Conf. dr. I. VĂDUVA
Bucharest University Computer Centre

G. CIOBANU
The Department of Economic Cybernetics
Academy of Economic Studies

The "Duration—Resource" Relation in PERT Type Problems with Resources

Relations between the durations and intensities of resources for the activities of a project considered as random variables, are represented by regression functions. These functions may be used in algorithms of Monte Carlo type for resource levelling

1. INTRODUCTION

We consider a project P represented by n distinct activities $a_1, a_2, \ldots a_n$ between which "precedence" relations $a_i \ll a_j$ exist, according to a given execution technology. For the carrying out of the project, m resources are consumed; their intensities $y_1, y_2, \ldots y_m$ for each activity are functions of duration X of the activity. Suppose these intensities remain always constant (Fig. 1) during the consumption of resources and, if not, they are approximated to constant sizes.

The durations and intensities of the resources for some or all the activities of project P can be considered as random variables which are conditioning each other. For example, the change in the number of installations used in activity a_i determines a change in the duration of execution and in the intensity of some resources (number of people working on these installations instruments, etc.); by reducing the number of installations, the number of people increases because the possibilities of each of them are limited.

Taking the real possibilities into account, it results that the durations and intensities of resources for each activity have maximum and minimum values. At the same time, the path of changing the duration or intensity of a resource may be considered as constant or variable.

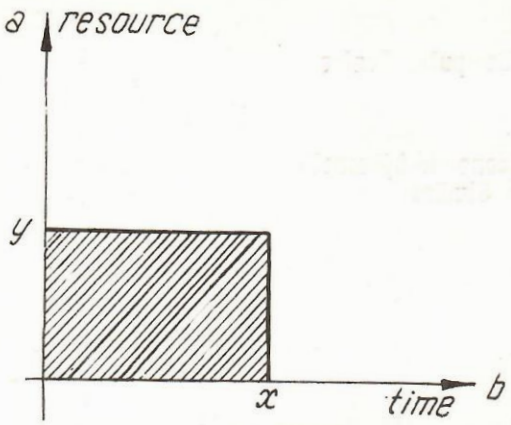

Fig. 1

In the problems of a PERT type, it is supposed in literature that the durations of the activities are random variables X stochastically independent and which satisfy the following properties (see Thomas Gilles [1]):

1° They should be limited, that is a finite time interval $[A, B]$ should exist, so that $P(X \notin [A, B]) = 0$.

2° $B - A = 6\sigma$ where $\sigma^2 = \text{var}\ (X)$.

3° They should be unimodal, that is a value $M \in [A, B]$, the most probable should exist.

Considering as random variables the durations and intensities of the resources for some or all the activities of a project, we shall say that we are treating problems of a PERT type with resources. These random variables must satisfy properties 1° and 2°.

The distribution functions for these random variables may be given analytically or by histograms. The problem arising is to determine values of duration and intensities by taking into consideration the fact that they condition each other, that is to establish a relation between durations and resources which should correspond to this inter-conditioning.

Goyal [4] proposes the duration-resource relation of the form:

$$x = x_{max} - b(y - y_{min}) \tag{1}$$

where $x \in [x_{min}, x_{max}]$, $y \in [y_{min}, y_{max}]$ represents, respectively, the duration and intensity of the resource (a single resource is defined for each activity), and b means the decrease of the duration when the intensity of the resource increases by a unit.

Langer [2] uses the analytic relation

$$y = Q \cdot x^{-1} \tag{2}$$

where $x \in [x_{min}, x_{max}]$, $y \in [y_{min}, y_{max}]$ and Q is the total quantity of the resource Y necessary for carrying out the activity.

Văduva and Ciobanu [3] based on the generation of conditioned multidimensional normal distribution generate the duration and intensities of resources for each activity. Suppose duration as a lognormal random variable and the intensities of the resources as lognormal and/or normal random variables.

Kondratev [5] considers that for each activity, a_k, a duration-resource selection is given: $[x_i, y_i)$ $i = 1, 2, \ldots n_{ak}$ and these are the single values which may be taken by the duration and the intensity of the resource.

In the PERT type problems with resources, the generation of the random variables representing durations and intensities of resources, is put under the following aspects:

i) the generation of the X random variables representing durations

ii) the generation of the random variables $Y_1, Y_2, \ldots Y_p$ representing intensities of resources conditioned by duration X.

iii) the generation of the X random variables representing durations conditioned by the intensities of resources $Y_1, Y_2, \ldots Y_p$.

Case i) is put also in the PERT problems, where for durations one may consider the Beta distribution lognormal, etc.

Generally, a duration—resource relation, which should express inter-conditioning, cannot be established. Each of the afore-mentioned proposals meets only some particular cases. This work establishes duration-resource relations as regression functions which should answer case ii).

In the following, the hypotheses will be made:

a) the durations X and the intensities of the resources Y (which we shall simply call resources) are random variables having the properties 1° and 2°;

b) the resources are continuous and homogenous;

c) during the carrying out of an activity its resources cannot be transferred to other activities;

d) by increasing the duration of an activity, the necessary resources $r(x)$ do not increase, but, they decrease or remain constant (see Fig. 2a) and the second derivate is positive ($r''(x) > 0$) during the interval $[x_1, x_2]$. The following results from figures 2 (b, c, d);

— if by increasing the duration of an activity the necessary resources increase (Fig. 2 b, d), by transformations of a symmetry type, to the straight line $x = \frac{x_1 + x_2}{2}$ and the straight line passing through points (x_1, y_1) and

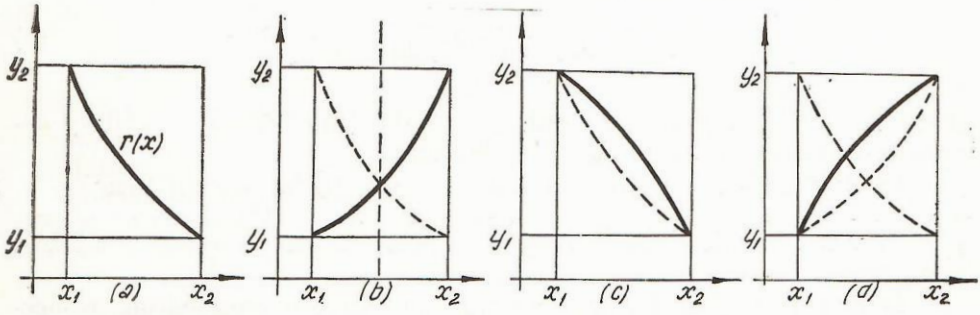

Fig. 2

(x_2, y_2) one comes to the representation in Fig. 2a (the transformations are given with a dotted line).

— if by increasing the duration of an activity, the necessary resources decrease slowly (Fig. 2c) by symmetry towards the straight line passing through points (x_1, y_2) and (x_2, y_1), one comes to the representation in Fig. 2a.

These transformations are easily inferred. Further, we shall consider that the decrease of the resource when the duration increases is obtained as in Fig. 2 a.

For simplifying the exposition, each activity uses only one resource, the consideration of several resources would return to the use of indices. For the project as a whole, several resources may be used and two distinct activities may use the same or different resources.

2. THE DURATION-RESOURCE RELATION.

Consider X and Y two random variables representing, respectively the duration and the resource for a certain activity. According to hypothesis a), the values of these variables belong to some intervals: $x \in [x_1, x_2]$ and $y \in [y_1, y_2]$.

If (x', y') and (x'', y'') are two "duration-resource" pairs for the same activity, taking into consideration the relation between duration and resource we have

$$\begin{aligned} y' \leqslant y'' &\Leftrightarrow x' \geqslant x'' \\ x' \leqslant x'' &\Leftrightarrow y' \geqslant y'' \end{aligned} \tag{3}$$

We note $z = y - y_1$. It results from (3) that the variation of the resource is a duration function $z = z(x)$ and, taking into account hypothesis a) that these random variables are limited, the following relations must be checked:

$$\lim_{\substack{x \to x_2 \\ x \in [x_1, x_2]}} z(x) = 0 \quad \text{and} \quad \lim_{\substack{x \to x_1 \\ x \in [x_1, x_2]}} z(x) = y_2 - y_1 \tag{4}$$

It can be easily noticed that there is a function $g(u)$ that

$$z(x) = \int_x^{x_2} g(u)\, du \tag{5}$$

to check relations (4); it easily results that the function $g(u)$ is slightly decreasing and has the following interpretation:

I.1 If $G(u)$ is a primitive of function $g(u)$, $[G'(u) = g(u)]$ then $G_\Delta = G(u + \Delta) - G(u)$ $u \in [x_1, x_2]$ represents (see fig. 3) the decrease of the resource when duration increase by Δ (Δ being a real number comprised between o and $x_2 - x_1$).

Lemma 1. If X and Y are two random variables representing, respectively the duration and resource for a certain activity and checking proper-

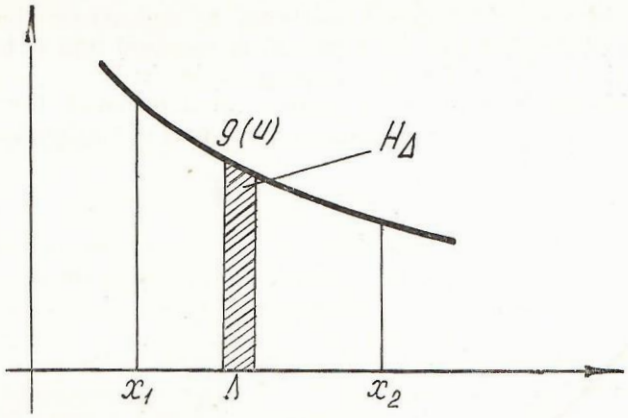

Fig. 3

a) — the surface represents the variation of the resource

ties 1° and 2° and $g(u)$ a monotonously decreasing function on $[x_1, x_2]$ with interpretation II, then

$$y = y_1 + \int_x^{x_2} g(u)\,du \tag{6}$$

Proof in relation $z = y - y_1$ is replaced $z(x)$ by expression (5) and formula (6) results.

In (6), the integral can be written as follows:

$$y = y_1 + \int_{x_1}^{x_2} g(u)\,du - \int_{x_1}^{x} g(u)\,du$$

and from (4)

$$y = y_1 + y_2 - y_1 - \int_{x_1}^{x} g(u)\,du$$

thus, another formula results

$$y = y_2 - \int_{x_1}^{x} g(u)\,du \tag{7}$$

If an activity uses only one resource in a similar way, we can express the duration in keeping with the resource.

$$x = x_1 + \int_y^{y_2} h(u)\,du \qquad \text{or} \quad x = x_2 - \int_{y_1}^{y} h(u)\,du$$

where $h(u)$ is a monotonously decreasing function on $[y_1, y_2]$ with the following interpretation:

I.2: if $H(u)$ is a primitive of function $h(u)$ $[H'(u) = h(u)]$,

then $H_\varepsilon = H(u + \varepsilon) - H(u)$, $u \in [x_1, x_2]$ represent the decrease of duration when the resources increase by ε (ε being a real number comprised between o and $y_2 - y_1$).

We shall further consider only formula (6), the conclusions being similar by using formula (7).

Taking into consideration hypotheses b) and d) on the resources it can be shown easily that the function $g(u)$ is constant and it has the second derivate positive $[g''(u) > 0]$ for the interval $[x_1, x_2]$.

We divide the interval $[x_1, x_2]$ into k equal parts of the length l: x_1, $x_1 + l$, $x_1 + 2l$, ... $x_1 + kl = x_2$ (Fig. 4). Each part of the length l corresponds

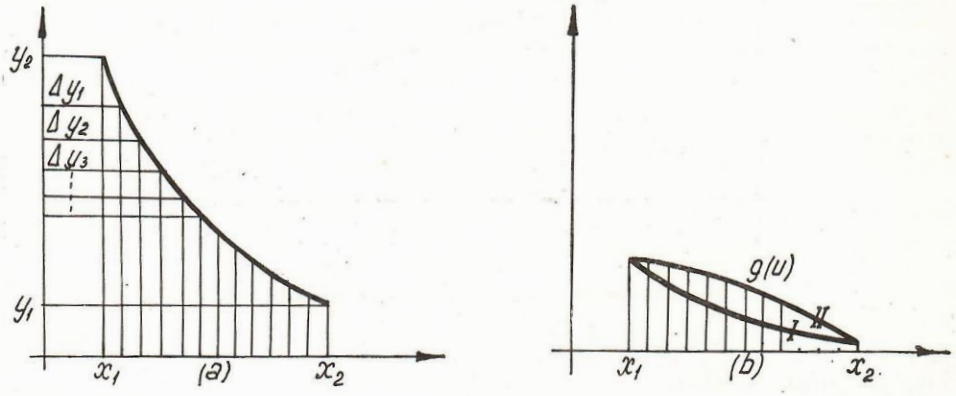

Fig. 4

to the variation of resource Δy_1, Δy_2,... Δy_k; as the resource function $r(x)$ is decreasing $\Delta y_1 > \Delta y_2 > \ldots \Delta y_k$ results; consider $Dy_{ij} = \Delta y_i - \Delta y_j$; the resource function has the second derivate positive $[r''(x) > 0]$; hence, $Dy_{12} > Dy_{23} > \ldots > Dy_{k-1,k}$ and taking into consideration interpretation I1(Δy_1, Δy_2,... Δy_k represents G_1, G_2, ... G_k) it results that function $g(x)$ (see fig. 4 bI) has the second derivate positive $[g''(x) > 0]$.

Consider $I = \int_{x_1}^{x_2} g(u)\ du$; $f(u) = \dfrac{g(u)}{I}$ is the distribution density $\left[\int_{x_1}^{x_2} f(u)\, du = 1\right]$ of a random variable U.

Thus function g: $[x_1, x_2]$ may be written as a product between a constant value I and a distribution density f

$$g(u) = I \cdot f(u) \tag{8}$$

The resource function $r(x)$ has positive values on the interval $[x_1, x_2]$. It results from relations (4) and (5) that $g(x)$ also has positive values on $[x_1, x_2]$ thus the constant value I is positive. It results from (8), as $g''(u) > 0$ and $I > 0$ that $f(u)$ has the second derivate positive $(f''(u) > 0)$.

Taking into consideration relations (4) we have: $\int_{x_1}^{x_2} g(u)\ du = y_2 - y_1$, thus (8) becomes: $g(u) = (y_2 - y_1)\, f(u)$.

Theorem 1. Consider X and Y two random variables representing the duration and resource of an activity and checking properties 1° and 2°. If U is a defined random variable on $[x_1, x_2]$ with the distribution density $f(u)$ and relation (8) is real, then

$$y = y_2 - (y_2 - y_1)\, F(x) \tag{9}$$

where F is the distribution function of variable U.

Proof. We replace $g(u) = (y_2 - y_1)\, f(u)$ in (6) and we have

$$y = y_1 + (y_2 - y_1)\int_x^{x_2} f(u)\,du = y_1 + (y_2 - y_1)\,(1 - F(x)) = y_2 - (y_2 - y_1)\,F(x)$$

Taking into consideration interpretation **I1**

$$G(x) = (y_2 - y_1)\,F(x)$$

Thus the random variable U may be given through the distribution function F which has the same interpretation as G, but on another scale. The distribution function F may be given in the form of a histogram.

Lemma 2. Consider X and Y two random variables representing the duration and resource of an activity and checking properties 1° and 2°. Consider U a defined random variable on $[x_1, x_2]$ with the distribution function F. If ψ and Φ are the distribution functions of the random variabiles X and Y, then

$$\Phi(y) = 1 - \psi\left(F^{-1}\left(\frac{y_2 - y}{y_2 - y_1}\right)\right) \tag{10}$$

Proof. From (9), we have:

$$\begin{aligned}
\Phi(y) &= P(Y < y) = P(y_2 - (y_2 - y_1)\,F(X) < y) \\
&= P(-(y_2 - y_1)\,F(X) < y - y_2) = \\
&= P\left(F(X) > \frac{y_2 - y}{y_2 - y_1}\right) = \\
&= P\left(X > F^{-1}\left(\frac{y_2 - y}{y_2 - y_1}\right)\right) = \\
&= 1 - \psi\left(F^{-1}\left(\frac{y_2 - y}{y_2 - y_1}\right)\right)
\end{aligned}$$

Thus, we can determine the distribution function of the resource when the distribution function of the duration is known.

Consider U a random variable with the distribution function $F(u, \theta)\theta$ being a parameter. This random variable is chosen so that the distribution density $f(u, \theta) = \dfrac{\partial F(u, \theta)}{\partial u}$ regardless of θ, on the interval $[x_1, x_2]$ should be monotonously decreasing, the second derivate positive $\left[\dfrac{\partial^2 f(u, \theta)}{\partial u^2} > 0\right]$ and a value θ_0 of parameter θ may be determined, so that

$$g(u) = \alpha f(u, \theta_0) \quad u \in [x_1, x_2] \tag{11}$$

where α is a real constant value different from zero.

Theorem 2. Consider X and Y two random variables representing, respectively, the duration and resource for a certain activity and checking

properties 1° and 2°. If U is a random variable for which relation (11) exists, then

$$y = y_1 + \alpha(F(x_2) - F(x) \tag{12}$$

Proof. In formula (6) we replace function $g(u)$ by expression (11) and results (12), so that:

$$y = y_1 + \alpha\int_x^{x_2} f(u, \theta_0)\,du = y_1 + \alpha(F(x_2\ \theta_0) - F(x, \theta_0))$$

Lemma 3. If X and Y are random variables which represent duration and resource, checking properties 1° and 2° and have the distribution functions ψ and Φ, then

$$\Phi(y) = 1 - \psi\left(F^{-1}\left(F(x_2) - \frac{y - y_1}{\alpha}\right)\right) \tag{13}$$

where F is the distribution function of the random variable U.

Proof. From (12) we have:

$$\begin{aligned}
\Phi(y) &= P(Y < y) = P(y_1 + \alpha(F(x_2) - F(X)) < y) \\
&= P\left(F(x_2) - F(X) < \frac{y - y_1}{\alpha}\right) \\
&= P\left(F(X) > F(x_2) - \frac{y - y_1}{\alpha}\right) \\
&= P\left(X > F^{-1}\left(F(x_2) - \frac{y - y_1}{\alpha}\right)\right) \\
&= 1 - \psi\left(F^{-1}\left(F(x_2) - \frac{y - y_1}{\alpha}\right)\right)
\end{aligned}$$

Thus, it results from (13) that we can determine the distribution function of the resource knowing the distribution function of the duration.

If the duration is a random variable X with a triangular distribution density,

$$\varphi(x) = \begin{cases} \dfrac{2(x - x_1)}{(x_2 - x_1)(m - x_1)}, & x \in [x_1,\ m] \\ \dfrac{2(x_2 - x)}{(x_2 - x_1)(x_2 - m)}, & x \in (m,\ x_2] \end{cases}$$

where m is the most probable value of duration $m \in [x_1,\ x_2]$ then:

$$\psi(x) = \begin{cases} \dfrac{(x - x_1)^2}{(x_2 - x_1)(m - x_1)}, & x \in [x_1,\ m] \\ \dfrac{m - x_1}{x_2 - x_1} + \dfrac{(x - m)(2x_2 - m - x)}{(x_2 - x_1)(x_2 - m)} & x \in (m,\ x_2] \end{cases}$$

Consider $F(u) = 1 - e^{-\lambda u}$ the exponential distribution function of the random value U. According to lemma 2

$$\Phi(y) = \begin{cases} \dfrac{1 - \left(x_1 + \dfrac{1}{\lambda}\ln\left(e^{-\lambda x_2} + \dfrac{y - y_1}{\alpha}\right)\right)^2}{(b-a)(m-a)}, & y \in [y_1, y_m] \\ 1 - \dfrac{m - x_1}{x_2 - x_1} + & \\ + \dfrac{\left(m + \dfrac{1}{\lambda}\ln\left(e^{-\lambda x_2} + \dfrac{y-y_1}{\alpha}\right)\right)\left(2x_2 - m + \dfrac{1}{\lambda}\ln\left(e^{-\lambda x_2} + \dfrac{y - y_1}{\alpha}\right)\right)}{(x_2 - x_1)(x_2 - m)} & \\ & y \in (y_m, y_2] \end{cases}$$

where from (12)

$$y_m = y_1 + \alpha(e^{-\lambda m} - e^{-\lambda x_2})$$

3. EXAMPLES OF DISTRIBUTIONS FOR THE RANDOM VARIABLE U.

For certain distributions known attributed to the random variable U which should meet the conditions in the previous paragraph, we shall particularize the duration-resource relation (12) and the estimation of the parameter α for relation (11) to exist.

3.1. *Exponential distribution.* The random variable U is of an exponential type, hence $f(u, \lambda) = \lambda e^{-\lambda u}$ and $F(u, \lambda) = 1 - e^{-\lambda u}$ For $\lambda > 0$, $f(u, \lambda)$ is monotonously decreasing and the second derivate is positive on the positive semistraight line $(0, \infty)$ thus relation (11) can be used. Formula (12) becomes:

$$y = y_1 + \alpha(e^{-\lambda x} - e^{-\lambda x_2}). \tag{14}$$

Proposition 1. Consider X and Y two random variables which represent, respectively, the duration and resource for a certain activity and checking properties 1° and 2°. If U is an exponential random variable, $f(u, \lambda) = \lambda e^{-\lambda u}$, then:

$$y = \beta\lambda e^{-\lambda x}, \quad x \in [x_1, x_2] \tag{15}$$

where β is a real constant value different from zero.

Proof. In (14) we replace $\alpha = \beta\lambda$, $\beta \neq 0$, $\lambda > 0$; thus

$$y = y_1 + \beta(\lambda e^{-\lambda x} - \lambda e^{-\lambda x_2})$$

or

$$y - \beta\lambda e^{-\lambda x} = y_1 - \beta\lambda e^{-\lambda x_2}. \tag{16}$$

Thus, β and λ must be determined for both members of relation (16) to be zero. For the same values of β and λ both members may be zero in virtue of relations (4). β and λ are found as follows: Consider $(x_i\ y_i)$ a duration-

resource selection of the volume n. We can thus determine the regression coefficients A and B from the linear form

$$\ln y = Ax + B$$

These regression coefficients are identified with the coefficients of relation

$$\ln y = -\lambda x + \ln(\beta\lambda)$$

obtained through the logarithmization of relation (15) and we have:

$$\lambda = -A, \quad \beta = \frac{1}{\lambda} e^{B}.$$

As the resource function has the second derivate positive (hypotheses d), it results that A is negative, hence λ is positive.

3.2. *The Weibull distribution.* The random variable U is of a Weibull type, with the distribution density $f(u, \gamma) = \gamma u^{\gamma-1} e^{-u^{\gamma}}$, $\gamma \geqslant 1$. The distribution function is $F(u) = 1 - e^{-u^{\gamma}}$; thus (12) becomes:

$$y = y_1 + \alpha(e^{-x^{\gamma}} - e^{-x_2^{\gamma}}) \tag{17}$$

The distribution density $f(u, \gamma)$ has a maximum point for $u' = \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma}}$. For relation (11) to be valid in the above-mentioned conditions, the interval $[x_1, x_2]$ must be on the right side of point u', that is $x_1 \geqslant \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma}}$ and thus, the Weibull density is decreasing on the interval $[x_1, x_2]$

$$x^{\gamma} = t$$

In (17) we make the change:

$$x = t^{\frac{1}{\gamma}}, \quad dx = \frac{1}{\gamma} t^{\frac{1}{\gamma}-1} dt$$

thus

$$y = y_1 + \alpha\left(\frac{1}{\gamma} t^{\frac{1}{\gamma}-1} e^{-t} - \frac{1}{\gamma} t_2^{\frac{1}{\gamma}-1} e^{-t_2}\right) \tag{18}$$

$$y = y_1 + \alpha\left(\Gamma\left(\frac{1}{\gamma}\right)\right)\left(f(t) - f(t_2)\right)$$

where

$$f(t) = \frac{1}{\Gamma\left(\frac{1}{\gamma}\right)} t^{\frac{1}{\gamma}-1} e^{-t}$$

is the gamma distribution density of a random variable T defined by transforming (18).

We note

$$\beta = \alpha \cdot \Gamma\left(\frac{1}{\gamma}\right)$$

thus

$$y = y_1 + \beta(f(t) - f(t_2)).$$

Taking into consideration (4) it results:

Proposition 2. If X and Y are random variables which represent the duration and resource and checking properties 1° and 2°, and U a Weibull random variable for which relation (11) exists, we have

$$y = \beta \cdot \frac{1}{\Gamma(\delta)} t^{\delta-1} e^{-t} = \beta f(t, \delta) \tag{19}$$

where

$$f(t, \delta) = \frac{1}{\Gamma(\delta)} t^{\delta-1} e^{-t}$$

is the distribution density of a random variable T defined by transforming

$$t = x^{\gamma}, \ \gamma = \frac{1}{\delta}$$

and $\beta \neq 0$ a real constant value.

3.3. *The Beta distribution* is defined by the distribution density

$$f(x; p, q) = \frac{1}{B(p, q)} x^{p-1}(1 - x)^{q-1} \quad p > 0 \ q > 0 \quad x \in (0, 1)$$

where

$$B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1} \, dx.$$

By substitution

$$x = \frac{1}{1 + u}$$

we obtain:

$$f(u; p, q) = \frac{1}{B(p, q)} u^{q-1}(1 + u)^{-(p+q)} \quad u \in (0, \infty) \tag{20}$$

where

$$B(p, q) = \int_0^{\infty} u^{q-1}(1 + u)^{-(p+q)} \, dy \quad p, q > 0$$

$f(u; p, q)$ has a maximum point for $u' = \frac{q-1}{p+1}$. Considering that the ran-

dom variable U has the distribution density (20), for relation (11) to be satisfied, thus the distribution density (20) to be decreasing on the interval $[x_1, x_2]$ it necessary for $x_1 \geqslant \frac{q-1}{p+1}$ thus, on the right side of the maximum point for density (20).

Further, we shall consider a particular case $p = q = 1$ and we obtain:

$$f(x) = (1 + x)^{-2} \quad x \in (0, \infty) \tag{21}$$

which is monotonously decreasing $(u' = 0)$;

$$F(x) = 1 - (1 + x)^{-1} \quad x \in (0, \infty) \tag{21'}$$

or

$$F(x) = 1 - x^{-1}, \quad x \in (1, \infty). \tag{21''}$$

Proposition 3. Consider X and Y two random variables which represent the duration and resource for a certain activity and checking properties 1° and 2°. If U is a random variable with the distribution density (21) and relation (11) exists, we have:

$$y = \alpha x^{-1} \quad x \in [x_1, x_2] \tag{22}$$

or

$$y = \alpha(1 + x)^{-1}$$

where α is a real constant value different from zero.

Proof — Relation (22) becomes:

$$y = y_1 + \alpha(x^{-1} - x_2^{-1}) \qquad \text{for (21 ″)}$$

$$y = y_1 + \alpha((1 + x)^{-1} - (1 + x_2)^{-1}). \qquad \text{for (21 ′)}$$

Thus, taking into consideration relations (4), one must determine α for (22) to be real. For this, we consider a solution (x_i, y_i) duration-resource of volume n and determine the A regression coefficient in the linear form

$$\ln y = -\ln x + A; \tag{23}$$

we logarithmize (22)

$$\ln y = -\ln x + \ln \alpha$$

and comparing it to (23), $\alpha = e^A$ results.

For this particular case (21) relations (22) are the same with relation (2) used by Langer in [2].

3.4. *Trapezoidal distribution.* The distribution density of the U random variation is of the form:

$$f(u) = mu + n, \quad u \in [x_1, x_2]. \tag{24}$$

We shall replace parameters m and n by other parameters μ_1 and μ_2 for making the connection with function $g(u)$ where

$$\mu_1 = g(x_1), \quad \mu_2 = g(x_2).$$

From system:

$$\mu_1 = \alpha(mx_1 + n)$$

$$\mu_2 = \alpha(mx_2 + n)$$

$$\int_{x_1}^{x_2} (mu + n)\, du = 1 .$$

results:

$$\alpha = \frac{(x_2 - x_1)(\mu_2 + \mu_1)}{2}; \quad m = \frac{2(\mu_2 - \mu_1)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)};$$

$$n = \frac{2(\mu_1 x_2 - \mu_2 x_1)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)} .$$

Thus:

$$f(u) = \frac{2(\mu_2 - \mu_1)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)}\, u + \frac{2(\mu_1 x_2 - \mu_2 x_1)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)} \tag{25}$$

$$F(u) = \frac{(\mu_2 - \mu_1)(u^2 - x_1^2)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)} + \frac{2(\mu_1 x_2 - \mu_2 x_1)(u - x_1)}{(x_2 - x_1)^2 (\mu_2 + \mu_1)} \tag{26}$$

$$g(u) = \frac{\mu_2 - \mu_1}{x_2 - x_1}\, u + \frac{\mu_1 x_2 - \mu_2 x_1}{x_2 - x_1} . \tag{27}$$

We note from (25) that for $f(u)$ to be decreasing, it is necessary that $\mu_1 \geqslant \mu_2$.

Proposition 4. Consider X and Y two random variables with properties 1° and 2° and which represent, respectively, the duration and resource for an activity. If U is a trapezoidal random variable for the interval $[x_1, x_2]$ and verifies relation (11), then:

$$y = y_1 + \frac{\mu_2 + \mu_1}{x_2 - x_1} \cdot \frac{x_2^2 - x^2}{2} + \frac{\mu_1 x_2 - \mu_2 x_1}{x_2 - x_1} \cdot (x_2 - x) \tag{28}$$

where μ_1 and μ_2 are parameters.

Proof. For the facility of the calculus, we use formula (6) in which we replace $g(u)$ given by relation (27) and obtain (28).

We note from (28) that y is of the form:

$$y = Ax^2 + Bx + C. \tag{29}$$

Thus μ_1 and μ_2 exist, finding the regression line (29) for a solution (x_i, y_i) of n volume.

If $\mu_1 = \mu_2 = \mu$ then $m = 0$; $n = \frac{1}{x_2 - x_1}$; $f(u) = \frac{1}{x_2 - x_1}$ (the distribution density uniform), $g(u) = \mu$, $g(u) = (y_2 - y_1) f(u)$ hence, $\mu = \frac{y_2 - y_1}{x_2 - x_1}$, $\alpha = y_2 - y_1$.

Proposition 5. Consider X and Y two random variables which represent the duration-resource for a certain activity and verify properties 1° and 2°. If U is a uniform random variable for $[x_1, x_2]$ which verifies relation (11), then:

$$y = y_1 + \mu(x_2 - x). \tag{30}$$

Proof. In (28) one considers $\mu_1 = \mu_2 = \mu$. Parameter μ is found by determining a regression straight line

$$y = Ax + B \tag{31}$$

for a selection (x_i, y_i) of n volume.

$$\mu = -A.$$

Because of hypothesis d), $A < 0$; thus $\mu > 0$.

Obs. — Relation (30) is the same with formula (2) proposed by Goyal [4].

4. APPLICATIONS.

4.1. *Estimation of durations and resources.*

Taking into consideration the random character of durations and resources, one can estimate for each activity of the project the duration and intensity of the resources so that the length of the critical path noted by T be approximate or equal to the planned lenght T_p, so we have:

$$|T - T_p| \leqslant \delta \tag{32}$$

where $\delta > 0$ is a real number.

Consider $a_1, a_2, \ldots a_n$ the activities of the project and for each activity a_i we consider duration X as a random variable assuming values in the interval $[x_1, x_2]$ and the resources $Y_1, Y_2, \ldots Y_m$ the intensities of which are random variables assuming values in the intervals $[y_{11}, y_{12}]$, $[y_{21}, y_{22}]$, $\ldots$ $[y_{m1}, y_{m2}]$. For each duration-resource couple (X, Y_j) $j = 1, 2, \ldots m$ we suppose as established a relation of the form (10) or (12). If activity a_i does not use resource Y_k then, the intensity of this resource will be zero and there is no question of finding such a relation.

In the further presented algorithm, one can easily follow the way of estimating the durations and resources for condition (32) to be satisfied.

The idea of the algorithm consists in simulating several critical paths or several lengths of the critical path, through the random generation of durations until a critical path is found which should verify condition (32).

Step 1. One sets $i = 1$.

Step 2. Activity a_i is considered for which the following is done:

2a. Duration $x \in [x_1, x_2]$ is generated according to the distribution function of the random variable X (for example, Beta, lognormal, etc.) which should meet conditions 1°—3°.

2b. Through relations (12), in keeping with the generated duration, the intensities of the resources are calculated:

$y_1 \in [y_{11}, y_{12}]$, $y_2 \in [y_{21}, y_{22}] \ldots$, $y_m \in [y_{m1}, y_{m2}]$.

Step 3. If $i = n$, then go to step 4, otherwise one sets $i = i + 1$ and go to step 2.

Step 4. The total duration of the project (the critical path) noted by T and the minimum and maximum, start and finish terms are calculated. One goes to step 5.

Step 5. If $|T - T_p| \leqslant \delta$, then algorithm has ended, otherwise if $T > T_p$ then go to step 6 and if $T < T_p$ then go to step 7.

Step 6. One sets $i = 1$.

6a. Activity a_i is considered; if this activity is not critical (the total time reserve equal to zero) one goes to step 6d.

6b. One sets $A_1 = x_1$, $A_2 = x$, where x was generated and a new duration $x \in [A_1, A_2]$ is generated according to the distribution function of the random variable X.

6c. The intensities of the resources with relations (12) are calculated.

6d. If $i = n$ then go to step 4, otherwise, set $i = i + 1$ and go to step 6a.

Step 7. One sets $i = 1$.

7a. Activity a_i is considered.

7b. One sets $A_1 = x$ and $A_2 = x_2$ where x was previously generated and a new duration $x \in [A_1, A_2]$ is generated according to the distribution function of the random variable X.

7c. The intensities of the resources with relations (12) are calculated.

7d. If $i = n$, then go to step 4, otherwise one sets $i = i + 1$ and go to step 7a.

4.2. *The uniform distribution of resources.*

Through the algorithm at § 4.1. for each activity $a_i (i = 1, 2, \ldots n)$, one determines duration x_i, the intensities of resources $y_{i1}, y_{i2}, \ldots y_{im}$ and, correspondingly, the minimum τ_i and maximum τ_i^* start terms.

We shall consider for the beginning, the start terms s_i of activities a_i uniformly generated for the intervals:

$$\left[\max_{a_k \ll a_i} (s_k + x_k),\ \tau_i^*\right],\ (i = 1, 2, \ldots n), \tag{33}$$

where by $a_k \ll a_i$ one notes that activity a_k is directly preceding activity a_i.

To obtain the intervals (33) it is necessary for activities $a_1, a_2, \ldots a_n$ to be topologically sorted, this implying to have $i < j$ if $a_i \ll a_j$.

We define the time reserve for activity a_i

$$R_{ai} = \min_{a_i \ll a_l} s_l - \max_{a_k \ll a_i} (s_k + x_k) - x_i \tag{34}$$

In keeping with this ordering of activities, we define the profiles of resources $Z_j = Z_j(t)$, $j = 1, 2, \ldots, m$, $t \in (0, T)$ by relation

$$Z_j(t) = \sum_{a_i \in Q_t} y_{ij} \tag{35}$$

where

$$Q_t = \{ a_i \mid s_i \leqslant t < s_i + x_i \}$$

t_c is called, a characteristic point, if at moment t_c at least an activity begins, that is there is at least an activity a_i with $s_i = t_c$.

Further, an algorithm for the uniform distribution of resources for duration T of the project is expounded.

The idea of the algorithm consists in the following: an scheduling is simulated through the uniform generation of the start term for each activity where the "duration-resource" couple was determined for each activity in algorithm 4.1. The maximum φ_i and minimum ψ_i of the profile for each resource are calculated. The durations are generated and correspondingly the resources and start terms for some activities are determined, pursuing the decrease of φ_i and the increase of ψ_i for each resource Y_i. The algorithm is convergent taking into consideration the limited number of activities and the condition $\varphi_i \geqslant \psi_i$, $i = 1, 2, \ldots m$.

At each stage, the solution is improved, namely the maximum top of the profiles has a decreasing tendency.

The algorithm is of a Monte-Carlo type.

Step 1. One determines $\varphi_i = \max\limits_t Z_i(t)$ and $\psi_j^* = \min\limits_t Z_i(t)$, $j = 1, 2, \ldots m$, $t \in (0, T)$ and correspondingly the characteristic points $t_{j1}, t_{j2}, \ldots, t_{jp_j}$ $j = 1, 2, \ldots, m$. One sets $\eta = 0$, and go to step 2.

Step 2. One forms $J = \{j \mid \varphi_j > \psi_j^*\}$. If $J = \varnothing$ the algorithm ended, otherwise go to step 3.

Step 3. Let $\Omega_M = \{a_i \mid s_i \leqslant t_{j_k} < s_i + x_i, R_{a_i} > 0, k = 1, 2, \ldots, p_j; j \in J\}$

3a. If $\Omega_M = \varnothing$ then go to step 5.

3b. Activities $a_i \in \Omega_M$ are ordered in the increasing order $a_1, a_2, \ldots, a_q$ according to the start terms s_i (first, the start terms are noted by s_i^*). One sets $i = 1$.

3c. One considers for activity a_i of duration $x \in [x_1, x_2]$ the interval $[A_1, A_2]$ where $A_1 = x$ and $A_2 = min\ (x_2, R_{a_i} + x)$. One generates the duration $x \in [A_1, A_2]$ according to the distribution function and with formulas (10)and (12) one calculates the intensities of resources $y_{i1}, y_{i2}, \ldots, y_{im}$.

3d. One generates uniformly the start term in the interval $\left[\max\limits_{a_k \ll a_i}(s_k + x_k), \min\limits_{a_i \ll a_l} s_l\right]$ and one calculates the profiles of resources with relation (35).

3e. One calcualtes $\psi_j = \min\limits_t Z_j(t)$, $j = 1, 2, \ldots m$ if there exists a j for which $\psi_j < \psi_j^*$ then go to step 3f. If for all $j \in J$, $\psi_j > \psi_j^*$ then, activity a_i is planned at the start term s_i generated at path 3d. One sets $\eta = 1$ and go to step 3f.

3f. If $i = q$, then go to step 4, otherwise one sets $i = i + 1$ and go to step 3c.

Step 4. If $\eta = 0$ then go to step 5 and if $\eta = 1$ go to 1.

Step 5. One determines $\psi_j = \min\limits_t Z_j(t)$, $\psi_j^* = \max\limits_t Z_j(t)$, $j = 1, 2, \ldots, m$ $t \in (0, T)$ and, correspondingly, the characteristic points $t_{j1}\ t_{j2}, \ldots, t_{jp_j}$ for $\psi_j > 0$. One sets $\xi = 0$ and go to step 6.

Step 6. One determines $J = \{j/\psi_j < \varphi_j^*\}$. If $J = \varnothing$ the algorithm ended, otherwise go to step 7.

Step 7. Let $\Omega_m = \{a_i \mid s_i \leqslant t_{j_k} < s_i + x_i, k = 1, 2, \ldots p_j, j \in J\}$.

7a. If $\Omega_m = \varnothing$ then go to step 9.

Steps 7b-7d are likewise with steps 3b-3d with the only difference that at step 7e one considers $A_1 = x_1$ and $A_2 = x$.

7e. One calculates $\varphi_j = \max\limits_{t} Z_j(t)$ $j = 1, 2, \ldots, m$

— if there exists a j for which $\varphi_j > \varphi_j^*$ then, go to step 7f.

— if for all $j \in J$, $\varphi_j \leqslant \varphi_j^*$ then, activity a_i is scheduled at the start term s_i generated at step 7d. one sets $\xi = 1$ and go to step 7f.

7f. If $i = q$ then go to step 8 and otherwise one sets $i = i + 1$ and go to step 3c.

Step 8. If $\xi = 0$, then go to step 9, if $\xi = 1$ then go to step 1.

Step 9. One sets $\varphi_j^+ = \varphi_j$. One determines $\varphi_j = max\ Z_j(t)$ $j = 1,2, \ldots, m$, $t \in (0, T)$ satisfying condition $\varphi_j < \varphi_j^+$ and, correspondingly, the characteristic points $t_{j1}, t_{j2}, \ldots t_{jp_j}$ $j = 1, 2, \ldots, m$. One sets $\eta = 0$ and go to step 2.

4.3. *Framing in a reserve of resources.*

We shall consider that each resource Y_k, $k = 1, 2, \ldots, m$ is limited by a available D_k $k = 1, 2, \ldots, m$ which may also be variable in the whole project execution period. Taking into consideration the duration of the project established at the duration found through the algorithm 4.1. we shall try to order the activities of the project and to establish the duration, respectively, the resources for each activity so that the excess over the available should be minimum. The algorithm is similar to the previous one and is of the Monte-Carlo type too.

Step 1. One sets $\varphi_j = \max\limits_{t} [Z_j(t) - D_j(t), 0]$ $j = 1, 2, \ldots, m$ and correspondingly determines the characteristic points $t_{j1}, t_{j2}, \ldots t_{jp_j}$, $j = 1, 2 \ldots, m$.

Step 2. One forms $J = \{j \mid \varphi_j > 0, j = 1, 2, \ldots m\}$. If $J = \varnothing$ then the algorithm has ended, otherwise go to step 3.

Step 3. Let $\Omega_M = \{ a_i \mid s_i < t_{j_k} < s_i + x_i,\ R_{a_i} > 0,\ k = 1,2, \ldots, p_j ; j \in I \}$.

3a. If $\Omega_M = \varnothing$ then go to step 4.

3b. Activities $a_i \in \Omega$ are ordered in the increasing $a_1, a_2, \ldots, a_q$ order according to the start terms s_i (first, the start terms are noted by s_i^*). One sets $i = 1$.

3c. One considers for activity a_i of the duration $x \in [x_1, x_2]$ the interval $[A_1, A_2]$ where $A_1 = x$ and $A_2 = min\ (x_2,\ R_{a_i} + x)$.

One generates duration $x \in [A_1, A_2]$ according to the distribution function and with the formulas (10) or (12) one calculates the intensities of resources $y_{i1}, y_{i2}, \ldots, y_{im}$.

3d. One uniformly generates the start term s_i for the interval $[\max\limits_{a_k \ll a_i} (s_k + x_k)\ \min\limits_{a_i \ll a_e} s_e]$ and calculates the profiles of resources with relation (35).

3e. If $i = q$ then go to step 1, otherwise set $i = i + 1$ and go to step 3c.

Step 4. One sets $\varphi_j^+ = \varphi_j$. One determines $\varphi_j = \max\limits_{t} [Z_j(t) - D_j(t), 0]$ $j = 1, 2, \ldots, m$. $t \in (0, T)$ which satisfy condition $\varphi_j < \varphi_j^+$ and, coorrespondingly, the characteristic points $t_{j1}, t_{j2}, \ldots, t_{jpj}$ $j = 1, 2, \ldots, m$ and go to step 2.

From the presentation of the algorithms their convergence results and, at each stage, the solution is improved.

REFERENCES

1. THOMAS GILLES: *Introduction de l'aléatoire dans les problèmes d'ordonnancement. Méthode de simulation.* Les méthodes de simulation, Dunod, Paris, 1968.
2. LANGER R: *Ein mehrprojekt plannungsmodell bei beschrankten Ressourcen und Variable Vorgangsdaner.* Rechentechnic Datenverarbeitung, no. 8, 1970.
3. VĂDUVA I., CIOBANU GH: *Aplicarea simulării digitale în rezolvarea problemelor de tip PERT cu resurse.* (Application of Digital Simulation in Solving PERT Type Problems with Resources). Advanced Summer Institute on Information Processing System, Mamaia, August, 1971.
4. GOYAL S. K: *How Much Manpower for Turnarounds?* Hydrocarbon Processing, May, 1969.
5. KONDRATEV V.P.: *Algoritm vtravnivania zagruski resursov v setevtx planax rabot.* Economika i matematiceskie metodi, no. 1, 1971.
6. CIOBANU GH: *Project Scheduling with Resource Allocation.* Economic Computation and Economic Cybernetics Studies and Research, no. 1/1971, Bucharest.

LA RELATION „DUREE-RESSOURCE" DANS LES PROBLEMES DE TYPE PERT A RESSOURCES

Les durées et les intensités des ressources pour quelques unes ou pour toutes les activités d'un projet peuvent être considérées comme des variables aléatoires qui se conditionnent réciproquement. Dans le présent article on établit les relations durées-ressources en tant que fonctions de régression à l'aide desquelles on détermine les intensités des ressources, tenant compte de l'engendrement aléatoire des durées. La fonction de régression est déterminée utilisant une variable aléatoire ayant la fonction de répartition connue et qui règle la correspondance entre la durée et l'intensité d'une ressource. Dans la partie finale on présente trois algorithmes utilisés dans le processus de nivellement des ressources employant cette sorte de relations.

ОТНОШЕНИЕ «ПРОДОЛЖИТЕЛЬНОСТЬ — РЕСУРС» В ЗАДАЧАХ ТИПА PERT С РЕСУРСАМИ

Продолжительность и интенсивность ресурсов по некоторым или по всем видам деятельности какого-либо проекта можно считать случайными взаимообусловленными переменными. В настоящей статье определяются отношения «продолжительность — ресурсы», как функции регрессии, при помощи которых определяется интенсивность ресурсов с учетом случайного возникновения продолжительности. Функция регрессии определяется с применением случайной переменной, которая имеет известную функцию распределения и регулирует соответствие между продолжительностью и интенсивностью данного ресурса. В заключение указываются три алгоритма, которые применяются в процессе выравнивания ресурсов, использующем эти отношения «продолжительность — ресурсы».