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Stochastic Programming with Multiple Objective Functions

Stochastic programming, particularly when there are several objective functions, is one of the most attractive and topical chapters of pure and applied mathematics.

1. INTRODUCTION

Many problems of mathematical programming related to several optimum criteria have a stochastic character because some of the data contained in such problems are random variables. In short, this occurs in at least three different instances [20]:

a) errors and variations of the parameters of the problem, which may sometimes be associated to probabilities;

b) risk and uncertainty, which renders possible in some cases a significant numerical representation of the utility function of a decision-maker (for instance, von Neumann's axiomatic system);

c) the need for developing optimal decision rules related to stochastic decision functions.

We consider the following problem of stochastic linear programming with multiple objective functions:

$$A(\omega)\, x \leqslant b(\omega) \tag{1}$$

$$x \geqslant 0 \tag{2}$$

$$max(min)\ Z_k(\omega) = c_k'(\omega)\ x \qquad k = 1, \ldots, r \tag{3}$$

where $A(\omega) = (a_{ij}(\omega))$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $b(\omega) = (b_i(\omega))$, $1 \leqslant i \leqslant m$ and $c_k(\omega) = (c_{kl}(\omega))$, $1 \leqslant k \leqslant r$, $1 \leqslant l \leqslant n$ are matrices whose elements are real-valued random variables defined on a probability space $\{\Omega, K, P\}$;

Ω is a Borel set, K is the σ — algebra of all Borel subsets of Ω and P is a probability measure.
$x = (x_1, \ldots, x_n)'$, is an n-dimensional, unknown column vector.
If

$$\gamma_k(\omega) = \max_x c_k'(\omega)\ x \quad (k = 1, \ldots, r),$$

then $\gamma_k(\omega)$ is a definite random variable in a probability space $\{\Omega_0, K_0, P_0\}$ where

$$\Omega_0 = \{\omega \mid D(\omega) \neq \varnothing\} \subseteq \Omega,$$

$$K_0 = K \cap \Omega_0,$$

P_0 is a probability induced in P by K_0, that is to say

$$P_0(A) = \frac{P(A)}{P(\Omega_0)}, \qquad A \in K_0,$$

$$D(\omega) = \{x \mid A(\omega)\ x \leqslant b(\omega),\ x \geqslant 0\}.$$

The difficulty of solving a problem of stochastic programming with multiple functions is due to the following two reasons:

1) When we consider one function only (for instance, the function of order k), that we assume to be a maximum, and the value of the optimum of $c_k'(\omega)\ x$ is a random variable, it is difficult to find an x^* so that

$$c_k'(\omega)\ x \leqslant c_k'(\omega)\ x^*$$

for any $\omega \in \Omega$, because for an $\omega_1 \in \Omega$ we may have $c_k'(\omega_1)\ x < c_k'(\omega_1)\ x^*$, while for anther $\omega_2 \in \Omega$ we may have $c_k'(\omega_2)\ x > c_k'(\omega_2)\ x^*$.

2) For certain $\omega \in \Omega$, the problem of deterministic programming with multiple objective functions has not a sole solution (various authors ascribe different meanings to vector $x \in R^n$ which should be "as good as possible" from the viewpoint of the whole lot of objective functions $Z_k(\omega)$).

Because the optimal values $\gamma_k(\omega)$ are random variables, they cannot be known in advance. However, we wish to know various characteristics of the random vector

$$\gamma(\omega) = (\gamma_1(\omega),\ \gamma_2(\omega), \ldots,\ \gamma_r(\omega))\ ^*)$$

before achieving coefficients $(A(\omega),\ b(\omega),\ c(\omega))$. On the analogy of the unidimensional case, such problems shall be termed distribution problems.

If random variables $Z_i(\omega)$, $1 \leqslant i \leqslant r$ are independent, then the distribution function $F_\gamma(x_1, \ldots, x_r)$, $(x_1, \ldots, x_r) \in R^r$, of the random vector $\gamma(\omega)$ is [11]:

$$F_\gamma(x_1, \ldots, x_r) = P(\{\omega : \gamma_1(\omega) < x_1, \ldots, \gamma_r(\omega) < x_r\}) =$$

$$= \prod_{i=1}^{r} P(\{\omega : \gamma_i(\omega) < x_i\}) = \prod_{i=1}^{r} F_i(x_i)$$

where $F_i(x)$ is the distribution function of $Z_i(\omega)$.

*) For instance, the joint distribution function of variables $\gamma_1(\omega), \ldots, \gamma_r(\omega)$.

This shall enable us to find the probability of those $\omega \in \Omega$ so that the objective functions shall range within certain given limits.

Theorem:

$$P(\{\omega : \underline{Z}_1 \leqslant Z_1(\omega) < \overline{Z}_1, \ldots, \underline{Z}_r \leqslant Z_r(\omega) < \overline{Z}_r\}) =$$

$$= \prod_{i=1}^{r} F_i(\overline{Z}_i) - \sum_{i=1}^{r} F_i(\underline{Z}_i) \prod_{\substack{k=1 \\ k \neq i}}^{r} F_k(\overline{Z}_k) + \sum_{i<j} F_i(\underline{Z}_i)\, F_j(\underline{Z}_j) \prod_{\substack{k=1 \\ k \neq i,j}}^{r} F_k(\overline{Z}_k) + \ldots +$$

$$+ (-1)^r \prod_{k=1}^{r} F_k(\underline{Z}_k).$$

Proof

Making use of the formula:

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_i P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) +$$

$$+ \ldots + (-1)^r P(A_1 \cap \ldots \cap A_r)$$

we get:

$$P(\{\omega : \underline{Z}_1 \leqslant Z_1(\omega) < \overline{Z}_1, \ldots, \underline{Z}_r \leqslant Z_r(\omega) < \overline{Z}_r\}) =$$

$$= F_Y(\overline{Z}_1, \ldots, \overline{Z}_r) - \sum_{i=1}^{r} F_Y(\overline{Z}_1, \overline{Z}_2, \ldots, \underline{Z}_i, \ldots, \overline{Z}_r) +$$

$$+ \sum_{i<j} F_Y(\overline{Z}_1, \overline{Z}_2, \ldots, \underline{Z}_i, \ldots, \underline{Z}_j, \ldots, \overline{Z}_r) + \ldots + (-1)^r F_Y(\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_r).$$

The theorem is proved if we take into account that

$$F_Y(x_1, \ldots, x_r) = \prod_{i=1}^{r} F_i(x_i)$$

for the independent $Z_i(\omega)$ random variables.

Before dwelling on the ways of solving problems of stochastic programming with multiple objective functions, we think fit to describe in short various attempts to solve the problem under the deterministic case.

2. VIEWPOINTS WITH REGARD TO THE SOLVING OF THE PROBLEM OF A DETERMINISTIC MATHEMATICAL PROGRAMMING WITH MULTIPLE OBJECTIVE FUNCTIONS

Consider the linear programming problem with multiple objective functions:

$$Ax \leqslant b$$

$$x \geqslant 0$$

$$optimum\ Z = CX$$

where:

$A = (a_{ij})\,(i = 1, ..., m,\ j = 1, ..., n)$ is a matrix $m \times n$;

$b = (b_1, ..., b_m)' \in R^m$ is a column vector;

$x = (x_1, ..., x_n)' \in R^n$ is an unknown column vector

$Z = (Z_1, ..., Z_r)'$ is a column vector having r components representing the objective functions;

$C = (c_{ij})\,(i = 1, ..., r;\ j = 1, ..., n)$ is a matrix $r \times n$.

We write:

$$D = \{x = (x_1, ..., x_n)' \mid Ax \leqslant b, x \geqslant 0\}.$$

The programming problem having multiple objective functions consists in finding a vector $x^* = (x_1^*, ..., x_n^*) \in D$ which shall be "as good as possible" from the viewpoint of the whole lot of objective functions $Z_h (h = 1, ..., r)$.

Because the definition of x^* is ambiguous (the expression "as good as possible" cannot be quantified, neither is it quite clear) each mathematical statement to this end has engendered a viewpoint and, consequently, a way of solving the linear programming problem with multiple objective functions.

The various attempts to define vector x^* have followed the subsequent main ways:

1. x^* is the vector optimizing a synthesis-function of the r efficiency functions:

$$h(Z) = h[Z_1, ..., Z_r].$$

Function h may be defined in various manners [10]:

a) $h[Z_1, ..., Z_r] = \underset{i=1, ..., r}{optimum} \{Z_i(x)\}$ [16], [23];

b) for $r = 2$, one may assume that $h[Z_1, Z_2] = \dfrac{Z_1(x)}{Z_2(x)}$, which leads us to an ordinary problem of fractional programming;

c) $h[Z_1, ..., Z_r] = \sum_{i=1}^{r} \alpha_i [Z_i(x)]^{\beta_i}, \qquad \alpha_i\, \beta_i \geqslant 0;$

d) $h[Z_1, ..., Z_r] = -\sum_{i=1}^{r} \alpha_i\, exp\,[-Z_i(x)], \qquad \alpha_i \geqslant 0;$

e) $h[Z_1, ..., Z_r] = \prod_{i=1}^{r} [Z_i(x)]^{\alpha_i}, \qquad \alpha_i \geqslant 0.$

2. x^* is the vector minimizing function [21]:

$$\Phi(x) = \min_{x \in \mathcal{D}} h(\Psi_1(x - X_1), ..., \Psi_r(x - X_r));$$

where $X_j = (x_{1j}, ..., x_{nj})'$ is the optimal solution to objective function Z_j. The form of function h may be:

a) $h(x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} (x_j - x_{jk})^2, \qquad \alpha_k \geqslant 0;$

b) $h(x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} |x_j - x_{jk}|;$

c) $h(x) = \max_{k=1, ..., r} \Psi_k(x - X_k);$

d) $h(x) = \sum_{k} \Psi_k(x - X_k);$

e) $h(x) = \prod_{k} \Psi_k(x - X_k),$

where Ψ_k is a distance function between vector $x \in D$ and the optimal solution X_k to function Z_k.

3. x^* is the vector belonging to a set of efficient points [7] that is defined like this:
$x^\circ \in D$ is efficient if for any $x \in D$ we have $Z_h(x) \leqslant Z_h(x^\circ)$ for $h = 1, ..., r$ and h_0 exists, so that $Z_{h_0}(x) < Z_{h_0}(x^\circ)$, (assuming that all the functions are maximum functions; if Z_j is a minimum function, we turn $\min Z_j(x) = -\max[-Z_j(x)]$).

3. OUTLINING SOME WAYS OF SOLVING STOCHASTIC PROGRAMMING PROBLEMS WITH MULTIPLE OBJECTIVE FUNCTIONS

Availing ourselves of the viewpoints described in the previous section, we shall try to outline a few possible ways for solving the stochastic programming problem with multiple objective functions.

Because with stochastic programming there is a great many variety of instances (only c to be random, vectors b and c to be random, etc.) and there are specific solving ways for each instance, we shall dwell further only on a few such instances.

3.1. Consider the stochastic programming problem with multiple objective functions:

$$A(\omega)\, x \leqslant b(\omega) \tag{4}$$

$$x \geqslant 0 \tag{5}$$

$$\max Z_k(\omega) = c_k'(\omega)\, x, \qquad k = 1, ..., r. \tag{6}$$

What does it mean to solve this stochastic programming problem with multiple objective functions?

Given the random variables $Z_k(\omega)$, $k = 1, 2, ..., r$, we draw up a new random variable as follows:

$$\xi(\omega) = \min_{k=1, ..., r} Z_k(\omega). \tag{7}$$

For a given elementary event $\omega \in \Omega$, to problem (4)—(6) is associated a deterministic programming problem with multiple objective functions. In order to solve it, a single function of the type 1° — *a* may be chosen.

We agree to find a solution to the stochastic programming problem (4)—(6) with multiple objective functions by solving the subsequent problem:

$$A(\omega)\, x \leqslant b(\omega) \tag{8}$$

$$x \geqslant 0 \tag{9}$$

$$\max_x \xi(\omega) \tag{10}$$

having a single objective function.

Let us find the distribution function $F_\xi(z) = P\{\omega \mid \xi(\omega) < z\}$ of the optimal value of random variable $\xi(\omega)$.

We consider the events:

$$A = \{\omega \mid \xi(\omega) \geqslant z\}$$

and

$$B_i = \{\omega \mid Z_i(\omega) \geqslant z\}, \qquad i = 1,2, ..., r.$$

It is obvious that there occurs relationship:

$$A = \bigcap_{i=1}^{r} B_i$$

that is:

$$\{\omega \mid \xi(\omega) \geqslant z\} = \{\omega \mid Z_1(\omega) \geqslant z, ..., Z_r(\omega) \geqslant z\}.$$

Because events B_i are independent, it follows that:

$$P(A) = P\{\omega \mid \xi(\omega) \geqslant z\} = P\left(\bigcap_{i=1}^{r} B_i\right) = \prod_{i=1}^{r} P(B_i) =$$

$$= \prod_{i=1}^{r} P\{\omega \mid Z_i(\omega) \geqslant z\}.$$

But

$$P\{\omega \mid \xi(\omega) < z\} = 1 - P\{\omega \mid \xi(\omega) \geqslant z\} =$$

$$= 1 - \prod_{i=1}^{r} P\{\omega \mid Z_i(\omega) \geqslant z\} = 1 - \prod_{i=1}^{r} [1 - P\{\omega \mid Z_i(\omega) < z\}].$$

Thus, we have proved the following:

Theorem:

The distribution function of the random variable $\xi(\omega)$ is given by:

$$F_\xi(z) = 1 - \prod_{i=1}^{r} [1 - F_{Z_i}(z)] \tag{11}$$

where F_{Z_i} is the distribution function of random variable Z_i.

If instead of problem (4)—(6) we consider problem:

$$A(\omega)\ x \geqslant b(\omega)$$

$$x \geqslant 0$$

$$min\ Z_k(\omega) = c_k'(\omega)\ x, \qquad k = 1, 2, ..., r,$$

we write, consequently:

$$\eta(\omega) = \max_{k=1,\ 2,\ ...,\ r} Z_k(\omega).$$

The problem under consideration is reduced to a problem with a single objective function:

$$A(\omega)\ x \geqslant b(\omega)$$

$$x \geqslant 0$$

$$\min_x [\eta(\omega) = \max_{k=1,\ ...,\ r} Z_k(\omega)].$$

In a similar manner we can prove the subsequent theorem.

Theorem:

The distribution function of variable $\eta(\omega)$ is given by:

$$F_\eta(\omega) = \prod_{i=1}^{r} F_{Z_i}(z). \tag{12}$$

C. E. Clark proves in [8] that if Z_1 and Z_2 are random variables with normal distributions $N(m_1, \sigma_1^2)$ and $N(m_2, \sigma_2^2)$ $(m_1 < m_2)$, then the random variable

$$\eta(\omega) = max(Z_1(\omega), Z_2(\omega))$$

has the mean

$$m = m_1\Phi(\alpha) + m_2\Phi(-\alpha) + \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2\pi}}\, e^{-\frac{\alpha^2}{2}}$$

and the variance

$$\sigma^2 = (m_1^2 + \sigma_1^2)\ \Phi(\alpha) + (m_2^2 + \sigma_2^2)\ \Phi(-\alpha) + (m_1 + m_2)\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2\pi}}\, e^{-\frac{\alpha^2}{2}} - m^2$$

where

$$\alpha = \frac{m_1 - m_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}$$

and $\Phi(\alpha)$ is the well-known Laplace's function.

3.2. Another way of approaching stochastic programming is to draw up a single function $h(Z_1, Z_2) = \dfrac{Z_1(x)}{Z_2(x)}$ (a function of the type $1^\circ - b$) instead of two functions. In this manner, the problem of stochastic programming with two objective functions is reduced to one with a single function.

$$Ax = b \tag{13}$$

$$x \geqslant 0 \tag{14}$$

$$\gamma(\omega) = \underset{x}{optimum} \frac{Z_1(\omega)}{Z_2(\omega)} \tag{15}$$

In order to find the distribution function of the random variable $\gamma(\omega)$ we shall consider the instance in which Z_1 and Z_2 are linear functions of some random variable $t(\omega)$, that is to say they are of the form:

$$Z_1(\omega) = (c' + t(\omega)\ c_1')\ x$$

$$Z_2(\omega) = (d' + t(\omega)\ d_1')\ x$$

where A is a matrix $m \times n$ $(m < n)$, c, c_1, d, d_1 are matrices $1 \times n$, while x and 0 are matrices $n \times 1$; $t(\omega)$ is a random variable having a distribution function $T(z)$ continuous and strictly increasing.

For the problem of stochastic fractional programming

$$Ax = b \tag{16}$$

$$x \geqslant 0 \tag{17}$$

$$\gamma(\omega) = \underset{x}{optimum} \frac{(c' + t(\omega)\ c_1')\ x}{(d' + t(\omega)\ d_1')\ x} \tag{18}$$

we make the following assumptions:

a) Set $D = \{x \mid Ax = b,\ x \geqslant 0\}$ is unvoid and bounded.

b) The denominator of the objective function preserves the same sign (let us assume it to be positive) with D, consequently $P\{\omega \mid (d' + t(\omega)\ d_1')\ x > 0\} = 1$.

c) Without restricting the generality, we assume that each possible basical solution is non-degenerate.

Indeed, save for some trivial instances, the denominator preserves the same sign with set D [22].

To find the distribution function of the random variable $\gamma(\omega)$ we follow a way similar to that used by B. Bereanu in [2], [3].

We assume $t(\omega) = \lambda$ to be a parameter varying within an interval $[\delta_1, \delta_2]$.

One proves that also under a fractional programming there are values of parameter λ [22].

$$\delta_1 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} \leqslant \delta_2$$

Consider $x^j (1 \leqslant j \leqslant p)$ to be the optimal solutions corresponding to intervals $(\lambda_{s-1}, \lambda_s)$ $s = 1, \ldots, p$. We write $\delta_1 = \lambda_0$ and $\delta_2 = \lambda_p$.

Theorem

Let $F(z)$ be the distribution function of random variable $\gamma(\omega)$ and $u_j(z) = = \dfrac{(zd' - c')\, x^j}{(c_1' - zd_1')\, x^j}$.

Then

$$F(z) = \sum_{j=1}^{p} H_j(z)$$

where:

$$H_j(z) = \begin{cases} T(\lambda_j) - T(\lambda_{j-1}), & \text{if } \begin{cases} (c_1' - zd_1')\, x^j > 0 \text{ and } u_j(z) \geqslant \lambda_j \\ (c_1' - zd_1')\, x^j < 0 \text{ and } u_j(z) \leqslant \lambda_{j-1} \\ (c_1' - zd_1')\, x^j = 0 \text{ and } c'x^j < zd'x^j \end{cases} \\ 0, & \text{if } \begin{cases} (c_1' - zd_1')\, x^j > 0 \text{ and } u_j(z) \leqslant \lambda_{j-1} \\ (c_1' - zd_1')\, x^j < 0 \text{ and } u_j(z) \geqslant \lambda_j \\ (c_1' - zd_1')\, x^j = 0 \text{ and } c'x^j \geqslant zd'x^j \end{cases} \\ \begin{matrix} T(u_j(z) - T(\lambda_{j-1}) \\ T(\lambda_j) - T(u_j(z)), \end{matrix} & \text{if } \lambda_{j-1} < u_j(z) < \lambda_j \text{ and } \begin{cases} (c_1' - zd_1')\, x^j > 0 \\ (c_1' - zd_1')\, x^j < 0 \end{cases} \end{cases}$$

$(1 \leqslant j \leqslant p)$

Proof

Consider events $A = \{\omega \mid \gamma(\omega) < z\}$ and $B_j = \{\omega \mid \lambda_{j-1} \leqslant t(\omega) < \lambda_j\}$ Because $\sum_{i=1}^{p} B_i = \Omega$ and $P(B_j) > 0$ $(1 \leqslant j \leqslant p)$ we can use the formula of total probability:

$$P(A) = \sum_{j=1}^{p} P(A \mid B_j) \cdot P(B_j)$$

that is

$$F(z) = P(A) = \sum_{j=1}^{p} P(A \cap B_j) = \sum_{j=1}^{p} H_j(z)$$

where:

$$H_j(z) = P\left\{\left[\omega \mid \frac{(c' + t(\omega)\, c_1')\, x}{(d' + t(\omega)\, d_1')\, x} < z\right] \cap \left[\omega \mid \lambda_{j-1} \leqslant t(\omega) < \lambda_j\right]\right\}.$$

a) If $(c_1' - zd_1')\, x^j > 0$,

then

$$\frac{(c' + t(\omega)\, c_1')\, x^j}{(d' + t(\omega)\, d_1')\, x^j} < z$$

when

$$t < \frac{(d'z - c')\, x^j}{(c_1' - zd_1')\, x^j}$$

so that:

$$H_k(z) = 0, \quad \text{if} \quad u_j(z) \leqslant \lambda_{j-1};$$
$$\mathrm{H}_k(z) = T(\lambda_j) - T(\lambda_{j-1}), \quad \text{if} \quad u_j(z) \geqslant \lambda_j$$

and

$$H_k(z) = T(u_j(z)) - T(\lambda_{j-1}), \quad \text{if} \quad \lambda_{j-1} < u_j(z) < \lambda_j.$$

b) If $(c_1' - zd_1')\, x^j < 0$,

then

$$\frac{(c' + t(\omega)\, c_1')\, x^j}{(d' + t(\omega)\, d_1')\, x^j} < z$$

when

$$t > \frac{(d'z - c')\, x^j}{(c_1' - zd_1')\, x^j}$$

so that

$$H_k(z) = 0, \quad \text{if} \quad u_j(z) \geqslant \lambda_j;$$
$$H_k(z) = T(\lambda_j) - T(\lambda_{j-1}), \quad \text{if} \quad u_j(z) \leqslant \lambda_{j-1}$$

and

$$H_k(z) = T(\lambda_j) - T(u_j(z)), \quad \text{if} \quad \lambda_{j-1} < u_j(z) < \lambda_j.$$

c) If $(c_1' - zd_1')\, x^j = 0$,

then

$$H_j(z) = 0, \quad \text{if} \quad c'x^j \geqslant zd'x^j$$

and

$$H_j(z) = T(\lambda_j) - T(\lambda_{j-1}), \quad \text{if} \quad c'x^j < zd'x^j.$$

The mean value of variable $\gamma(\omega)$ can be calculated as in [2]:

$$E\gamma(\omega) = \sum_{j=1}^{p} \int_{\lambda_{j-1}}^{\lambda_j} \frac{(c' + tc_1')\, x^j}{(d' + td_1')\, x^j}\, \mathrm{d}T(\lambda)$$

3.3. Let us consider the following stochastic programming problem with multiple objective functions:

$$Ax \geqslant b$$

$$x \geqslant 0$$

and r efficiency functions

$$min\ Z_k(\omega) = c_k'(\omega)\ x, \qquad k = 1, ..., r.$$

where only vectors c_k are random variables with known distributions.

If vector c_k has a joint normal distribution with means m_{kj} and covariance matrix $\sum_k$, then Z_k observes a law normal with mean $M_k = m_k' x$ and variance $D_k^2 = x' \sum_k x$.

We can choose for each function as an objective the minimization of its mean value, which means that the problem may be replaced by a deterministic one having multiple objective functions, and for its solving one can use one of the methods described in [5], [18].

Instead of function $Z_k(\omega)$ we can use a utility function $U(Z_k)$ whose mean value must be minimized [9].

If we choose: [9]

$$U(Z_k) = 1 - exp\ (-aZ_k)$$

where a is a positive constant that is a measure for the aversion to risk, we shall point out that

$$E\ U(Z_k) = 1 - exp\left[-aM_k + \frac{1}{2}a^2D_k^2\right]$$

It follows that the minimization of the mean value of $U(Z_k)$ is equivalent to the minimization of the quadratic function

$$aM_k - \frac{1}{2}a^2D_k^2.$$

In this way, the problem under consideration is reduced to:

$$Ax \geqslant b$$

$$x \geqslant 0$$

$$min\left(aM_k - \frac{1}{2}a^2D_k^2\right) \qquad k = 1, 2, ..., r$$

which is a quadratic programming problem with multiple objective functions and for whose solving we can use methods described in [10], [13].

The case of a single objective function is considered by Freund [9].

3.4. Functions $Z_k(\omega)$ can be turned into utility functions in von Neumann-Morgenstern's sense. In this sense, each function $Z_k(\omega)$ will undergo a transformation of the form:

$$Z_k'(\omega) = \alpha_k Z_k(\omega) + \beta_k$$

where α_k and β_k are found out of the system

$$\alpha_k X_k + \beta_k = 1$$
$$\alpha_k Y_k + \beta_k = 0$$

where

$$X_k = \underset{\substack{\omega \in \Omega \\ x \in D}}{infimum}\ Z_k(\omega)$$

and

$$Y_k = \underset{\substack{\omega \in \Omega \\ x \in D}}{supremum}\ Z_k(\omega)$$

Lemma:

The distribution function of the random variable Z'_k is

$$F_{Z'_k}(z) = \begin{cases} F_{Z_k}\left(\dfrac{z - \beta_k}{\alpha_k}\right), & \text{if } \alpha_k > 0 \\ 1 - F_{Z_k}\left(\dfrac{z - \beta_k}{\alpha_k} + 0\right), & \text{if } \alpha_k < 0 \end{cases}$$

The proof is immediate [11].

The r utility functions Z'_k can be algebraically summed up, so getting a new function

$$Z^*(\omega) = \sum_{k=1}^{r} Z'_k(\omega)$$

The distribution function of the random variable $Z^*(\omega)$ will be:

$$F_{Z^*}(y) = \int_{D_r} \ldots\ldots \int dF_{Z'_1} \ldots dF_{Z'_r}$$

where:

$$D_r = \{(x_1, \ldots, x_r) \mid (x_1, \ldots, x_r) \in R^r,\ x_1 + x_2 + \ldots x_r < y\}$$

Practically, if variables Z'_k belong to the same distribution class, then Z^* shall belong to the same class only with stable distributions.

3.5. Following the viewpoint 1° — e we can reduce problem (4)—(6) to:

$$A(\omega)\, x \leqslant b(\omega)$$
$$x \geqslant 0$$
$$(max)\ Z(\omega) = \prod_{k=1}^{r} [Z_k(\omega)]^{\alpha_k}, \qquad \alpha_k \geqslant 0$$

with a single objective function.

Let be:

$$m_k = M\left[Z_k^{\alpha_k}\right]$$

$$\sigma_k^2 = D^2\left[Z_k^{\alpha_k}\right]$$

The stochastic problem can be reduced to a deterministic one by considering either the mean, or the variance of $Z(\omega)$, but these have got very intricate expressions. For instance, in [19] is pointed out that:

$$D^2\left[\prod_{i=1}^{r} Z_i^{\alpha i}\right] = \sum_{i=1}^{r} m_1^2 m_2^2 \ldots m_{i-1}^2 \sigma_i^2 m_{i+1}^2 \ldots m_r^2 +$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{r} m_1^2 \ldots m_{i-1}^2 \sigma_i^2 m_{i+1}^2 \ldots m_{j-1}^2 \sigma_j^2 m_{j+1}^2 \ldots m_r^2 +$$

$$+ \sum_{\substack{i,j,k=1 \\ i<j<k}}^{r} m_1^2 \ldots m_{i-1}^2 \sigma_i^2 m_{i+1}^2 \ldots m_{j-1}^2 \sigma_j^2 m_{j+1}^2 \ldots m_{k-1}^2 \sigma_k^2 m_{k+1}^2 \ldots m_r^2 + \ldots + \prod_{k=1}^{r} \sigma_k^2.$$

Owing to its intricate form, in practice it can be used only in very particular instances.

3.6. Consider the problem:

$$max\ Z_k = c_k' x \qquad k = 1, \ldots, r. \tag{19}$$

subject to constraints:

$$P\left(\sum_{j=1}^{n} a_{ij}(\omega)\ x_j \leqslant b_i(\omega)\right) \geqslant \alpha_i; \qquad i = 1, \ldots, m$$

$$x_j \geqslant 0 \qquad j = 1, \ldots, n \tag{20}$$

having r efficiency functions and the restraints checked only to a certain amount of instances.

If a_{ij} and b_i are independent random variables (a_{ij} of the type $N(\mu_{ij}, \sigma_{ij}^2)$ and b_i of the type $N(\eta_i, \theta_i^2)$) and $\alpha_i \geqslant 0.5$, in [14] it is stated that the probabilistic restraints are equivalent to the following deterministic constraints:

$$[\Phi^{-1}(\alpha_i)]^2\left(\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2 + \theta_i^2\right) - \left(\eta_i - \sum_{j=1}^{n} \mu_{ij} x_j\right)^2 \leqslant 0; \qquad i = 1, \ldots, m \tag{21}$$

where:

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{t^2}{2}}\, dt$$

Problem (19)—(20) is thus reduced to a deterministic problem with multiple objective functions, having functions (19) and restraints (21).

If only b_i is random, consequently $\mu_{ij} = a_{ij}$ and $\sigma_{ij}^2 = 0$, the problem is reduced to:

$$max\ Z_k = c_k' x, \qquad k = 1, \ldots, r$$

subject to constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant \eta_i - \theta_i \Phi^{-1}(\alpha_i); \qquad i = 1, \ldots, m.$$

$$x_j \geqslant 0, \qquad j = 1, \ldots, m$$

which is a deterministic problem with multiple objective functions.

3.7. Further, we consider the stochastic programming problem having single objective function:

$$Ax \leqslant b \tag{22}$$

$$x \geqslant 0 \tag{23}$$

$$\max_{x \in D} c'(\omega)\, x \tag{24}$$

having deterministic coefficients in the constraints and c, an n-dimensional random vector whose distribution is known.

If we assume that c is a random variable which observes a multi-normal, non-degenerate distribution, having $\bar{c}$ as vector of the mean values and Σ as covariance matrix, then $c'(\omega)\, x$ is a normal random variable, with mean $\bar{c}'x$ and square mean deviation $\sigma = \sqrt{x' \Sigma x}$.

In this instance, the problem shall be stated again, in deterministic terms, and one of the following functions shall be associated to it:

1) maximization of the mean value of $c'(\omega)x$,

$$\max_{x \in D} F_1 = \max_{x \in D} \bar{c}'x, \tag{25}$$

2) maximization of the probability, so that the value of the function shall exceed a certain level k (we shall be contended if the objective function assumes at least value k):

$$\max_{x \in D} F_2 = \max_{x \in D} P\{\omega | c'(\omega)\, x \geqslant k\}. \tag{26}$$

This is the minimal risk model, considered independently by Charnes and Cooper [7], as well as by B. Bereanu [4] · A. M. Geoffrion calls it "aspiration criteria".

3) $max\ F_3 = max\ f$, for which:

$$P\{\omega \mid c'(\omega)\ x \geqslant f\} \geqslant \alpha. \tag{27}$$

This is the model considered by Kataoka [12], which A.M. Geoffrion called "fractile criteria".

We are going to point out that these viewpoints with regard to the solving of problem (22)—(24) are a natural consequence of the the different viewpoints in the deterministic programming with multiple objective functions.

For problem (22)—(24) we must choose vector x before knowing the values of c, so as to simultaneously take into account the mean and the mean square deviation of $c'x$, as follows:

$$\widetilde{F}_1 = \max_{x \in D} \bar{c}'x$$

and

$$\widetilde{F}_2 = \min_{x \in D} \sqrt{x' \Sigma x}$$

Function $\widetilde{F}_1$ is equivalent to function

$$\widetilde{F}_1^* = \min_{x \in D} (k - \bar{c}'x),$$

where k is a definite positive constant.

We are confronted with the deterministic case with two objective functions, one of which is linear while the other is quadratic.

If we observe the viewpoint in [10], by the compromise solution x^* between the two functions $\widetilde{F}_1^*$ and $\widetilde{F}_2$ we understand the solution for which we have:

$$\frac{\widetilde{F}_1^*(x^*)}{\widetilde{F}_2(x^*)} = \min_{x \in D} \frac{\widetilde{F}_1^*(x)}{\widetilde{F}_2(x)} = \min_{x \in D} \frac{k - \bar{c}'x}{\sqrt{x' \Sigma x}}$$

which is a non-linear function. In this manner we find again the result achieved by B. Bereanu [4] and Charnes and Cooper [7].

A "compromise solution" may be assumed to be the solution maximizing a linear combination of $\widetilde{F}_1$ and $\widetilde{F}_2$ [10]

$$\max_{x \in D} \alpha_1 \bar{c}'x + \alpha_2 (x' \Sigma x)^{1/2}.$$

For $\alpha_1 = 1$ and $\alpha_2 = -\Phi^{-1}(\alpha)$, where Φ is Laplace's function, we find again Kataoka's model, because:

$$F_\alpha(c'x) = \bar{c}'x + \Phi^{-1}(\alpha)(x' \Sigma x)^{1/2}$$

where $F_\alpha(c'x)$ is α — fractile of the distribution function of $c'x$.

3.8. Let us consider now a particular case of problem (4)—(6). The restraints of the problem are assumed to be deterministic:

$$Ax \leqslant b \tag{28}$$

$$x \geqslant 0$$

and r efficiency functions

$$(\max) Z_k = \sum_{j=1}^{r} c_{kj} x_j \qquad k = 1,2, \ldots, r \tag{29}$$

where coefficients c_{kj} are discrete random variables with s values

$$c_{kj} \in \{c_{kj}^1, c_{kj}^2, ..., c_{kj}^s\}$$

corresponding to the s states of nature $N_1, N_2, ..., N_s$.

The most general case that may occur is the one where we do not know which efficiency function will be chosen for optimization, nor what the state of nature will be, that is to say the effective values of coefficients c_{kj}.

When the state of nature (let it be N_l) and the objective function to be maximized (let it be Z_h) are known, the problem turns out to be a deterministic problem with a single objective function

$$max\ Z_h(x) = \sum_{j=1}^{n} c_{hj}^l x_j.$$

But at the moment that we make our decision we do not know the decision that will be made, nor the state of nature. Consequently, we must make a provision that the solution vector x shall consider all the functions and all the states of nature.

Let $p_k (k = 1, 2 ..., r)$ be the probability for function Z_k to be chosen as an objective function and $q_u (u = 1, 2 ..., s)$ the probability of occurring the state of nature N_u.

If the r objective functions and the s states of nature are simultaneously considered, $r.\ s$ efficiency functions will follow.

$$max\ Z_{hk} = \sum_{j=1}^{n} c_{hj}^k x_j. \tag{30}$$

For a certain state of nature N_h, a synthesis function of the r functions is set:

$$Z_h^* = \sum_{k=1}^{r} p_k Z_{hk} = \sum_{k=1}^{r} p_k \sum_{j=1}^{n} c_{hj}^k x_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{r} p_k c_{hj}^k \right) x_j = \sum_{j=1}^{n} c_j x_j \tag{31}$$

where:

$$c_j = \sum_{k=1}^{r} p_k c_{hj}^k.$$

If functions Z_{hk} express concrete economic values, distinct and irreducible to a common value, then they are substituted by utility functions Z'_{hk} in sense von Neumann-Morgenstern and thereupon function Z^* is ascertained with the aid of formula (31).

Depending on the available information on the states of nature, the problem is solved as follows:

a) Under certainty conditions.

(In this case it is known that a single state of nature N_1, $q_l = 1, q_t = 0\ t \neq l$ shall be achieved).

The problem of stochastic programming with multiple objective functions has been reduced to an ordinary stochastic programming problem with a single function having a random vector c.

b) Under risk conditions

(Probabilities q_j of achieving the states of nature are known).

For each state of nature N_h one synthesis function $Z_h^*(h = 1, 2, ..., s)$ is ascertained and s problems of linear programming are solved, with as many solutions $S_1, S_2, ..., S_s$.

We calculate $Z_{uk}(u = 1, ..., s;\ k = 1, 2, ..., s)$, the value of function Z_i associated to solutions S_u and the states of nature N_k, achieving the decision matrix:

States of nature / Solutions	N_1	N_2	...	N_s
S_1	Z_{11}	Z_{12}		Z_{1s}
S_2	Z_{21}	Z_{22}		Z_{2s}
.....	.....	.....	.....	.....
S_s	Z_{s1}	Z_{s2}		Z_{ss}

Some of the S_i solutions may be discarded. For instance, if for solutions S_i and S_j we have

$$Z_{ik} \leqslant Z_{jk} \qquad k = 1, ..., s$$

we discard solution S_i. In this case we say that solution S_i is ruled out by solution S_j. Thus, having assumed that the ruled solutions, have been discarded out of solutions $S_1, ..., S_s$ we shall choose that solution $S^* \in \{S_1, ..., S_s\}$ maximizing the weighted mean of values Z_{ij}; probabilities p_k are the weights.

$$max \sum_{k=1}^{s} p_k Z_{uk} \qquad u = 1, ..., s.$$

c) Under uncertainty conditions

(We do not know the probabilities of occurring the states of nature).

In this case, the decision matrix is applied to one of the Wald, Hurwicz, Savage or Laplace's diagrams. [15].

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LA PROGRAMMATION STOCHASTIQUE A PLUSIEURS FONCTIONS — OBJECTIF

Le but de l'auteur est celui d'esquisser les directions menant à la résolution de certains problèmes de programmation stochastique à plusieurs fonctions-objectif. Dans la première partie on définit le problème relevant les difficultés que sa résolution présente. Dans la seconde partie on donne un aperçu assez bref sur les diverses modalités adoptées par la littérature de spécialité pour aborder la question de la programmation linéaire à plusieurs fonctions-objectif au cas déterministe. La troisième partie renferme l'examen de plusieurs cas possibles pour le problème considéré, ainsi que quelques solutions.

СТОХАСТИЧЕСКОЕ ПРОГРАММИРОВАНИЕ С НЕСКОЛЬКИМИ ЦЕЛЕВЫМИ ФУНКЦИЯМИ

Предметом настоящей статьи является обрисовывание направлений решения задач стохастического программирования с несколькими целевыми функциями. В первой части статьи дается определение задачи с указанием встречающихся при ее решении трудностей. Во второй части статьи делается краткий обзор различных способов освещения в специальной литературе вопроса линейного программирования с несколькими целевыми функциями в определенном случае. Третья часть статьи содержит рассмотрение нескольких возможных случаев исследуемого вопроса, а также и некоторые решения.

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