V. MIERLEA
State Committee of Planning

An Algorithm for Finding the Minimal Length Hamiltonian Path in a Graph

Operational research is a very useful instrument for the optimum decision-making action having a main class of elements — algorithms of mathematical models.

The problem of finding a minimal length Hamiltonian path in a graph is well known in the specialty literature. Because it arises in many activities, such as equipment work planning, vehicle routes, etc., several algorithms have been put forth with a view to achieving a solution. However, owing to rather the long computing time required, this problem is still open to further approach.

To make easier the algorithm programming work we shall somewhat schematize the problem and the algorithm. To this end we shall use the following notations:

DP = problem data
D = definition
PR = proposition, assertion
P = postulate, that is a theorem assumed to be known, which is being used for proving purposes
CR = criterion
C = conclusion
O = remark.

We shall also use the IBM symbols for drawing up a flow chart.

1. MATHEMATICAL STATEMENT OF THE PROBLEM

D1 = graph definition

DP1: Consider a symmetrical, full, loopless graph $\mathbf{G} = (X, U)$. In other words $\forall\, x_i,\ x_j \in X,\ i \neq j,\ \exists (x_i, x_j) \in U$, $(x_j, x_i) \in U$ and $(x_i,\ x_i) \notin U$ (Fig. 1).

DP2: To each arc $u_{i,j} = (x_i, x_j) \in U$ a finite number $l_{i,j} \geqslant 0$, $(i \neq j)$ and $l_{i,i} = \infty$; {**DP1**} is associated.

D2: We call $l_{i,j}$ — the "length" of $u_{i,j}$; {**DP2**}.

D3: The definition of the Hamiltonian path in a graph; {**D1**}.

D4: Consider $\mathcal{C}$ — to be the set of Hamiltonian paths μ in **G**; {**DP1**}.

O1: Further, we shall cut out indexes for $u_{i,j}$, $l_{i,j}$ etc. each time that no confusions are possible; {**DP2**}.

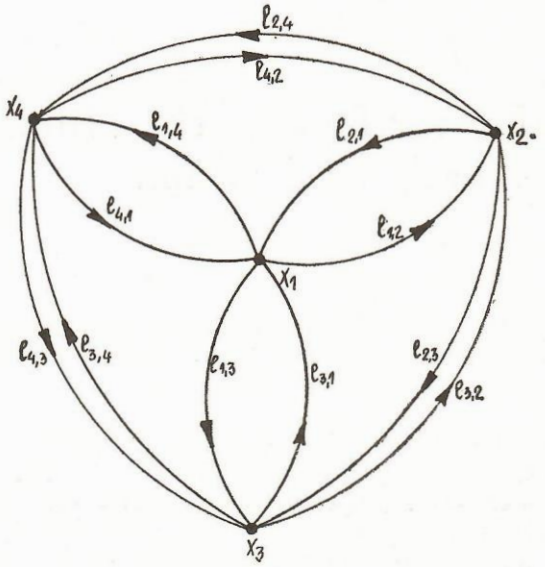


Fig. 1

D5: We shall call $l(\mu)$ — the "length" of Hamiltonian path μ, where $l(\mu) = \sum_{u \in \mu} l(u)$; {**O1**, **D4**,**D2**}.

DP3: It is required to find μ so that

$$\underline{l}(\mu) = \min_{\mu \in \mathcal{C}} \sum_{u \in \mu} l(u); \quad \{\mathbf{D5}, \mathbf{O1}\}.$$

2. DESCRIPTION OF THE ALGORITHM

The algorithm through which the minimal length Hamiltonian path in a graph is to be found is given in the flow chart (Fig. 2) containing the necessary notions. We give these notions further.

D6: Let n be the number of vertices $x \in X$; {**DP1**}.

P1: In **G**, $\exists n!$ Hamiltonian paths {**D3**, **DP1**}.

O2: The idea underlying the algorithm is to find a number of arcs $u_{i,j} \in U$ which certainly do not belong to minimal length Hamilto-

nian path. This is achieved in agreement with a certain criterion which is described further. By leaving out the arcs found in this way, the number of Hamiltonian paths in G will decrease.

D7: Consider matrix $L = [l_{i,j}]$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ where

$$l_{i,i} = \infty; \quad \{\mathbf{D6}, \mathbf{DP2}\}.$$

D8: $N = \{1, 2, \ldots n\}$; $\{\mathbf{D6}\}$.

D9: $L_i = \{l_{i,1}, l_{i,2}, \ldots l_{i,n}\}$, $(i \in N)$; $\{\mathbf{D8}, \mathbf{D7}\}$.

D10: $L^j = \{l_{1,j}, l_{2,j}, \ldots, l_{n,j}\}$, $(j \in N)$; $\{\mathbf{D8}, \mathbf{D7}\}$.

D11: Consider ω to be an increasing order operator of elements

$$l_{i,j} \in L_i, \ (1 \leqslant j \leqslant N); \quad \{\mathbf{D9}, \mathbf{D8}\}.$$

D12: $L_{0,i} = \omega L_i$, $(i \in N)$; $\{\mathbf{D11}, \mathbf{D9}, \mathbf{D8}\}$.

03: It follows from $\{\mathbf{D12}\}$ that

$$L_{0.i} = \{l_{i,j_{i,1}}; \ l_{i,j_{i,2}}, \ldots, l_{i,j_{i,n}}\}$$

where: $l_{i,j_{i,1}} \leqslant l_{i,j_{i,2}} \leqslant \ldots \leqslant l_{i,j_{i,n}}$; $\{\mathbf{D11}\}$.

D13: $L_0 = [l_{i,j_{i,k}}]$, $l_{i,j_{i,k}} \in L_{0.i}$, $1 \leqslant i \leqslant n$. $1 \leqslant k \leqslant n$; $\{\mathbf{D12}, \mathbf{D6}\}$.

04: $\{\mathbf{D12}\}$ can also be written:

$$L_0 = \omega L; \quad \{\mathbf{D13}, \mathbf{D11}, \mathbf{D7}\}.$$

D14: $L_0^k = \{l_{1,j_{1,k}}, l_{2,j_{2,k}}, \ldots, l_{n,j_{n,k}}\}$,

$$l_{i,j_{i,k}} \in L_0, \ (i \in N); \quad \{\mathbf{D13}, \mathbf{D8}\}.$$

D15: $d = \sum_{i=1}^{n} l_{i,j_{i,1}}$ where $l_{i,j_{i,1}} \in L_0^1$, $(i \in N)$; $\{\mathbf{D14}, \mathbf{D8}\}$.

PR1: $d \leqslant \underline{l}(\mu)$ $\{\mathbf{D15}, \mathbf{DP3}\}$.

05: $\{\mathbf{PR1}\}$ becomes obvious if we take into account $\{\mathbf{DP3}\}$ and $\{\mathbf{D15}\}$. Indeed by assuming

$$\underline{l}(\mu) = \min_{\mu \in \varrho} l(\mu) \quad \text{where} \quad \underline{l}(\mu) = \sum_{\substack{i=1 \\ k \in N \setminus \{n\}}}^{u} l_{i,j_{i,k}}$$

$\{\mathbf{PR1}\}$ can consequently be written

$$\sum_{i=1}^{n} l_{i,j_{i,1}} \leqslant \sum_{\substack{i=1 \\ k \in N \setminus \{n\}}}^{n} l_{i,j_{i,k}} \qquad \text{or}$$

$$\sum_{\substack{i=1 \\ K \in N \setminus \{n\}}}^{n} (l_{i,j_{i,1}} - l_{i,j_{i,k}}) \leqslant 0$$

This relationship holds true if $\{\mathbf{03}\}$ is taken into account. We have assumed $k \in N \setminus \{n\}$, because it follows from $\{\mathbf{03}, \mathbf{DP2}\}$ that $l_{i,j_{i,n}} = \infty$. For this reason we shall take $u_{i,j_{i,n}} \notin \mu$ for which $\{\mathbf{DP3}\}$ is complied with.

D16: $m_i = max\,\{k \in N / l_{i,\,j_{i,k}} \in L_{0,i},\ u_{i,\,j_{i,k}} \in \mu,\ \mu \in \mathcal{C}$
and we do not know whether μ verifies {**DP3**}, $i \in N$; {**D12, D8 D4**},

D17: $\mathcal{L} = \{l_{i,\,j_{i,k}} \in L_0 / 1 \leqslant k \leqslant m_i,\ i \in N\}$; {**D16, D13, D8**}.

D18: $\mathcal{J}$ is a counter for Hamiltonian paths $\mu \in \mathcal{C}$ to be found; {**D4**}.

D19: $l^{\mathcal{J}} = l(\mu^{\mathcal{J}})$, $\mu^{\mathcal{J}} \in \{\mathcal{C} / l(\mu^{\mathcal{J}}) \in \mathcal{L}\}$; {**D18, D17, D4**}.

CR1: If $l(\mu^{\mathcal{J}}) \leqslant d - l_{i,\,j_{i,1}} + l_{i,\,j_{i,m_i}}$ (1)

where $l_{i,\,j_{i,1}} \in L_0^1$, $l_{i,\,j_{i,m_i}} \in L_0^m i$ then

$\forall \mu \in \mathcal{C}$ so that $l(\mu) \in \mathcal{L}$, $\mu \ni u_{i,\,j_{i,m_i}}$, we shall have

$$l(\mu) \geqslant l(\mu^{\mathcal{J}});\ \{\mathbf{D18, D16, DP2, D17}\}. \tag{2}$$

Indeed, it follows from {**PR1**} that

$$d - l_{i,\,j_{i,1}} \leqslant l(\mu) - l_{i,\,j_{i,m_i}} \tag{3}$$

or

$$l(\mu) \geqslant d - l_{i,\,j_{i,1}} + i_{i,\,j_{i,m_i}} \tag{4}$$

(1) and (4) lead to (2); {**03**}.

D20: $n_i = max\,\{k \in N / l_{i,\,j_{i,k}} \in L_{0,i}$ for which $u_{i,\,j_{i,k}}$ are considered in order to find $\mu^{\mathcal{J}}\}$; {**D18, D12, D8**}.

06: n_i, $i \in N$, defined in {**D20**} shows the number of elements

$l_{i,\,j_{i,n_i}} \in L_{0,i}$ where $l(\mu^{\mathcal{J}})$ is to be found.

D21: $N_i = \{t_i \in N / t_i \leqslant n_i\}$, $(i \in N)$; {**D20, D8**}.

D22: $\mathcal{L}_r = \{l_{I,\,j_{I,t}} \in L_{0,I} / 1 \leqslant t \leqslant n_I,\ 1 \leqslant I \leqslant n\}$; {D20, 06, D8, D12}

D23: $\mu^{\mathcal{J}}_{n_I} = [u_{I,\,j_{I,n_I}}, \ldots]$ so that $l(u_{i,\,j}) \in \mathcal{L}_r$, $u_{i,\,j} \in \mu^{\mathcal{J}}_{n_I}$ $i \in N$; {**D22, D20, D18, D8**}.

D24: l — the name of a variable to which we shall attach $\min\limits_{\mathcal{J}} l(\mu^{\mathcal{J}}_{I,\,j_{I,n_I}})$, where $l(\mu^{\mathcal{J}}_{I,\,j_{I,n_I}}) \in \mathcal{L}_r$; {**D22, D18**}.

We shall originally assume $l = \sum\limits_{i=1}^{n} l_{i,\,j_{i,m_i}}$ because it is easy to verify that $\sum\limits_{i=1}^{n} l_{i,\,j_{i,m_i}} \geqslant \sup\limits_{\mathcal{J}} \{l(\mu^{\mathcal{J}}_{I,\,jI,\,n_I})\}$, $I \in N$, $n_I \leqslant m_I$

D25: $\bar{\mu}$ the name of a variable to which we attach Hamiltonian path

$\bar{\mu} = \mu^{\mathcal{J}}_{n_I}$, for which $l > l(\mu^{\mathcal{J}}_{n_I})$; {**D24, D23**}.

07: The cycling **11 — 12 — 13** in the flow chart is used in order to leave out $l_{i,\,j_{i,m_i}}$ in $\mathcal{L}$ for which {**CR1**} is verified: {**D17, D16**}. (1)

08: The cycling **15—16—10** in the flow chart is carried out in order to make use of {**CR1**} for all the values $i \in N$; {**D8**}.

O9: The cycling **17—7—8** in the flow chart is used in order to find $l = \min_{\mathcal{J}} l(\mu^{\mathcal{J}}_{I, j_{I, n_I}})$; {**D24**, **D23**}.

D26: $r = \max_{i \in N} \{n_i\}$; {**D20**, **D8**}.

D27: V_t is a variable containing information to be used in order to find $\mathcal{L}_{r+1}$ by increasing variable n_t; {**D22**}.

O10: The signification of using variable V_t in {**D27**} stands out more clearly if cyclings **25 — 26 — 27 — 31** and **25 — 30 — 27 — 31** are followed up.

D28: k a counter for variations of V_t; {**D27**, **O10**}.

D29: $t \in N$, t a counter

D30: $I \in N$, I a counter

C1: The algorithm described can also be used for finding the minimal length Hamiltonian path in a graph whatever. This is possible if in {**D7**} we assume

$$l_{i, j} = \infty \text{ if } \mu_{i, j} \notin U; \quad \{\mathbf{DP1}\}.$$

C2: This algorithm can also be used for an increasing ordering of set $\{l(\mu)/\mu \in \mathcal{C}\}$ if in {**CR1**} $\min_{\mathcal{J}} \{l(\mu^{\mathcal{J}}_{I, j_{I,1}})\}$ is not assumed to be minimum, but the second, the third, the... therm of the increasing succession $\{l(\mu^{\mathcal{J}^K}_{I, j_{I, 1}})\}$ $\mathcal{J}_K \in \mathcal{J}\}$.

ILLUSTRATION

To render evident the importance of such a solution for the economy, we shall illustrate it through a problem which arises in finding out the minimal length Hamiltonian path in a graph.

Find the succession in which activities a_1, a_2, a_3, a_4, are to be carried out on a machine so that the unemployed time of the machine throughout the work process shall be minimal.We assume that the machine is not being employed during the adaptation time for performing a new activity and that this time depends on the activity carried out and on the activity to be carried out. The adaptation time is assumed to be known.

Matrix L of the adaptation times is given in Table I, where an element on a line, for instance on line 3, stands for the adaptation time of the machine in order to carry out one of the four activities, when the third one had been performed.

It is assumed that, if the third activity is followed by the same third activity, element $l(u_{33}) = \infty$.

In Table II, elements $l_{i, j_{i, k}} \in L_0$, and index $j_{i, k}$; {**D13**} are rendered evident.

Thus, following up the flow chart (Fig. 2), we find $\underline{l}(\mu)$ ({**DP3**})

$$d = 6; \mathbf{3} \ , \ d \ngeq l(\mu); \ \mathbf{4}$$

$$m_i := 3; \ n_i := 2; \ r := 2; \ V_i := r; \ I := 1; \ N_i := \{1,2\}; \ I := 1;$$

Table I

i \ j	1	2	3	4
1	∞	17	12	3
2	5	∞	0	4
3	8	3	∞	10
4	4	0	5	∞

Table II

i \ $j_{i,k}$	$k=1$	$k=2$	$k=3$	$k=4$
1	$3_{(4)}$	$12_{(3)}$	$17_{(2)}$	∞
2	$0_{(3)}$	$4_{(4)}$	$5_{(1)}$	∞
3	$3_{(2)}$	$8_{(1)}$	$10_{(4)}$	∞
4	$0_{(2)}$	$4_{(1)}$	$5_{(3)}$	∞

$$I \in Nl := 37,\ \mathbf{5},\ \mathbf{6}.$$

$$\mu^{\mathcal{J}}_{n_I} = [u_{1,4}\ \ u_{4,2}\ \ u_{2,3},\ \ u_{3,1}];\ \mathbf{7}$$

$$l(\mu^{\mathcal{J}}_{n_I}) = 3 + 0 + 0 + 8 = 11$$

$$37 > 11;\ \mathbf{8}$$

$$l := 11,\ \bar{\mu} := [u_{1,4},\ u_{4,2'},\ u_{2,3'},\ u_{3,1}];\ \mathbf{9}$$

For: i: $+1$

$11 \leqslant 6 - 3 + 17 \Rightarrow m_1 := 2$

$11 \leqslant 6 - 3 + 12 \Rightarrow m_1 := 1;\ n_1 := 1$

$11 > 6 - 3 + 3$; (the cycling **10 — 11 — 12 — 13**).

For: $i := 2$

$11 \leqslant 6 - 0 + 5 \Rightarrow m_2 := 2$

$11 > 6 - 0 + 4$

For: $i := 3$

$11 \leqslant 6 - 3 + 10 \Rightarrow m_3 := 2$

$11 \leqslant 6 - 3 + 8 \Rightarrow m_3 := 1;\ n_3 := 1$

$11 > 6 - 3 + 3$

For: $i := 4$

$11 \leqslant 6 - 0 + 5 \Rightarrow m_4 := 2$

$11 > 6 - 0 + 4$

Since there is no other new $\mu \subset \mathcal{L}(\{\mathbf{D}17\})$, it follows that $l(\mu) = 11$, $\mu = [u_{1,4},\ u_{4,2}\ u_{2,3},\ u_{3,1}]$.

REFERENCES

1. C. BERGE: *Teoria grafurilor şi aplicaţiile ei.* (Graph Theory and its Applications). Editura tehnică, Bucharest 1969. (Translated from French language).
2. J. D. C. LITTLE, G. N. KATTA, W. A. DURA, C. KAREL, *An Algorithm for the Salesman Problem.* Operations Research, 11, 6 (1963).
3. I. NĂDEJDE, C. ZIDĂROIU, C. BERTHALLER, S. SBURLAN *Probleme de cercetare operaţională.* (Problems of Operational Research). Editura Academiei R.S.R., Bucharest, 1971.

ALGORITHME POUR DETERMINER LE CIRCUIT HAMILTONIEN DE „LONGUEUR" MINIME DANS UN GRAPHE

L'algorithme de calcul présenté porte sur la détermination de la solution optime de l'ordonnancement séquentiel de n travaux sur m machines, ayant un temps minime entre le démarrage et l'arrêt, de telle façon que le temps de non-utilisation des machines m_i (entre les opérations des travaux L^j) soit minime.

Mettant en évidence certains rapports entre x_i^j (le temps non-utilisé par les machines pour le travail L^j, entre les opérations $O_{i-1}^{j,1}$ et O_i^j considérant que $O_i^j = O_i^{j,1} = + O_i^{j,2}$), y_{i-1}^j (le temps non-utilisé par la machine m_i entre les opérations O_i^{j-1} et O_i^j), O_i^j (le temps d'obtenir un stock minime d'opérations O_{i-1}^j, que nous noterons $O_{i-1}^{j,1}$ et qui sont nécessaires pour commencer l'opération O_i^j) et b_i^j (le temps minime nécessaire pour terminer l'opération O_i^j après avoir terminer l'opération O_{i-1}^j) on obtient un algoritqme pour déterminer M de la relation:

$$M = \min_{p \in P} \sum_{i=2}^{m} \sum_{j=2}^{n} y_{i,p}^j$$

où P représente la foule des permutations des travaux L^j.

АЛГОРИТМ ДЛЯ ОПРЕДЕЛЕНИЯ ГАМИЛЬТОНОВОЙ ЦЕПИ НАИМЕНЬШЕЙ «ДЛИНЫ» В ГРАФЕ

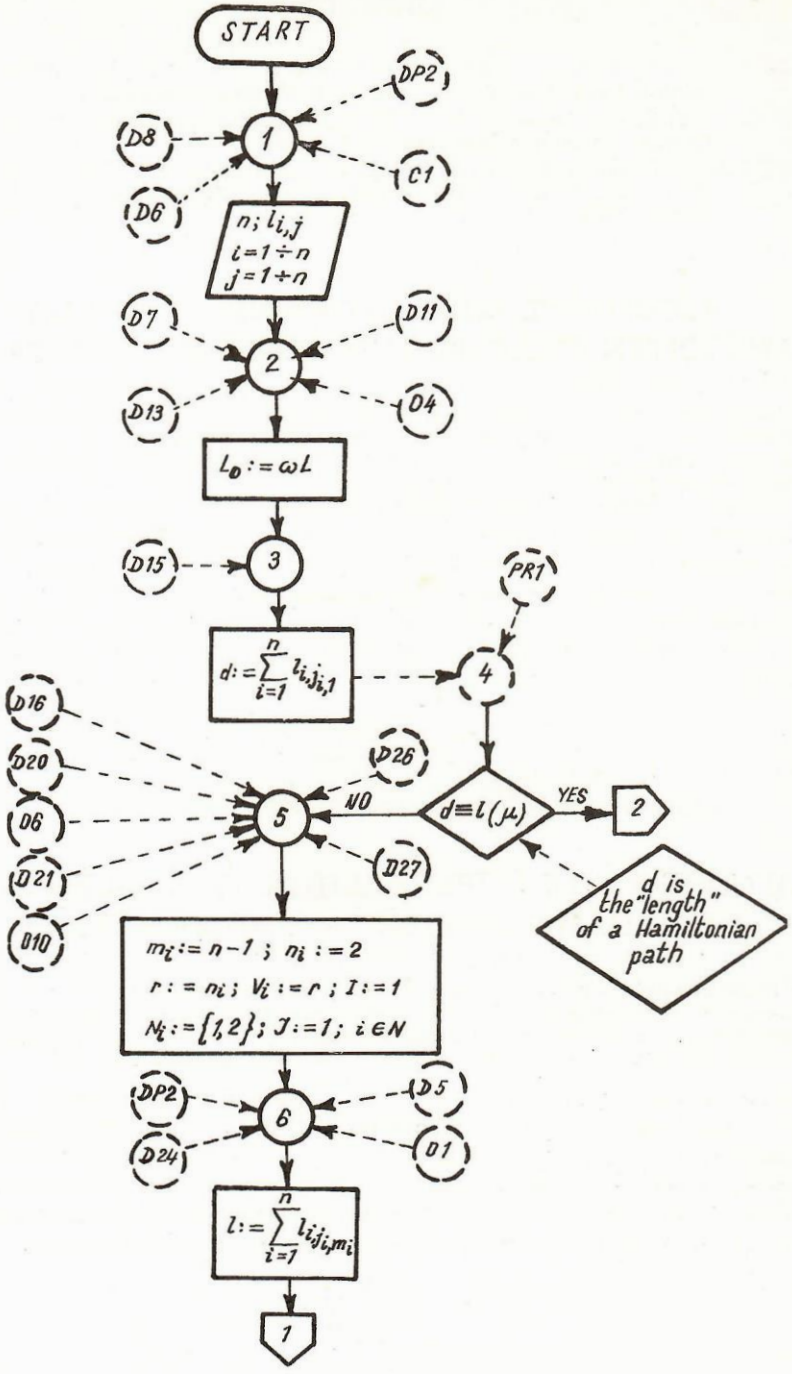
Представляемый алгоритм расчета относится к определению оптимального решения секвенционного составления документов для выполнения n работ на m машинах с наименьшим межоперационным временем при пуске и остановке таким образом, чтобы время неиспользования машин m_i (между операциями работ L^j) было наименьшим.

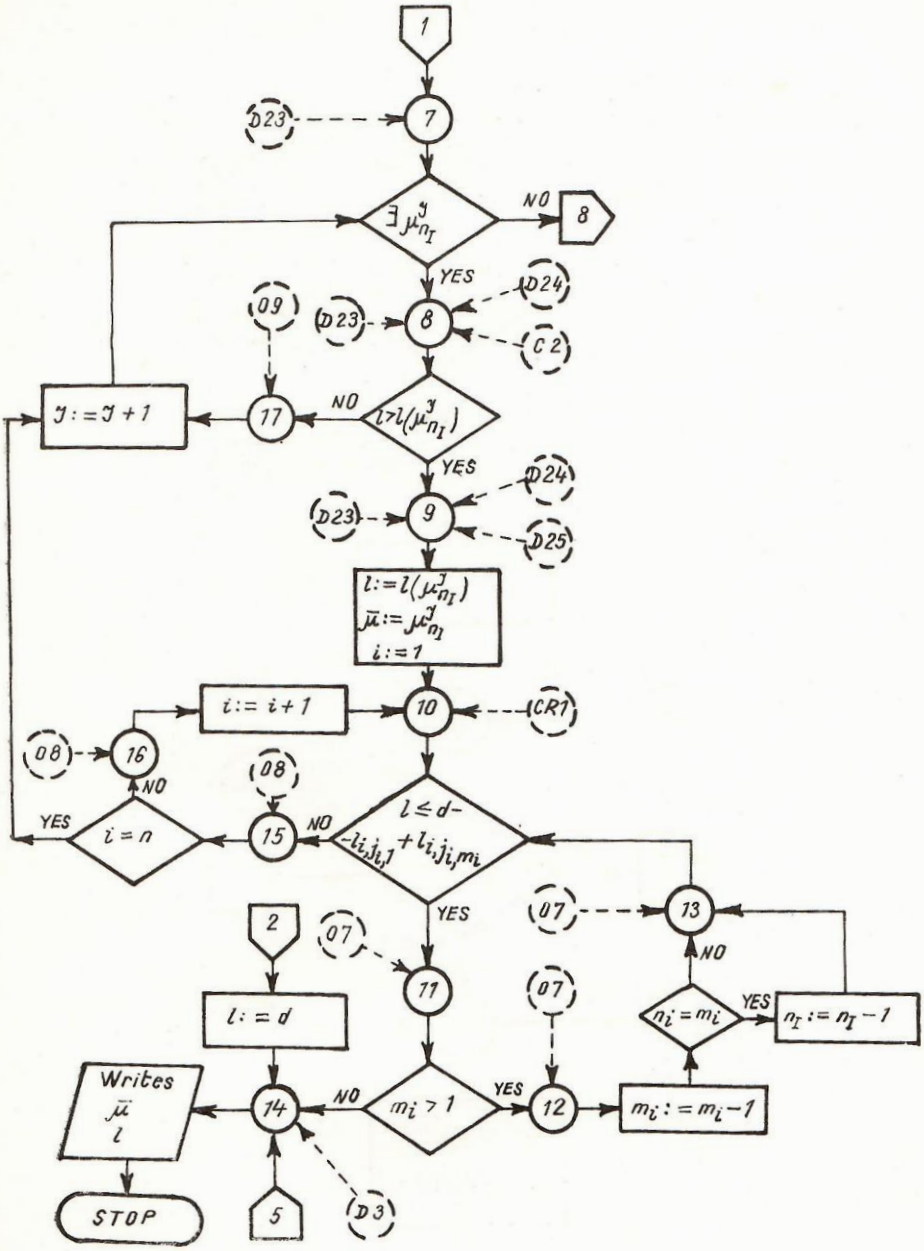
Выявляя некоторые отношения между x_i^j (временем неиспользования машин для работы L^j между операциями $O_{i,1}^{j-1}$ O_j^i, считая, что $O_i^j = O_i^{j,1} + O_i^{j,2}$), y_{i-1}^j (временем неиспользования машины m_i между операциями O_i^{j-1} и O_i^j), O_i^j (временем получения наименьшего запаса операций O_{i-1}^j, обозначаемых $O_{i-1}^{j,1}$ и необходимых для начала операции O_i^j) и b_i^j (наименьшим временем, необходимым для окончания операции O_i^j после окончания операции O_{i-1}^j), автор получает алгоритм для определения величины M отношения:

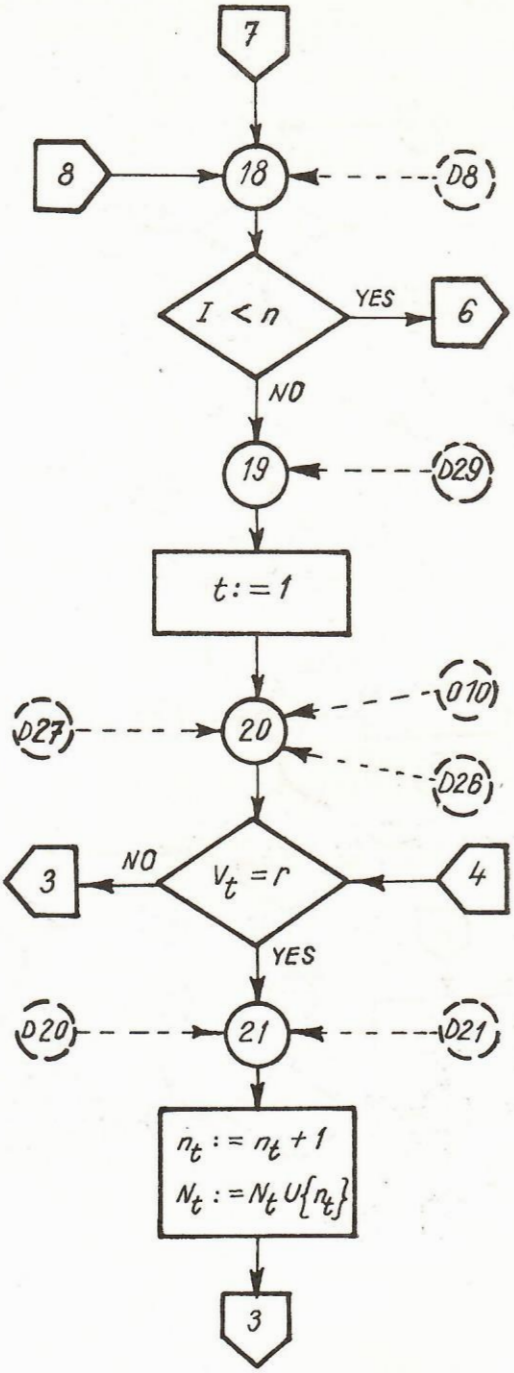
$$M = \min_{p \in P} \sum_{i=2}^{m} \sum_{j=2}^{n} y_{i,p}^j,$$

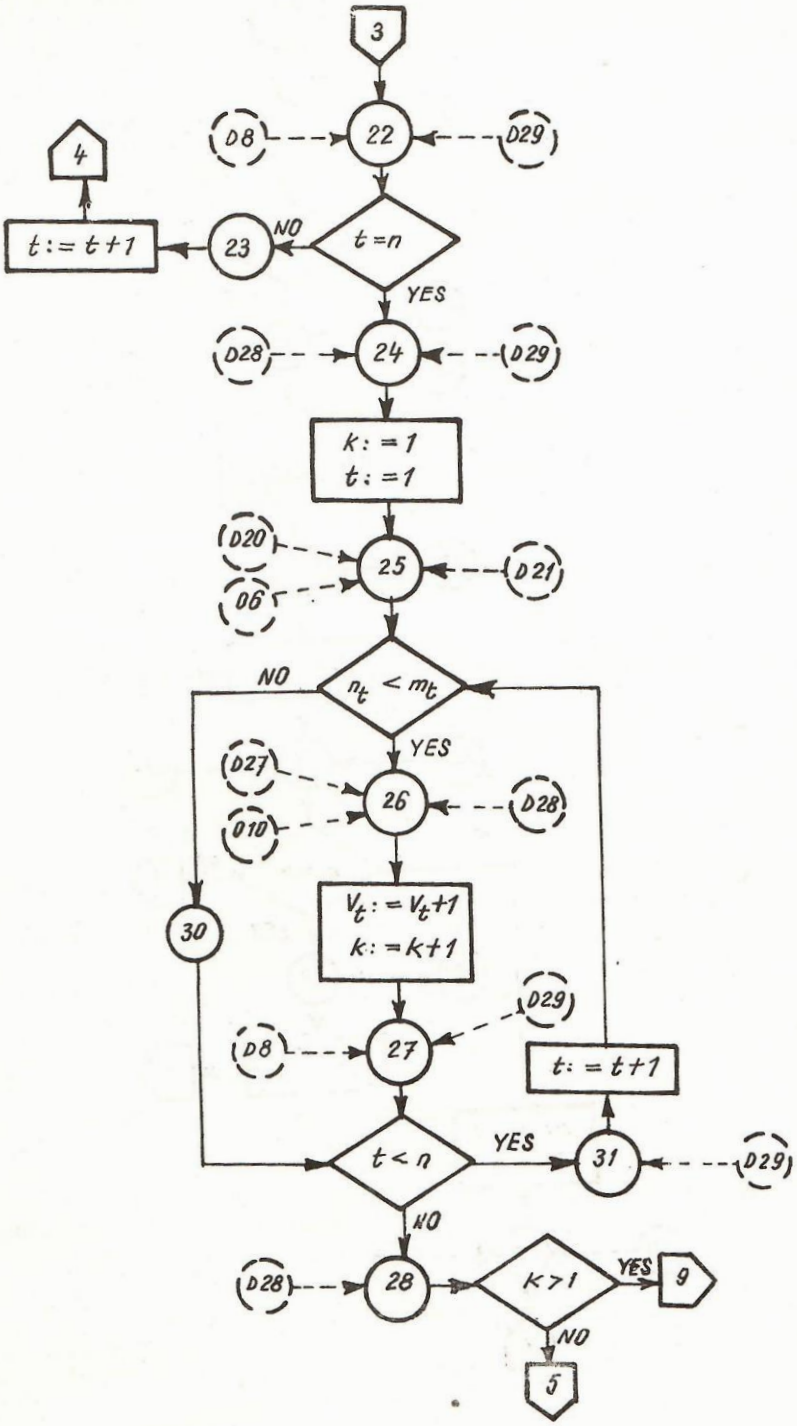
где:

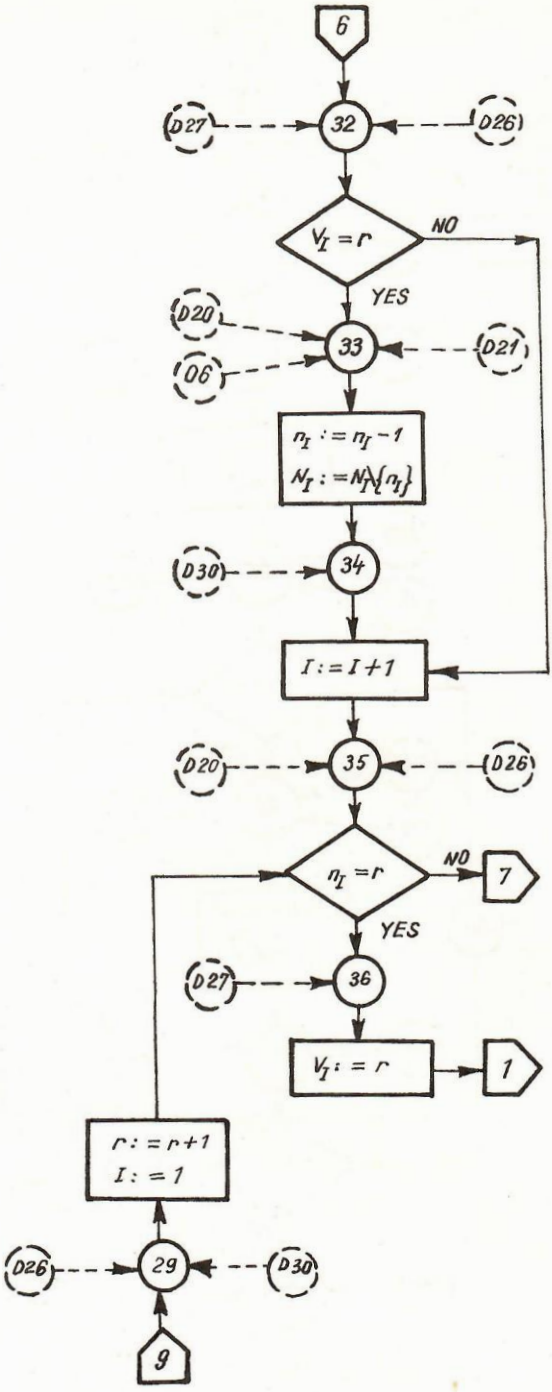
P — множество перестановок работ L^j.


START
DP2
D8
1
C1
D6
n; li,j
i=1÷n
j=1÷n
D7
D11
2
D13
D4
L0 := ωL
D15
3
PR1
d := Σ li,ji,1
4
D16
D20
D26
D6
5
NO
d≡l(μ)
YES
2
D21
D27
D10
d is the "length" of a Hamiltonian path
mi := n-1 ; ni := 2
r := ni ; Vi := r ; I := 1
Ni := {1,2} ; J := 1 ; i ∈ N
DP2
D5
6
D24
D1
l := Σ li,ji,mi
1


1
D23
7
$\exists\, \mu^{\jmath}_{n_I}$
NO
8
YES
D24
09
D23
8
C2
$\jmath := \jmath + 1$
77
NO
$l > l(\mu^{\jmath}_{n_I})$
YES
D24
D23
9
D25
$l := l(\mu^{\jmath}_{n_I})$
$\bar{\mu} := \mu^{\jmath}_{n_I}$
$i := 1$
$i := i + 1$
10
CR1
08
16
NO
08
YES
$i = n$
15
NO
$l \le d - l_{i,j_{i,1}} + l_{i,j_i,m_i}$
YES
07
13
NO
2
07
11
07
$n_i = m_i$
YES
$n_I := n_I - 1$
$l := d$
Writes
$\bar{\mu}$
l
14
NO
$m_i > 1$
YES
12
$m_i := m_i - 1$
STOP
5
D3


7
8
18
D8
I < n
YES
6
NO
19
D29
t := 1
D10
D27
20
D26
3
NO
$V_t = r$
4
YES
D20
21
D21
$n_t := n_t + 1$
$N_t := N_t \cup \{n_t\}$
3


3
D8
22
D29
4
t := t+1
23
NO
t = n
YES
D28
24
D29
k := 1
t := 1
D20
06
25
D21
NO
$n_t < m_t$
YES
D27
D10
26
D28
30
$V_t := V_t+1$
k := k+1
D29
D8
27
t := t+1
t < n
YES
31
D29
NO
D28
28
K > 1
YES
9
NO
5


6
D27
32
D26
$V_I = r$
NO
YES
D20
33
D21
06
$n_I := n_I - 1$
$N_I := N_I \setminus \{n_I\}$
D30
34
$I := I + 1$
D20
35
D26
$n_I = r$
NO
7
YES
D27
36
$V_I := r$
1
$r := r + 1$
$I := 1$
D26
29
D30
9