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An Application of the Modulus Distribution of Some Normal Variables to Production Quality Control

To investigate production and industrial systems means in a way to investigate human life itself

A. When investigating some production processes we may meet with quality characteristics X whose values are affected by the sign "plus" or "minus". If during the quality control the sign of the different entered values of the characterictic under study is lost without any possibility of identification, then we obviously have to examine the modulus distribution of the characteristic in question because we are ignorant of that sign.

We shall consider further the case that most frequently occurs in practice, where the quality characteristic X is a random variable having a normal distribution. We are to deal with the modulus distribution of the random normal variable in view of applying this very distribution to the production quality control.

Let Y be a normal variable having mean m and dispersion σ^2; its distribution is:

$$f(y;\, m;\, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(y-m)^2}{\sigma^2}} \quad ; \quad y \in R \tag{1}$$

We write $X = |\, Y\,|$. The distribution $\varphi(x,\ m,\ \sigma)$ of variable X is ascertained as follows:

$$\varphi(x,\ m,\ \sigma) = P\{x \leqslant X \leqslant x + dx\} = P\{y \leqslant Y \leqslant y + dy\} + P\{-y - dy \leqslant Y \leqslant -y\} = f(y)\, dy + f(-y)\, dy$$

or

$$\varphi(x;\, m;\, \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{(x-m)^2}{2\sigma^2}} + e^{-\frac{(x+m)^2}{2\sigma^2}}\right] \tag{2}$$

where we have substituted y by x because $y \geqslant 0$.

Relationship (2) can be written successively under the form:

$$\varphi(x;\, m,\, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2-2xm+m^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2+2xm+m^2}{2\sigma^2}}$$

$$\varphi(x;\, m,\, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2+m^2}{2\sigma^2}} \left[e^{\frac{xm}{\sigma^2}} + e^{-\frac{xm}{\sigma^2}} \right],$$

or, amplifying by 2,

$$\varphi(x;\, m;\, \sigma) = \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{x^2+m^2}{2\sigma^2}} ch\, \frac{xm}{\sigma^2} \tag{3}$$

If in (3) we write:

$$\lambda = \frac{m}{\sigma} \tag{4}$$

then:

$$\varphi(x;\, m;\, \sigma) = \frac{1}{\sigma\sqrt{\frac{\pi}{2}}} e^{-\frac{1}{2}\left[\lambda^2+\left(\frac{x}{\sigma}\right)^2\right]} ch\left(\frac{\lambda x}{\sigma}\right) \tag{5}$$

where λ and σ are the distribution parameters.

We are going to dwell upon the production quality control through cumulative sum charts in case the quality characteristic is a random variable whose distribution is yielded by relationship (5).

We shall describe the cumulative sum control charts for the mean and the mean square deviation of this random variable which is met with in workmanship operations of processing metallurgy.

B. THE MEAN CONTROL

We can see in relationship (4) that the distribution of the modulus variable depends on parameters λ and σ where $\lambda = \frac{m}{\sigma}$. We assume σ to be known because the hypotheses regarding mean m are, because of relationship (4), hypotheses concerning parameter λ.

We assume the case that the process is under control and the mean of the quality charachteristic X for the whole population C of products is $m = m_0 = 0$; hence $\lambda = \lambda_0 = \frac{m_0}{\sigma} = 0$ (hypothesis H_0: $\lambda = 0$).

We have set the target of detecting through control an increase of the mean from value 0 to value m_1, that is an increase of λ from value 0 to value λ_1 (hypothesis H_1: $\lambda = \lambda_1 > 0$).

In order to get the sequential testing of the probabilities ratio underlying the drawing up of the cumulative sum control charts we make use of the

observations $x_1, x_2, ..., x_n$ in a volume selection n extracted from population C of products. The selection variables $x_1, x_2, ..., x_n$ are independent variables and their distribution has been yielded by relationship (5).

The sequential testing of probabilities ratio comparing hypotheses H_0: $\lambda = 0$ and H_1: $\lambda = \lambda_1$, following the n observations is obtained by finding the ratio of probabilities p_{1n} and p_{on} where

$$\left.\begin{aligned} p_{1n} &= \prod_{i=1}^{n} \sigma^{-1} \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}\left[\lambda_1^2 + \left(\frac{x_i}{\sigma}\right)^2\right]}\, ch\left(\frac{\lambda_1 x_i}{\sigma}\right) \\ p_{on} &= \prod_{i=1}^{n} \sigma^{-1} \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}\left[\frac{x_i}{\sigma}\right]^2} \end{aligned}\right\} \tag{6}$$

Taking into account values p_{1n} and p_{on}, the probabilities ratio is:

$$\rho = \frac{p_{1n}}{p_{on}} = e^{-\frac{1}{2} n\lambda_1^2} \prod_{i=1}^{n} ch\left(\lambda_1 x_i/\sigma\right). \tag{7}$$

Wald's restrictions yield us:

$$\frac{\alpha_1}{1-\alpha_0} < \rho < \frac{1-\alpha_1}{\alpha_0} \tag{8}$$

where α_0 and, respectively, α_1 are the risks of type I and II, that is to say α_0 is the probability of detecting a variation of λ from value 0 to value λ_1 without its existence in reality, and α_1 is the probability of not detecting such a variation although it really exists.

Applying logarithms to relationship (8) we have:

$$\ln \frac{\alpha_1}{1-\alpha_0} < -\frac{1}{2} n\lambda_1^2 + \sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) < \ln \frac{1-\alpha_1}{\alpha_0} \tag{9}$$

or

$$\ln \frac{\alpha_1}{1-\alpha_0} + \frac{1}{2} n\lambda_1^2 < \sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) < \ln \frac{1-\alpha_1}{\alpha_0} + \frac{1}{2} n\lambda_1^2 \tag{10}$$

If in relationship (10) we assume $\alpha_1 \approx 0$ and consider only the right side of the inequality, that is the inequality corresponding to the hypothesis assumed, we have:

$$\sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) < \frac{1}{2} n\lambda_1^2 - \ln(\alpha_0) \tag{11}$$

Considering equality

$$\sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) = \frac{1}{2} n\lambda_1^2 - \ln \alpha_0 \tag{12}$$

instead of inequality (11), we obtain the equation of a straight line on a plane where the ordinate and the abscissa of a point are, respectively,

$$\sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) \quad \text{and} \quad n.$$

It follows that on a cumulative sum control chart where we have marked points $\left(1, ch\frac{\lambda_1 x}{\sigma}\right)$, $\left(2, ch\frac{\lambda_1 x_1}{\sigma} + ch\frac{\lambda_1 x_2}{\sigma}\right)$ the straight line (12) is a control limit straight line; its parameters are obviously yielded by the relationships:

$$d = -\frac{2\ln \alpha_0}{\lambda_1^2}; \quad \theta = \operatorname{arctg}\frac{1}{2}\lambda_1^2 \tag{13}$$

With the sizes d and θ yielded by formulas (13), the cumulative sum chart having a control limit for the mean is illustrated by Fig. 1.

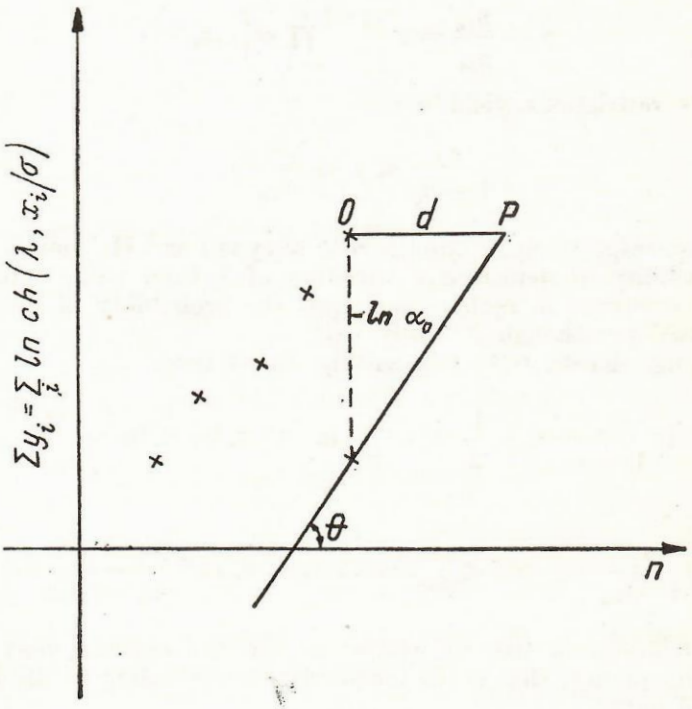

Fig. 1.

We see that, because y_i is not negative for any i, sum $\sum_i y_i$ is undecreasing.

If we consider values

$$y_i' = y_i - \frac{1}{2}\lambda_1^2 \tag{14}$$

instead of sizes y_i, then, beside the fact that the chart we have obtained has smaller sizes, we have a possibility for immediately drawing up the control limit line by connecting point P with the point located on the vertical at a distance equal to $-\ln \alpha_0$ under the latest marked point.

For computing values $y_i = \ln ch(\lambda_1 x_i/\sigma)$ we make use of tables which give us the values y_i contingent upon the values $X = \frac{\lambda_1 x_i}{\sigma}$.

Table I contains the corresponding values of Y for a few values of X.

If the X values are larger, then we may use the approximate formula $y \approx \lambda_1 x/\sigma - \ln 2 \approx X - 0.693$.

Values for ln *chX*

Table I

X	ln *ch* X	X	ln *ch* X
		1.1	0.512
0.0	0.000	1.2	0.594
0.1	0.005	1.3	0.678
0.2	0.020	1.4	0.766
0.3	0.044	1.5	0.855
0.4	0.078	1.6	0.947
0.5	0.120	1.7	1.040
0.6	0.170	1.8	1.134
0.7	0.227	1.9	1.229
0.8	0.291	2.0	1.325
0.9	0.360	2.1	1.422
1.0	0.434	2.2	1.519
		2.3	1.617
		2.4	1.715
		2.5	1.814

Here is an example: A compound machine turns out in its two sections pairs of parts so that each pair may be coupled by superposition through a rotary motion of 180° around an axis Δ as shown in Fig. 2.

If with their superposition one of the parts turns to be larger, the material in excess is automatically removed by a stamping device. The coupling and stamping are performed by the machine without enabling us to know from which section of the machine proceeds the removed material. This material is weighed each time that ten pairs of parts have been turned out. The respective values are entered into the first column of Table II.

We want to know if there occurs a systematical deviation in the size of the manufactured parts.

To this end we shall assume $\lambda_0 = 0$ and the standard deviation of the weight of parts from the normal weight to be 0.125 g each of the machine sections. The square mean deviation for both sections of the machine will consequently be $0.125 \cdot \sqrt{2} = 0.177$ g.

Suppose now that we mean to detect through control an increase in the mean from 0 to 0.125 g. Then:

$$\lambda_1 = \frac{0.125}{0.177} = 0.707$$

$$y_i = \ln ch 4x_i$$

and

$$y'_i = \ln ch 4x_i - \frac{1}{4}$$

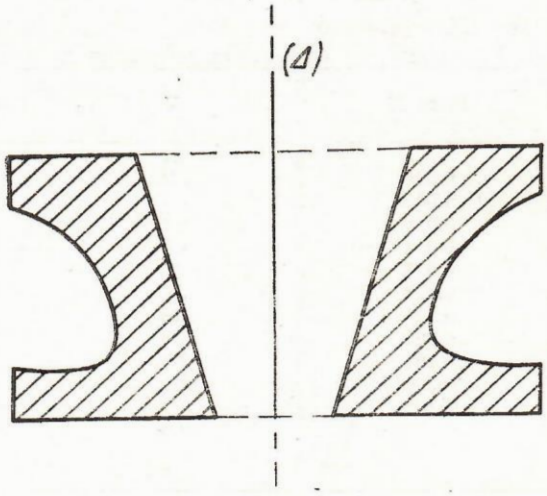

Fig. 2.

Table II

x_i grams	$y_i = \ln ch\, 4x_i$	$y'_i = y_i - 0.25$	$\Sigma y'_i$
0.21	0.32	0.07	0.07
0.17	0.22	—0.03	0.04
0.10	0.08	—0.17	—0.13
0.08	0.05	—0.20	—0.33
0.27	0.50	0.25	0.08
0.25	0.43	0.18	0.10
0.20	0.29	0.04	0.14
0.20	0.29	0.04	0.18
0.17	0.22	—0.03	0.15
0.17	0.22	—0.03	0.12
0.33	0.69	0.44	0,56
0.25	0.43	0.18	0.74

With the aid of the values in Table II we draw up the chart illustrated in Fig. 3, where on the ordinate-axis we write the values $\sum_{i=1}^{n} y_i'$.

Taking into account that $y_i' = y_i - \frac{1}{2}\lambda_1^2$, we have, for the general case:

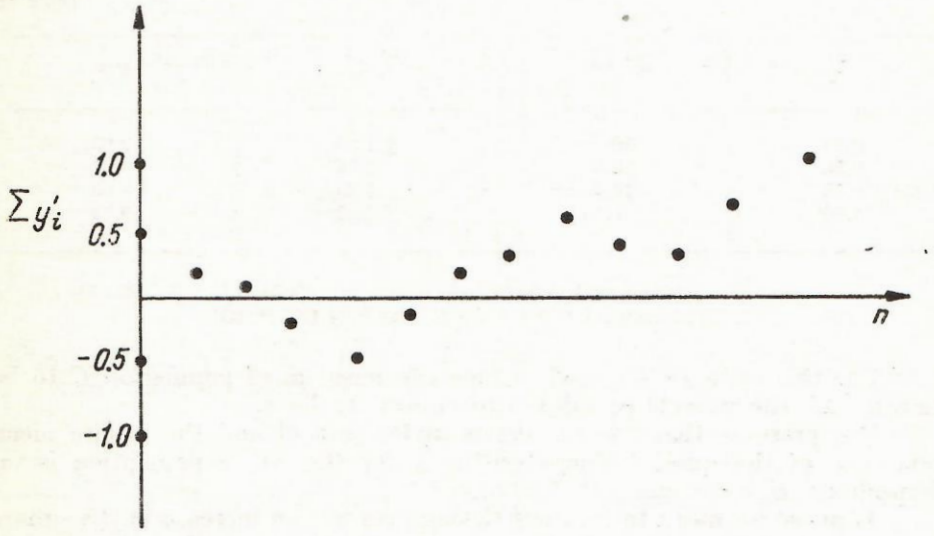

Fig. 3.

$$\sum_{i=1}^{n} y_i' = \sum_{i=1}^{n} \ln ch(\lambda_1 x_i/\sigma) - \frac{1}{2} n\lambda_1^2 < \frac{1}{2} n\lambda_1^2 - \frac{1}{2} n\lambda_1^2 - \ln \alpha_0 \qquad (15)$$

or:

$$\sum_{i=1}^{n} y_i' = -\ln \alpha_0. \qquad (16)$$

It follows that the control limit straight line is parallel to the abscissa-axis located under the latest point marked on the chart at a distance equal to $-\ln \alpha_0$.

In the example considered, the control limit straight line will consequently be a parallel to the abscissa-axis at a distance of $-\ln 0.01 = 4.6$ if $\alpha_0 = 0.01$, or at a distance of $-\ln 0.05 = 3$ if $\alpha_0 = 0.05$. We see that no point is located under the control limit, which means that the manufacturing of the parts is proceeding normally.

The mean length of the observation sequence required in view of reporting on the control chart any failure or tendency in the processing has the approximate value $(-\ln \alpha_0) E^{-1}$, where E is yielded by relationship:

$$E \approx \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_1^2}\right)\left(2\lambda_1 + \frac{\pi - 3}{\lambda_1}\right) + \lambda_1^2\left[2F(\lambda_1) - \frac{3}{2}\right]\ln 2 + 1 - F(\lambda_1) \qquad (17)$$

and

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{1}{2}y^2} dy$$

Table III shows a few values of E^{-1} contingent upon λ_1.

Table III

λ_1	E^{-1}	λ_1	E^{-1}
0.25	980	1.25	2.95
0.50	73.5	1.50	1.67
0.75	16.6	1.75	1.06
1.00	6.13	2.00	0.73

C. THE SQUARE MEAN DEVIATION CONTROL

For this instance we shall assume the mean m of population C to be known and the parameter subject to control to be σ.

We presume that the process is under control and the square mean deviation of the quality characteristic X for the whole population is σ_0 (hypothesis $H_0 : \sigma = \sigma_0$).

Suppose we mean to find out through control an increase in the square mean deviation from value σ_0 to value $\sigma_1 > \sigma_0$ (hypothesis $H_1 : \sigma = \sigma_1 > \sigma_0$).

In order to know the probability sequential ratio we make use of the observations $x_1, x_2, \ldots, x_n$ in a volume selection x extracted from population C of products. Variables $x_1, x_2, \ldots, x_n$ are independent with the distribution yielded by relationship (5).

The probability sequential ratio comparing hypotheses $H_0 : \sigma = \sigma_0^0$ and $H_1 : \sigma = \sigma_1$ following the n observations is found out by solving the probabilities ratio:

$$p_{1n} = \prod_{i=1}^{n} \sigma_1^{-1} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}\left[\left(\frac{m}{\sigma_1}\right)^2 + \left(\frac{x_i}{\sigma_1}\right)^2\right]} ch\left(\frac{m x_i}{\sigma_1^2}\right)$$

$$p_{0n} = \prod_{i=1}^{n} \sigma_0^{-1} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}\left[\left(\frac{m}{\sigma_0}\right)^2 + \left(\frac{x_i}{\sigma_0}\right)^2\right]} ch\frac{m}{\sigma_0^2} x_i \qquad (18)$$

Taking into account values p_{1n} and p_{0n}, the probability sequential ratio is:

$$\rho = \frac{p_{1n}}{p_{0n}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\frac{1}{2}(nm^2 + \sum_{i=1}^{n} x_i^2)(\sigma_0^{-2} - \sigma_1^{-2})} \prod_{i=1}^{n} \left[\frac{ch\frac{m x_i}{\sigma_1^2}}{ch\frac{m x_i}{\sigma_0^2}}\right] \qquad (19)$$

Wald's restrictions yield:

$$\frac{\alpha_1}{1-\alpha_0} < \rho < \frac{1-\alpha_1}{\alpha_0} \tag{20}$$

α_0 and α_1 being the risks of type I and II.

Applying logarithms to the right member of inequality (20) where $\alpha_1 \approx 0$, we get:

$$n\left[\frac{1}{2}m^2(\sigma_0^{-2}-\sigma_1^{-2}) - \ln\frac{\sigma_1}{\sigma_0}\right] + \left[\frac{1}{2}\sum_{i=1}^{n} x_i^2\right](\sigma_0^{-2}-\sigma_1^{-2}) +$$

$$+\sum_{i=1}^{n} \ln\left[\frac{ch\frac{x_i m}{\sigma_1^2}}{ch\frac{x_i m}{\sigma_0^2}}\right] < -\ln\alpha_0. \tag{21}$$

Inequality (21) may be written under the form:

$$\sum_{i=1}^{n} \ln\frac{ch(x_i m/\sigma_1^2)}{ch(x_i m/\sigma_0^2)} + \frac{1}{2}\sum_{i=1}^{n} x_i^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right) < n\left[\ln\frac{\sigma_1}{\sigma_0} - \frac{1}{2}m^2(\sigma_0^{-2}-\sigma_1^{-2})\right] - \ln\alpha_0.$$

or:

$$\left(\sum_{i=1}^{n} \ln ch(x_i m/\sigma_1^2) - \frac{1}{2}\sum_{i=1}^{n} x_i^2/\sigma_1^2\right) - \left(\sum_{i=1}^{n} \ln ch\, m x_i/\sigma_0^2 - \right.$$

$$\left. - \frac{1}{2}\sum_{i=1}^{n} x_i^2/\sigma_0^2\right) < -\ln\alpha_0 + n\left[\ln\frac{\sigma_1}{\sigma_0} - \frac{1}{2}m^2(\sigma_0^{-2}-\sigma_1^{-2})\right] \tag{22}$$

Writing $\ln ch\left(\frac{x_i m}{\sigma_1^2}\right) - \frac{1}{2}\sum\frac{x_i^2}{\sigma_1^2} = y_i^{(1)}$

$$\ln ch\left[\left(\frac{x_i m}{\sigma_0^2}\right) - \frac{1}{2}\left(\frac{x_i}{\sigma_0}\right)^2\right] = y_i^0$$

we have:

$$\sum_{i=1}^{n}(y_i^{(1)} - y_i^{(0)}) < -\ln\alpha_0 + n\left[\ln\frac{\sigma_i}{\sigma_0} - \frac{1}{2}m^2(\sigma_0^{-2}-\sigma_1^{-2})\right] \tag{23}$$

Considering equality

$$\sum_{i=1}^{n}(y_i^{(1)} - y_i^{(0)}) = -\ln\alpha_0 + n\left[\ln\frac{\sigma_1}{\sigma_0} - \frac{1}{2}m^2(\sigma_0^{-2}-\sigma_1^{-2})\right] \tag{24}$$

instead of inequality (23), we find the equation of a straight line on a plane where the ordinate and the abscissa of a point are, respectively, $\sum_i(y_i^{(1)} - y_i^{(0)})$ and n.

It follows, then, that on a cumulative sum control chart on which we have marked points $(1, y_1^{(1)} - y_1^{(0)}, (2, y_2^{(1)} - y_2^{(0)})$.... the straight line (24) is a control limit straight line (Fig. 4).

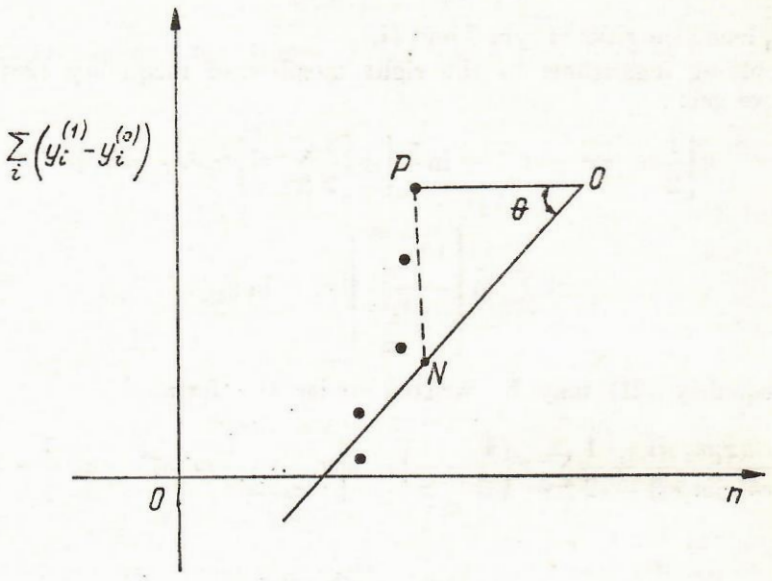

Fig. 4.

The control limit ON is determined by the control parameters

$$\theta = \operatorname{arctg}\left[\ln\frac{\sigma_1}{\sigma_0} - \frac{1}{2}m^2(\sigma_0^{-2} - \sigma_1^{-2})\right]$$

and

$$ON = -\ln \alpha_0$$

which values are derived from relationship (24).

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UNE APPLICATION DE LA RÉPARTITION DU MODULE DE CERTAINES VARIABLES NORMALES AU CONTROLE DE LA QUALITÉ DE LA PRODUCTION

Dans cet article on présente la répartition du module de certaines variables aléatoires normales dans les paramètres m et σ.

On étudie ensuite le contrôle au moyen de cartes à sommes cumulées dans le cas de la moyenne et de l'écart moyen quadratique, la caractéristique de qualité X du preccesus ayant une répartition du type présenté.

ПРИМЕНЕНИЕ РАСПРЕДЕЛЕНИЯ МОДУЛЯ НОРМАЛЬНЫХ ПЕРЕМЕННЫХ ПРИ КОНТРОЛЕ КАЧЕСТВА ПРОДУКЦИИ

В настоящей статье указывается распределение нормальных случайных переменных в параметрах m и σ.

Затем рассматривается контроль при помощи карт с кумулированными суммами в случае средней и среднеквадратического отклонения, характеристика качества X процесса, имеющего распределение представленного типа.