ST. MITITELU

Flows Saturating Entrances or Exits in Transport Networks

This study presents the existence of flows in transport networks with authorized intervals, with two or several main nodes which on the entrance and respectively exit arces assume maximum or minimum values.

The results obtained are transposed for transport networks with limited capacities as well.

Consider a graph $G = (X, U)$ in which $X = \{a_1, a_2, \ldots a_n, a, b\}$ is the set of nodes, $U = \{1, 2 \ldots, m\}$ is the set of arces and $1 = (b, a)$ is the return arc.

Graph G in which the interval $[b_i, c_i]$ with $-\infty \leqslant b_i \leqslant c_i \leqslant \infty$ is associated to each arc $i \neq 1$, is called the transport network with authorized intervals. We shall note this network with $\mathcal{R}$ and a flow into arc i with φ_i.

The generalized problem of the maximum flow can be enunciated as follows:

"Find in the network $\mathcal{R}$ a flow $(\varphi_1, \ldots, \varphi_m)$ so that:

$$b_i \leqslant \varphi_i \leqslant e_i, \quad i = 2, \ldots, m, \tag{1}$$

$$\varphi_1 = maximum, \tag{2}$$

where φ_1 is the maximum flow passing through the network.

If:

$$\varphi_1 = minimum$$

in conditions (1), we can say that we have the generalized problem of the minimum flow.

The study presents aspects of these two problems.

For a set of nodes $A \subset X$, the set of arces
is called co-cycle:

$$\omega(A) = \omega^{-}(A) \cup \omega^{+} + (A)$$

where the set of incident arces inwards A, $\omega^{-}(A)$, or the set of incident arces outwards A, $\omega^{+}(A)$, is non-vacuum.
Note by:

$$\omega = (\omega_1, \ldots, \omega_m),$$

where

$$\omega_i = \begin{cases} 0, \text{ if arc } i \notin \omega(A), \\ +1, \text{ if arc } i \in {}^{\prime}(V)_{+}{}^{\omega} \\ -1, \text{ if arc } i \in \omega^{-}(A), \end{cases}$$

the vector associated to co-cycle $\omega(A)$.

Take

$$B_1 = max\left\{ \sum_{i \in \omega^{+}(a)} b_i, \sum_{i \in \omega^{-}(b)} b_i \right\}.$$

THEOREM 1 — The necessary and sufficient condition for the existence in the network $\mathcal{R}$ with authorized intervals $[b_i, c_i]$ of a flow which should saturate the exit arces is that the relations:

$$\sum_{i \in \omega^{-}(A)} c_i \geqslant \sum_{i \in \omega^{+}(A) \bigcap \omega^{-}(b)} c_i + \sum_{i \in \omega^{+}(A) \setminus \omega^{-}(b)} b_i \tag{3}$$

$$\sum_{\in \omega(A)} \omega_i b_i \geqslant 0, \tag{4}$$

should take place for any set $A \subseteq \{a_1, a_2, \ldots, a_n\}$.

Proof. In fact, a flow $(\varphi_1 \ldots \varphi_m)$ is sought so that:

$$B_1 \leqslant \varphi_1 \leqslant \infty,$$

$$b_i \leqslant \varphi_i \leqslant c_i, \; i \neq 1, \; i \notin \omega^{-}(b),$$

$$c_i = \varphi_i = c_i, \; i \in \omega^{-}(b).$$

According to Hoffman's theorem [1], for the existence of a flow in a network with authorized intervals, it is necessary and sufficient that the relations:

$$\sum_{i \in \omega^{-}} c_i \geqslant \sum_{i \in \omega^{+}} b_i \tag{5}$$

$$\sum_{i \in \omega^{+}} c_i \geqslant \sum_{i \in \omega^{-}} b_i, \tag{6}$$

should take place, regardless of the elementary co-cycle ω.

If

$$A = \{a_{i_1} a_{i_2}, \ldots, a_{i_p}\}, \ (p \leqslant n)$$

then

$$\omega(A) = \bigcup_{K=1}^{P} \omega(a_{i_k}),$$

where $\omega(a_{i_k})$ is the elementary co-cycle determined by point a_{i_k}.

With this observation from relations (5) and (6), the following relations result:

$$\sum_{i \in \omega^-(A)} c_i \geqslant \sum_{i \in \omega^+(A)} b_i \tag{7}$$

$$\sum_{i \in \omega^+(A)} c_i \geqslant \sum_{i \in \omega^-(A)} b_i \tag{8}$$

We shall refer to relation (7)

If $a \in A, b \in A$, it results that $1 \in \omega^+(A)$ and as $c_1 = \infty$, it results that $\sum_{i \in \omega^-(A)} c_i = \infty$.

If $b \in A$ it results that $1 \in \omega^+(A)$ and

$$\sum_{i \in \omega^+(A)} b_i = \sum_{i \in \omega^+(A)} b_i + B_1$$

So in this case, in the second member of inequality (7), we cannot make appear the numbers c_i, $i \in \omega^-(b)$, for which $\varphi_i = c_i$.

In this way, supposing that $a \in A$ and $b \in A$ relation (7) and from relations:

$$b_i = c_i, \ \text{if} \ i \in \omega^-(b),$$

it results that:

$$\sum_{\in \omega^-(A)} c_i \geqslant \sum_{i \in \omega^+(A) \cap \omega^-(b)} b_i + \sum_{i \in \omega^+(A) \setminus \omega^-(b)} b_i = \sum_{i \in \omega^+(A) \cap \omega^-(b)} c_i + \sum_{i \in {}^+\omega(A) \setminus \omega^-(b)} b_i$$

that is relation (3).

Relation (8) cannot be decomposed in this way. But the sufficiency of the theorem is ensured because relation (4) implies relation (8). Indeed, it is also written:

$$\sum_{i \in \omega^+(A)} b_i - \sum_{i \in \omega^-(A)} b_i \geqslant 0,$$

or

$$\sum_{i \in \omega^+(A)} b_i \geqslant \sum_{i \in \omega^-(A)} b_i \tag{9}$$

and as $b_i < c_i$, relation (8) results from relation (9).

Corollary 1 (Gale's Theorem). The necessary and sufficient condition for the existence of a flow $(\varphi_1, \ldots, \varphi_m)$ in a transport network with limited

capacities $c_i \geqslant 0$, so that $0 \leqslant \varphi_i \leqslant c_i$ $(i \neq 1)$, a flow which should saturate the exit arces is that the relation

$$\sum_{i \in \omega^-(A)} c_i \geqslant \sum_{i \in \omega^+(A) \cap \omega^-(b)} c_i \tag{3'}$$

should exist regardless of the set $A \subseteq \{a_1, \ldots, a_n\}$.

Proof, it results from theorem (1) for $b_i = 0$

In this case, relation (4) loses its significance.

The same considerations can be made also for the entrance arces.

THEOREM 2. The necessary and sufficient condition for the existence of a flow in the network $\mathcal{R}$ with the authorized intervals $[b_i, c_i]$ which should saturate the entrance arces is that the relations:

$$\sum_{i \in \omega^+(A)} c_i \geqslant \sum_{i \in \omega^-(A) \cap \omega^+(a)} c_i + \sum_{i \in \omega^-(A) \omega^+(a)} b_i \tag{10}$$

$$\sum_{i \in \omega(A)} \omega_i b_i \leqslant 0, \tag{11}$$

should take place for any set $A \subseteq \{a_1, a_2, \ldots, a_n\}$.

Proof. Flow $(\varphi_1, \ldots, \varphi_m)$ satisfies the conditions:

$$B_1 \leqslant \varphi_1 \leqslant \infty;$$

$$b_i \leqslant \varphi_i \leqslant c_i, \quad i \neq 1, \; i \notin \omega^+(a);$$

$$c_i = \varphi_i = c_i, \quad i \in \omega^+(a)$$

Let us confine ourselves to relation (8). If $a \in A$, $b \notin A$, it results that:

$$1 \in \omega^-(A) \text{ and } \sum_{i \in \omega^-(A)} b_i = \sum_{\substack{i \in \omega^-(A) \\ i \neq 1}} b_i + B_1$$

So that we cannot outline the numbers c_i, $i \in \omega^-(b)$. If $a \notin A$, $b \in A$, it results that $1 \in \omega^+(A)$ and $\sum_{i \in \omega^+(A)} c_i = \infty$. If $a \notin A, b \notin A$ and knowing that $b_i = c_i$ for $i \in \omega^+(a)$, from (8), it results that:

$$\sum_{i \in \omega^+(A)} c_i \geqslant \sum_{i \in \omega^-(A)} b_i = \sum_{i \in \omega^-(A) \cap \omega^+(a)} b_i + \sum_{i \in \omega^-(A) \setminus \omega^+(a)} b_i =$$

$$= \sum_{i \in \omega^-(A) \cap +(a)} c_i + \sum_{i \in \omega^-(A) \setminus \omega^+(a)} b_i$$

that is relation (10).

The sufficiency of the theorem is ensured, because relation(11) implies relation (7) in the same way in which relation (4) implied relation (8) in theorem 1.

Corollary 2. The necessary and sufficient condition for the existence of a flow $(\varphi_1, \ldots, \varphi_m)$ in a transport network with limited capacities $c_i \geqslant 0$,

so that $0 \leqslant \varphi_i \leqslant c_i$ $(i \neq 1)$, a flow which should saturate the entrance arces is the existence of relation:

$$\sum_{i\omega\in^{+}(A)} c_i \geqslant \sum_{i\in\omega^{-}(A)\cap\omega^{+}(a)} c_j \tag{12}$$

Proof results from theorem 2 for $b_i = 0$.

For any flow φ_1 we have $\varphi_1 \geqslant \sum_{i\in\omega^{+}(a)} b_i$ and $\varphi_1 \geqslant \sum_{i\in\omega^{-}(b)} b_i$.

THEOREM 3. A necessary and sufficient condition for the existence of a minimum flow in the network $\mathfrak{R}$ with the authorized intervals $[b_i, c_i]$, so that $\varphi_1 = \sum_{i\in\omega^{-}(b)} b_i$, is that relations:

$$\sum_{i\in\omega^{-}(A)} b_i \leqslant \sum_{i\in\omega^{+}(A)\cap\omega^{-}(b)} b_i + \sum_{i\in\omega^{+}(A)\setminus\omega^{-}(b)} c_i, \tag{13}$$

$$\sum_{i\in\omega(A)} \omega_i b_i \leqslant 0, \tag{14}$$

should take place for any set $A \subseteq \{a_1, a_2, \ldots, a_n\}$.

Proof. A flow $(\varphi_1, \ldots, \varphi_m)$ is sought, so that:

$$B_1 \leqslant \varphi_1 \leqslant \infty;$$

$$b_i \leqslant \varphi_i \leqslant c_i,\ c \neq 1,\ i \notin \omega^{-}(b);$$

$$b_i = \varphi_i = b_i,\ i \in \omega^{-}(b).$$

We shall confine ourselves to relation (8).

If $a \in A$, $b \notin A$, it results that $1 \in \omega(A)$ and (8) remains unchanged. If $b \in A$, it results that $1 \in \omega^{+}(A)$ and (8) is real, because $\sum_{i\in\omega^{+}(A)} c_i = \infty$. Finally, if $a, b \in A$ and having $c_i = b_i$ for $i \in \omega(b)$ we obtain from (8):

$$\sum_{i\in\omega^{-}(A)} b_i \leqslant \sum_{i\in\omega^{+}(A)} c_i = \sum_{i\in\omega^{+}(A)\cap\omega^{-}(b)} c_i + \sum_{i\in\omega^{+}(A)\setminus\omega^{-}(b)} c_i =$$

$$= \sum_{i\in\omega^{+}(A)\cap\omega^{-}(b)} b_i + \sum_{i\in\omega^{+}(A)\setminus\omega^{-}(b)} c_i,$$

that is relation (13).

Relation (7) is not decomposed in this way but its existence is ensured by relation (14) which implies it.

THEOREM 4. A necessary and sufficient condition for the existence of a minimum flow in the network $\mathfrak{R}$ with the authorized intervals $[b_i, c_i]$, so that $\varphi_1 = \sum_{i\in\omega^{+}(a)} b_i$, is that relations:

$$\sum_{i\in\omega^{+}(A)} b_i \leqslant \sum_{i\in\omega^{-}(A)\cap\omega^{+}(a)} b_i + \sum_{i\in\omega^{-}(A)\setminus\omega^{+}(a)} c_i, \tag{15}$$

$$\sum_{i\in\omega(A)} \omega_i b_i \geqslant 0, \tag{16}$$

should take place for any set $A \subseteq \{a_1, a_2, \ldots, a_n\}$.

Proof. A flow $(\varphi_1, \ldots, \varphi_m)$ is sought for which $\varphi_i = b_i$ for $i \in \omega^+(a)$.

Referring to relation (7), if $a \in A$, $b \notin A$, or $b \in A$, it is verified in a banal manner, or, the figures b_i, $i \in \omega^+(a)$ do not appear, for which $\varphi_i = b_i$.

So, if a, $b \notin A$, from relation (7), we obtain:

$$\sum_{i \in \omega^+(A)} b_i \leqslant \sum_{i \in \omega^-(A)} c_i = \sum_{i \in \omega^-(A) \cap \omega^+(a)} c_i + \sum_{i \in \omega^-(A) \setminus \omega^+(a)} c_i =$$

$$= \sum_{i \in \omega^-(A) \cap \omega^+(a)} b_i + \sum_{i \in \omega^-(A) \setminus \omega^+(a)} c_i,$$

that is relation (15).

Relation (8) is not decomposed in this way, but is implied by relation (16), thus ensuring the sufficiency of the theorem.

The previous results can be applied also to the special transport networks [4] $\widetilde{R}$.

Such a network contains points $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots a_p (P > n)$, and a traffic of materials on the arces of the network is implied between nodes $a_1, a_2, \ldots, a_n$. The network has the authorized intervals $[b_i, c_i]$ $i = 1, 2, \ldots m$. We shall consider network $\mathfrak{R}_{kl}$ $(1 \leqslant k,\ l \leqslant n,\ k \neq l)$, of all roads going from node k to node l. We suppose that for each transport direction $(k,\ l)/[b^{(kl)},\ c^{(kl)}]$ the interval is affected to arc i which is found on a road going from k to l, an interval where:

$$\sum_{\substack{k,l=1 \\ k \neq l}}^{n} b_i^{(kl)} = b_i; \tag{17}$$

$$\sum_{\substack{k,l=1 \\ k \neq l}}^{n} c_i^{(k,l)} = c;$$

$$b^{(kl)} \leqslant c_i^{(kl)};$$

If between any two main nodes a_k, a_l there exists a traffic of materials, we say that there exists a flow in network $\widetilde{\mathcal{R}}$.

Corollary 3. The necessary and sufficient condition for the existence of a flow in network $\widetilde{\mathcal{R}}$, with the main nodes $a_1, \ldots, a_n$ and with the authorized intervals $[b_i, c_i]$ a flow which should saturate the exit arces, is that the relations:

$$\sum_{i \in \omega^-(A)} c_i \geqslant \sum_{\substack{k,l=(A) \\ k \neq l}}^{n} \sum_{\substack{i \in \omega^+(A) \\ i \in \omega^-(al)}} c_i^{(k,l)} + \sum_{\substack{K,l=1 \\ K \neq l}}^{n} \sum_{i \in \omega^+(A)/\omega^-(al} b^{(k,l)},$$

$$\sum_{i \in \omega(A)} \omega_i b_i \geqslant 0$$

should take place, regardless of the set $A \subseteq X \setminus \{a_1, a_2, \ldots, a_n\}$.

Corollary 4. The necessary and sufficient condition for the existence in network $\widetilde{\mathfrak{R}}$ with the main nodes $a_1, a_2, \ldots, a_n$ and with the authorized intervals $[b_i, c_i]$ of a flow which should saturate the entrance arces is that relations:

$$\sum_{i\in\omega^+(A)} c_i \geqslant \sum_{\substack{k,l=1\\k\neq l}}^{n} \sum_{\substack{i\in\omega^-(A)\\ i\in\omega^+(a_k)}} c_i^{(k,l)} + \sum_{\substack{k,l=1\\k\neq l}}^{n} \sum_{i\in\omega^+(A)/\omega^+(a_k)} b_i^{(k,i)},$$

$$\sum_{i\in\omega^-(A)} \omega_i b_i \leqslant 0,$$

should take place, regardless of the set $A\subseteq X\setminus\{a_1, a_2, \ldots, a_n\}$.

Corollary 5. The necessary and sufficient condition for the existence in a transport network with limited capacities $c_i \geqslant 0$ and with n main nodes $a_1, a_2, \ldots, a_n$ of a flow which should saturate the exit arces is that the relation:

$$\sum_{i\in\omega^-(A)} c_i \geqslant \sum_{\substack{k,l=1\\k\neq l}}^{n} \sum_{\substack{i\in\omega^+(A)\\ i\in\omega^-(al)}} c_i^{(k,l)},$$

should take place regardless of the set $A\subseteq X\setminus\{a_1, a_2, \ldots, a_n\}$.

Corollary 6. The necessary and sufficient condition for the existence in a network with n main nodes $a_1, a_2, \ldots, a_n$ and with limited capacities $c_i \geqslant 0$ of a flow which should saturate the entrance arces is that relation:

$$\sum_{i\in\omega^+(A)} c_i \geqslant \sum_{\substack{k,l=1\\k\neq l}}^{n} \sum_{\substack{i\in\omega^-(A)\\ i\in\omega^+(a_k)}} c_i^{(k,l)}$$

should take place regardless of the set $A\subseteq X\setminus\{a_1, a_2, \ldots, a_n\}$

The proves of corollaries 3 and 4 result from theorems 1 and 2 and the demonstrations of corollaries 5 and 6 result from corollaries 1 and 3, respectively 2 and 4, taking into consideration relations (17).

Example 1. The earth is evacuated from a ground in view of the foundation of a building, with the aid of a dump truck. There are two exit points $b_j (j = 1, 2)$ and for a rapid traffic, acces to the site is made through other three points $a_i (i = 1, 2, 3)$.

Knowing that maximum $c_1 = 20$, respectively $c_2 = 23$ dump trucks per hour can pass through the exit points and owing to work on the site on routes $(a_i - b_j)$, $(i = 1, 2, 3;\ j = 1, 2)$, maximum $c_{ij} (i = 1, 2, 3,;\ j = 1, 2)$ dump trucks per hour can pass, c_{ij} being given through matrix:

$$(c_{ij})_{\substack{1\leqslant i\leqslant 2\\ 1\leqslant j\leqslant 3}} = \begin{pmatrix} 10 & 7 \\ 8 & 8 \\ 4 & 8 \end{pmatrix};$$

1) determine the minimum number of dump trucks which must pass through entrances $a_i (i = 1, 2, 3)$ for an optimum functioning of exits $b_j (j = 1, 2)$;

2) establish whether the exists $b_j (j = 1, 2)$ have an optimum functioning if $c'_1 = 18$, $c'_2 = 15$ and $c'_3 = 10$ dump trucks per hour pass through the three entrances $a_i (i = 1, 2, 3)$.

Solution 1). We shall associate to the problem a transport network as follows: $a_i (i = 1, 2, 3)$, $b_j (j = 1, 2)$ will be the inner nodes between which we shall consider arces (a_i, b_j), $(i = 1, 2, 3; j = 1, 2)$ with the capacities $c_{ij} (i = 1, 2, 3; j = 1, 2)$. Consider a node b (exit) and arces $(b_j\ b,\ )$, $(j = 1, 2)$ with the capacities $c_j (j = 1, 2)$ and another node a (entrance) and arces (a, a_i), $(i = 1, 2, 3)$ with the capacities $c'_i (i = 1, 2, 3)$.

The exits $b_j (j = 1, 2)$ have an optimum functioning if the exit arces of the associated transport network are saturated by a flow passing through the network. According to Gale's theorem relation (3') must take place, where $A \subseteq \{a_1, a_2, a_3, b_1, b_2\}$.

In keeping with the set A, relation (3') is leading to several situations; for example, if $A = \{b_1, b_2\}$, $\sum_{i \in \omega^-(A)} c_i = \sum_{i=1}^{3} c_{i_1} + \sum_{i=1}^{3} c_{i2} = (10 + 8 + 4) + (7 + 8 + 8) = 45$ and $\sum_{i \in \omega^+(A) \cap \omega^-(b)} c_i = < c_1 + c_2 = 43$, are obtained, so $45 > 43$. All situations are included in table 1.

Examining the last column of table 1, results:

$$\begin{cases} c'_1 \geqslant 15 \\ c'_2 \geqslant 14 \\ c'_3 \geqslant 10 \\ c'_1 + c'_2 \geqslant 31 \\ c'_1 + c'_3 \geqslant 27 \\ c'_2 + c'_3 \geqslant 26 \\ c'_1 + c'_2 + c'_3 \geqslant 43 \end{cases} \tag{18}$$

Since the exits have an optimum functioning equivalent to the passing of 43 dump trucks per hour, it results that:

$$c'_1 + c'_2 + c'_3 = 43, \tag{19}$$

because a surplus of dump trucks either might not enter the network or, even if they would enter, they should station in the network.

Taking into consideration relation(19) and from the first six inequations of system (18), it results:

$$\begin{aligned} 15 &\leqslant c'_1 \leqslant 17 \\ 14 &\leqslant c'_2 \leqslant 16 \\ 10 &\leqslant c'_3 \leqslant 12 \end{aligned} \tag{20}$$

The number systems (c_1', c_2', c_3') which verify relations (19) and (20) are the solutions to the problem. They are: (15, 16, 12), (16, 15, 12), (16, 16, 11), (17, 14, 12), (17, 15, 11) and (17, 16, 10).

The problem having six solutions, practically that very one is chosen for which some of the economic coefficients depending on routes and on the condition of the dump trucks and which are not taken into consideration by the problem, have greater efficiency in the carrying out of work with a very low cost.

2) Although $c_1' + c_2' + c_3' = 18 + 15 + 10 = 43$, system (18, 15, 10) is not a solution to the problem because the distribution of the 43 dump trucks for the three entrances $a_i (i = 1, 2, 3)$ is not made correctly.

Example 2. A town is crossed by a continuos flow of dump trucks. Access is taking place through two entrances $a_i (i = 1, 2)$ of capacities $c_i' (i=1, 2)$ of dump trucks per hour, evacuation is made through two exits $b_j (j = 1, 2)$ of capacities $c_j (j = 1, 2)$ of dump trucks per hour and the crossing is made on routes (a_i, b_j), $(i, j = 1, 2)$ of capacities $c_{ij} (i, j = 1, 2)$ of dump trucks per hour. Determine capacities $c_i' (i = 1, 2)$ if the flow of dump trucks is optimum at the entrance.

A particular case:

$$\begin{matrix} c_1 = 250 \\ c_2 = 230 \end{matrix} \qquad (c_{ij})\ 1 \leqslant ij \leqslant 2 = \begin{pmatrix} 150 & 125 \\ 200 & 220 \end{pmatrix}$$

Solution Problem is associated a transport network with the inner nodes $a_i (i = 1, 2)$, $b_j (j = 1, 2)$ linked between them through arces (a_i, b_j), $(i, j = 1, 2)$ with capacities $c_{ij} (i = 1, 2)$, with entrance a and arces (a, a_i), $(i = 1, 2)$ of capacities $c_i' (i = 1, 2)$ and with exit b and arces (b_j, b), $(j = 1, 2)$ of capacities $c_j (j = 1, 2)$.

The conditions of the problem require that the flow passing through the network should saturate the entrance arces into the network and in this case relation (8) is taking place with the set $A = \{a_1, a_2, b_1, b_2\}$.

The various situations of the set are comprised in table 2.

It results from the last column:

$$\begin{matrix} c_1' \leqslant 275 \\ c_2' \leqslant 420 \\ c_1' + c_2' \leqslant 480 \end{matrix}$$

The flow being optimum at the entrance, $c_1' + c_2' = 480$, $c_1' \leqslant 275$, $c_2' \leqslant 420$. So, we can consider $c_1' = 200$ and $c_2' = 280$.

Table 1

Nr.	A	$\sum_{i\in\omega^-(A)} c_i$	$\sum_{i\in\omega^+(A)\cap\omega^-(b)} c_i$	$\sum_{i\in\omega^-(A)} c_i \geqslant \sum_{i\in\omega^+(A)\cap\omega(b)} c_i$	Conclusions
1	$\{b_1\}$	22	20	$22 \geqslant 20$	—
2	$\{b_2\}$	23	23	$23 = 23$	—
3	$\{b_1\ b_2\}$	45	43	$45 \geqslant 43$	—
4	$\{a_1,\ b_1\}$	$c'_1 + 12$	20	$c'_1 + 12 \geqslant 20$	$c'_1 \geqslant 8$
5	$\{a_1,\ b_2\}$	$c'_1 + 16$	23	$c'_1 + 16 \geqslant 23$	$c'_1 \geqslant 7$
6	$\{a_2,\ b_1\}$	$c'_2 + 14$	20	$c'_2 + 14 \geqslant 20$	$c'_2 \geqslant 6$
7	$\{a_2\ b_2\}$	$c'_2 + 15$	23	$c'_2 + 15 \geqslant 23$	$c'_2 \geqslant 8$
8	$\{a_3,\ b_1\}$	$c'_3 + 18$	20	$c'_3 + 18 \geqslant 20$	$c'_3 \geqslant 2$
9	$a_3,\ b_2\}$	$c'_3 + 15$	23	$c'_3 + 15 \geqslant 23$	$c'_3 \geqslant 8$
10	$\{a_1,\ b_1,\ b_2\}$	$c'_1 + 28$	43	$c'_1 + 28 \geqslant 43$	$c'_1 \geqslant 15$
11	$\{a_2,\ b_1\ b_2\}$	$c'_2 + 29$	43	$c'_2 + 29 \geqslant 43$	$c'_2 \geqslant 14$
12	$\{a_3,\ b_1,\ b_2\}$	$c'_3 + 33$	43	$c'_3 + 33 \geqslant 43$	$c'_3 \geqslant 10$
13	$\{a_1,\ a_2,\ b_1\}$	$c'_1 + c'_2 + 4$	20	$c'_1 + c'_2 + 4 \geqslant 20$	$c'_1 + c'_2 \geqslant 16$
14	$\{a_1,\ a_3,\ b_1\}$	$c'_1 + c'_3 + 8$	20	$c'_1 + c'_3 + 8 \geqslant 20$	$c'_1 + c'_3 \geqslant 12$
15	$\{a_2,\ a_3,\ b_1\}$	$c'_2 + c'_3 + 10$	20	$c'_2 + c'_3 + 10 \geqslant 20$	$c'_2 + c'_3 \geqslant 10$
16	$\{a_1,\ a_2,\ b_2\}$	$c'_1 + c'_2 + 8$	23	$c'_1 + c'_2 + 8 \geqslant 23$	$c'_1 + c'_2 \geqslant 15$
17	$\{a_1,\ a_3,\ b_2\}$	$c'_1 + c'_3 + 8$	23	$c'_1 + c'_3 + 8 \geqslant 23$	$c'_1 + c'_3 \geqslant 15$
18	$\{a_2,\ a_3,\ b_2\}$	$c'_2 + c'_3 + 7$	23	$c'_2 + c'_3 + 7 \geqslant 23$	$c'_2 + c'_3 \geqslant 16$
19	$\{a_1, a_2, b_1\ b_2\}$	$c'_1 + c'_2 + 12$	43	$c'_1 + c'_2 + 12 \geqslant 43$	$c'_1 + c'_2 \geqslant 31$
20	$\{a_1,\ a_2,\ b_1\ b_2\}$	$c'_1 + c'_3 + 16$	43	$c'_1 + c'_3 + 16 \geqslant 43$	$c'_1 + c'_3 \geqslant 27$
21	$\{a_2, a_3, b_1,\ b_2\}$	$c'_2 + c'_3 + 17$	43	$c'_2 + c'_3 + 17 \geqslant 43$	$c'_2 + c'_3 \geqslant 26$
22	$\{a_1,\ a_2,\ a_3,\ b_1\}$	$c'_1 + c'_2 + c'_3$	20	$c'_1 + c'_2 + c'_3 \geqslant 20$	$c'_1 + c'_2 + c'_3 \geqslant 20$
23	$\{a_1,\ a_2,\ a_3,\ b_2\}$	$c'_1 + c'_2 + c'_3$	23	$c'_1 + c'_2 + c'_3 \geqslant 23$	$c'_1 + c'_2 + c'_3 \geqslant 23$
24	$\{a_1,\ a_2,\ a_3,\ b_1,\ b_2\}$	$c'_1 + c'_2 + c'_3$	43	$c'_1 + c'_2 + c'_3 \geqslant 43$	$c'_1 + c'_2 + c'_3 \geqslant 43$

Table 2

Nr.	A	$\sum_{i\in\omega^+(A)} c_i$	$\sum_{i\in\omega^-(A)\cap\omega^+(a)} c_i$	$\sum_{i\in\omega^+(A)} c_i \geqslant \sum_{i\in\omega^-(A)\cap\omega^+(a)} c_i$
1	$\{a_1\}$	275	c'_1	$275 \geqslant c'_1$
2	$\{a_2\}$	420	c'_2	$420 \geqslant c'_2$
3	$\{a_1, b_1\}$	375	c'_1	$375 \geqslant c'_1$
4	$\{a_1, b_2\}$	380	c'_1	$380 \geqslant c'_1$
5	$\{a_2, b_1\}$	470	c'_2	$470 \geqslant c'_2$
6	$\{a_2, b_2\}$	430	c'_2	$430 \geqslant c'_2$
7	$\{a_1, b_1, b_2\}$	480	c'_1	$480 \geqslant c'_1$
8	$\{a_2, b_1, b_2\}$	480	c'_2	$480 \geqslant c'_2$
9	$\{a_1, a_2, b_1\}$	595	$c'_1 + c'_2$	$595 \geqslant c'_1 + c'_2$
10	$\{a_1, a_2, b_2\}$	580	$c'_1 + c'_2$	$580 \geqslant c'_1 + c'_2$
11	$\{a_1, a_2, b_1, b_2\}$	480	$c'_1 + c'_2$	$480 \geqslant c'_1 + c'_2$

REFERENCES

1. Berge C., Ghoulia-Houri A.: *Programmes, jeux et réseaux de transport.* Dunod, Paris, 1962.
2. Berge C.: *Teoria grafelor şi aplicaţiile ei.* (The Theory of Graphs and its Applications). Editura ştiinţifică. Bucharest, 1969.
3. Mititelu S.: *Determinarea cantităţii maxime de materiale care trece printr-o reţea de transport şi distribuţia ei pe arce.* (Determination of the Maximum Quantity of Materials Passing Through a Transport Network and its Distribution on Arces). Mathematical Studies and Research, nr. 5, volume 21. Editura Academiei R. S. România, Bucharest, 1969.
4. Mititelu S.: *Extinderea unei probleme de maxim la reţele oarecare de transport.* (Extension of a Problem of Maximum to Certain Transport Networks). Mathematical Studies and Research, nr. 7, volume 21, Editura Academiei R. S. România, Bucharest,1969.
5. Kaufmann A.: *Metode şi modele ale cercetării operaţionale.* (Operational Research Methods and Models). Vol. 2, Editura Ştiinţifică, Bucharest, 1967.
6. Ionescu H., Dinescu C., Burlacu V.: *Teoria grafelor cu unele aplicaţii în economie.* (The Theory of Graphs with Certain Applications in Economy). Editura ştiinţifică, Bucharest, 1969.

FLUX QUI SATISFONT LES ENTREES ET LES SORTIES DANS LES RÉSEAUX DE TRANSPORT

On considère un réseau de transport à intervalles autorisés $[b_i, c_i]$, $(i \neq 1)$ où $1 = (b, a)$ est l'arc de retour.

On étudie dans cet article l'existence des flux maximums qui vont saturer les arc-d'entrée ou de sortie du réseau, ainsi que l'existence de certains flux maximums qui génès

rent, dans le réseau, des flux égaux à la somme des capacités minimums des arcs d'entrée respectivement de sortie du réseau.

Pour $b_i = 0$ on obtient des résultats correspondants pour les réseaux de transport à capacités limitées. Les résultats obtenus sont appliqués aux réseaux de transport ayant plusieurs entrées et sorties.

ПОТОКИ, НАСЫЩАЮЩИЕ ВХОДЫ И ВЫХОДЫ В ТРАНСПОРТНЫХ СЕТЯХ

Дается транспортная сеть с авторизованными пределами $[b_i, c_i]$, $(i \neq 1)$, где $1 = (b, a)$ — возвратная дуга. В статье изучается наличие некоторых потоков максимального значения, которые порождают в сети потоки, равные сумме наименьших емкостей входных дуг, соответственно выходных дуг сети.

Для $b_i = 0$ получаются соответствующие результаты для транспортных сетей с ограниченными емкостями. Полученные результаты применяются в транспортных сетях с большими входами и выходами.