	4.69	0.142	17.86
7	1.207	5.50	0.128	21.08
8	1.184	6.26	0.118	24.11
9	1.164	6.99	0.110	27.01
10	1.146	7.69	0.103	29.82

For the N mean number of observations on the selection frequency necessary to detect the failures in the process in a sense or another, we make use of the approximation formula

$$N \approx \frac{d}{K\left[\left(\frac{\sigma_1}{\sigma_0}\right)^2 \operatorname{cotg}\,\theta - 1\right]} n.$$

where $k = n - 1$, n being the volume of each of the selections making up the sequency.

In table III we enter the values of N contingent upon the selection volume n and the ratio σ_1/σ_0 for the values 0.005 and 0.001 of risk α_0.

The table compares the values of N, for the standard charts and cumulative sum charts for dispersion and amplitude.

We can see in table III that the cumulative sum charts for dispersion and amplitude are more convenient than standard charts if $\alpha_0 = 0.001$, and also that for the volume selection $n = 6$ the mean number of observations for amplitude is equal to that for the selection volume $n = 8$, which accounts for the use of selections having a smaller volume ($n = 6$).

All this makes clear the general method for drawing up cumulative sum charts for a control through dispersion and amplitude, without giving details on different ways of choosing the scale. A convenient choice of the relationship between the units of measure on the two axes is most important, since it renders easier both the drawing up and the use of the chart in various practical instances.

We also stress that this type of charts is more economical as compared to the standard charts as regards the control of precision — one of the essential technical factors of an industrial process — which is investigated through the study of the variability of the respective process with a view to securing the production homogeneity.

Table III

σ_1/σ_0	n	$\alpha_0 = 0.005$			$\alpha_0 = 0.001$		
		Standard charts and dispersion	Cumulative sum charts		Standard charts and dispersion	Cumulative sum charts	
			Dispersion	Amplitude		Dispersion	Amplitude
1.2	4	131	187	191	388	244	248
	6	149	169	180	423	220	234
	8	161	161	180	440	210	234
1.3	4	72.6	85.5	86.9	181	111	113
	6	77.1	77.0	82.0	182	100	107
	8	79.5	73.3	82.0	179	96.5	107
1.4	4	45.6	49.2	50.0	99.3	64.2	65.2
	6	46.6	44.3	47.2	95.1	57.8	61.5
	8	47.0	42.2	47.2	90.9	55.0	61.5
1.5	4	31.5	32.2	32.7	61.6	42.0	42.6
	6	31.6	29.0	30.9	57.4	37.8	40.2
	8	31.8	27.6	30.8	54.	36.0	40.2
2.0	4	11.1	8.8	8.9	15.7	11.4	11.6
	6	11.5	7.9	8.4	15.0	10.3	10.9
	8	12.3	7.5	8.4	15.1	9.8	10.9
2.5	4	7.1	4.1	4.2	8.8	5.4	5.5
	6	8.0	3.7	4.0	9.1	4.9	5.2
	8	9.3	3.5	4.0	10.1	4.6	5.2

The approximations we have made as regards both the type of distribution of the statistics used and the formulas yielding the control limits (d and θ) prove to be acceptable in practice, from the standpoint of the efficaciousness of this type of checking.

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LE CONTROLE DES PROCESSUS DE PRODUCTION, PAR DES CARTES A SOMMES CUMULEES A L'AIDE DE LA DISPERSION DE L'AMPLITUDE

L'article s'occupe de la construction des cartes (fiches) de contrôle à sommes cumulées dans l'étude de la variabilité des processus de production, le paramètre poursuivi étant la dispersion de la caractéristique de qualité, étudiée.

En guise de statistiques (fonctions de sélection) on prend la dispersion de sélection s^2 et l'amplitude de sélection W.

On établit les formules qui donnent, dans les deux cas, les limites de contrôle et on expose la manière de construire et d'interpréter, en géneral, une telle carte.

On présente également, dans l'article, les cas dans lesquels ce type de contrôle est préférable, en ce qui concerne son efficacité, par rapport aux cartes standard classiques.

КОНТРОЛЬ ПРОИЗВОДСТВЕННЫХ ПРОЦЕССОВ ПРИ ПОМОЩИ КАРТ С КУМУЛИРОВАННЫМИ СУММАМИ, СОСТАВЛЕННЫХ НА ОСНОВЕ ДИСПЕРСИИ АМПЛИТУДЫ СЕЛЕКЦИИ

В настоящей статье рассматривается вопрос построения контрольных карт (карточек) с кумулированными суммами при изучении переменности производственных процессов, причем наблюдаемым параметром является дисперсия изучаемой качественной характеристики.

В качестве статистических данных (селекционных функций) берутся селекционная дисперсия и амплитуда селекции.

Устанавливаются формулы, дающие в обоих случаях пределы контроля, и указывается, как строится и как толкуется, в общем, такая карта.

В статье указываются также случаи, когда этот тип контроля является предпочтительным в отношении эффективности по сравнению с классическими стандартными картами.