V. DUMITRU
The Academy of Economic Studies
Department of Economic Cybernetics

Function Optimization Without Evaluating Derivatives (II)

*All the methods regarding the optimization of a function of several variables without using the derivative, as well as the complex method for unconstrained non-linear programming, as described in the first part of this paper *, were programmed and numerically experimented on the computer. Particular attention was granted to the Simplex algorithm for unconstrained non-linear programming, for which several alterations have been suggested, some of which seem to be justified by the numerical experiments carried out. The numerical results achieved are indicated in the tables appended to this paper.*

A) ON THE SIMPLEX ALGORITHM FOR UNCONSTRAINED OPTIMIZATION

While analyzing the simplex algorithm for unconstrained optimization, such as described in [2] or [8], we have considered several modifications which have lead to the following variants:

1. The simplex method with unidimensional optimization

The reflection point x_r^k is computed according to formula

$$x_r^k = min\ f[x_h^k + \lambda(x_k^0 - x_h^k)]$$

using an unidimensional optimization algorithm. [2], [10], [22], [24]. Beside the interest in itself, this idea enables a considerable simplification of the algorithm:

(i) Compute:

$$x_o^k = \frac{\sum_{i=1}^{n+1} x_i^k - x_h^k}{n}$$

*) In "Economic Computation and Economic Cybernetics Studies and Research, no. 2, 1973".

Find: $x_r^k = min\ f[x_h^k + \lambda(x_o^k - x_h^k)]$.

(ii) a) If $f(x^k) \leqslant f(x_s^k)$ then:

$$x_i^{k+1} := x_i^k, \quad i \neq h,\ i = 1, \ldots, n+1;$$
$$x_h^{k+1} := x_r^k,$$

b) If $f(x_r^k) > f(x_s^k)$, then a total contraction is effected:

$$x_i^{k+1} := \frac{x_l^k + x_i^k}{2},\ i = 1, 2, \ldots, n+1.$$

(iii) The stop criterion is checked.

2. **The simplex method with weighted centroid**

The ascertainment of x_o^k in the original method as being

$$x_o^k = \frac{\sum_{i=1}^{n+1} x_i^k - x_h^k}{n}$$

is based on the intuitive idea that if x_o^k is found to be the mean of the points in which the function has a value less than in x_h^k, then we can expect to obtain on the $x_o^k - x_h^k$ direction a point where the value of the function will be less than in x_h^k. On broadening this idea, it is natural to consider a mean of the points weighted by the decrease of the function per distance unit on the $x_i^k - x_h^k$ direction, so that:

$$x_o^k = \sum_{\substack{i=1 \\ i \neq h}}^{n+1} \delta_i^k x_i^k,$$

where:

$$\delta_i^k = \frac{\Delta f_i^k}{\sum_{i=1}^{n+1} \Delta f_i^k},\ i \neq h$$

and:

$$\Delta f_i^k = \frac{f_h^k - f_i^k}{\| x_h^k - x_i^k \|},\quad f_i^k = f(x_i^k).$$

3. **The maximal simplex**

Given the simplex generated by the points x_i^k, $i = 1, \ldots, n+1$, for each iteration or at every m iterations, the simplex is extended to "the maximum" on the $x_i^k - x_h^k$, $i = 1, \ldots, n+1$; $i \neq h$, directions, computing

$$x_{im}^k = min\ f[x_h^k + \lambda_i^k (x_i^k - x_h^k)],\ i \neq h,\ i = 1, \ldots, n+1$$
$$x_i^k := x_{im}^k.$$

Obviously, not always x_{im}^k is farther than the initial x_i^k from x_h^k, but this seems probable with the first iterations.

4. The weighted simplex with unidimensional optimization

Of course, these ideas can be combined. Such a combination is the weighted simplex with unidimensional optimization, which has the following steps:

1) a) Given for iteration k the simplex generated by the points x_i^k, $i = 1, \ldots, n+1$, determines:

x_h^k — the vertex where $f(x_h^k) = \max\limits_i f(x_i^k)$, $i = 1, \ldots, n+1$;

x_s^k — the vertex where $f(x_s^k) = \max\limits_i f(x_i^k)$, $i \neq h$, $i = 1, \ldots, n+1$;

x_l^k — the vertex where $f(x_l^k) = \min\limits_i f(x_i^k)$, $i = 1, \ldots, n+1$.

b) Check the stop criterion. If it is satisfied, then x_l^k is the point of minimum.

2) Compute:

a) $$\Delta f_i^k = \frac{f_h^k - f_i^k}{\| x_h^k - x_i^k \|}, \quad i \neq h,$$

b) $$\delta_i^k = \frac{\Delta f_i^k}{\sum\limits_{\substack{i=1 \\ i \neq h}}^{n+1} \Delta f_i^k};$$

c) $$x_o^k = \sum_{\substack{i=1 \\ i \neq h}}^{n+1} \delta_i^k x_i^k.$$

where: $f_i^k = f(x_i^k)$.

3) Find out, by means of unidimensional optimization subroutine:

$$x_r^k = \min f\,[x_h^k + \lambda\,(x_o^k - x_h^k)].$$

4)

a) If

$$f(x_r^k) \leqslant f(x_s^k)$$

then

$$x_i^{k+1} := x_i^k, \quad i = 1, \ldots, n+1; \quad i \neq h,$$
$$x_h^{k+1} := x_r^k.$$

b) If

$$f(x_r^k) > f(x_s^k)$$

then $$x_i^{k+1} := \frac{x_l^k + x_i^k}{2}, \quad i = 1, \ldots, n+1.$$

5) Repeat from 1)

5. **The simplex algorithm with cyclical optimization, a mixed type algorithm (SIMIXC)**

The mixed type algorithms are algorithms in which the optimization process combines several algorithms with a view to speeding the convergence process by using each algorithm in that part of the process of searching for the optimum where it is more efficient. Moreover, we can draw up a mixed type algorithm by combining an algorithm with a good behaviour in the numerical experiment in problems solving, but for which we have not a proved convergence, with a convergent algorithm. A convergence theorem for the mixed type algorithms is Zangwill's convergence theorem **B** [27].

In view of clarifying our ideas, assume we have a mixed algorithm M, for minimum, as a composition map of two algorithms, $M = A \cdot B$, of which B is convergent. Zangwill's theorem states that if:

1) $\forall\ x^{k+1} = M(x^k)$ we have $f(x^{k+1}) \leqslant f(x^k)$;

2) all the generated points belong to a compact set;

3) for $k \in K_1 \subset K$, K_1 an infinite sub-set of the set of indexes K, $x^k = M(x^{k-1}) \equiv B(x^{k-1})$, B beeing a convergent algorithm, then the mixed algorithm M is convergent.

Taking into account all the above stated, we suggest the following mixed type algorithm: **the Simplex algorithm with cyclical optimization on the axes (SIMIXC).**

i) Given the point x_i^k, the simplex is generated by a single iteration of the cyclical optimization algorithm (CYCO) by finding the points:

$$\{x_{i+1}^k \mid f(x_{i+1}^k) = min\ f\ [x_i^k + \lambda_i^k e_i],\ i = 1, \ldots, n\}$$

e_i beeing the unit vectors.

In case we make use of the cyclical optimization algorithm on the axes and the diagonal (CYCOD), then compute

$$\{x_{n+2}^k \mid f(x_{n+2}^k) = min\ f\ [x_{n+1}^k + \lambda_{n+1}^k\ (x_{n+1}^k - x_1^k)]\}$$

and assign to x_{n+1}^k the components of x_{n+2}^k:

$$x_{n+1}^k := x_{n+2}^k.$$

ii) Check the stop criterion.

If, for instance

$\| x_{n+1}^k - x_1^k \| < \varepsilon$ the procedure ends and x_{n+1}^k is the approximation of the point of optimum.

iii) Check if the generated simplex is degenerate. We assume that the simplex is degenerate if the displacement on one of the axes is quite small, which is to say that at that step of iteration the corresponding variable has remained nearly constant. We suggest the following test for checking the degeneration:

$$\exists_i\ [\| x_{i+1}^k - x_i^k \| / \| x_{n+1}^k - x_1^k \| < \varepsilon.$$

In case of degeneration, $x_1^{k+1} := x_{n+1}^k$ and the algorithm is resumed from **i)**.

iv) The simplex generated by CYCO or CYCOD becomes the original simplex for the simplex algorithm (with a weighted centroide or not, or in any of the above mentioned variants).

The iterations of the simplex algorithm are continued until the optimum criterion for simplex is satisfied and the procedure is resumed from **i)**. On this occasion we suggest one more modification of the simplex algorithm, namely that the procedure shall be repeated from **i)** also in case that $f(x_r) > < f(x_s)$, in other words the total contraction operation is discarded. In both cases, the procedure is repeated from **i)**, having as an initial point the point where the function has had its best value in the optimum sense.

According to Zangwill's theorem, the suggested mixed algorithm is, through its construction, convergent, at least for the class of functions for which the cyclical algorithm is convergent. As shown by the tests carried out, it is more efficient than the cyclical optimization algorithm and more reliable in solving problems than both algorithms that it includes.

We can also think out a variant of the mixed algorithm suggested above, that we shall call a **Simplex Cyco Mixed Algorithm with Variable Dimension.**

If the simplex degenerates, supposing the degeneration criterion is met for r directions (in other words, r variables remain constant within a CYCO iteration), then the procedure of the mixed algorithm is not resumed from **i)** but a step IV is effected in the subvariety with $n - r$ dimensions, that is to say the simplex algorithm is applied to the $n + 1 - r$ "different" points which have been achieved as long as the simplex criterion is met or there occurs $f(x_r) > f(x_s)$ in this simplex with a small number of dimensions, after which the procedure is resumed from **i)**, generating a simplex made up again of $n + 1$ vertices. As long as the simplex algorithm is applied to the reduced simplex the r variables corresponding to the directions, on which the displacement was not possible, have adequate constant values in the function. In order to make independent the above description of algorithms we will descibe briefly **the cyclical optimization algorithm on the axes of co-ordinates (CYCO).**

The cyclical optimization algorithm on the axes of co-ordinates is so simple that it is described by its own name [27], [28], [29] [30] [31].

Given the point x_1^k at iteration k, with the aid of an algorithm of unidimensional optimization we determine

$$x_{i+1}^k = min\ (max)\ f\ [x_i^k + \lambda_i^k e_i],\ i = 1, 2, ..., n,$$

where e_i is a unit coordonate vector with 1 on position i.

If a certain stop criterion, for instance $\| x_{n+1}^k - x_1^k \| < \varepsilon$, is not verified, then $x_1^{k+1} := x_{n+1}^k$ and the procedure continues.

Although so simple, this algorithm, in case $f(x)$ is differentiable and pseudo-concave, is convergent [31]. However, Fletcher in [20] characterized it as having a strong oscillatory behaviour and many a time is inefficacious in attaining the optimum point.

In view of increasing the efficiency of this algorithm [5], [7], after performing an iteration like the one described above we go on with a unidimensional optimization on the diagonal $[x_1^k,\ x_{n+1}^k]$ of the hyperparallelepipedon

generated by the points so that: **the cyclical optimization algorithm on the axes of co-ordinates and diagonal (CYCOD)** has with iteration k the following steps:

(i) Given x_1^k, find:

$$x_{i+1}^k = min\ (max)\ f[x_i^k + \lambda_i^k\ e_i],\ i = 1, 2,, ..., n$$

where e_i is a vector unitary with 1 on position i.

(ii) Find

$$x_{n+2}^k = min\ \ (max)\ \ f[x_1^k + \lambda_{n+2}^k\ \ (x_{n+1}^k - x_1^k)].$$

(iii) Assign x_{n+2}^k, to x_{n+1}^k:

$$x_{n+1}^k := x_{n+2}^k.$$

(iv) If the stop criterion, for instance $\| x_{n+1}^k - x_1^k \| < \varepsilon$ is not verified, go to **(i)**.

It is obvious that through the suggested alteration the algorithm remains convergent and the numerical tests thereby confirm that the algorithm becomes, generally speaking, more efficient.

B) NUMERICAL RESULTS

a) Numerical experiments

The numerical experiments carried-out had as an object the study of the behaviour of the different algorithms in finding the optimum of some functions having a higher or lower complex degree of non-linearity. What is persued, in general, is the efficiency of the algorithm, as measured in its capacity of solving problems with a certain given accuracy and a minimal time expenditure for man and machine. In an attempt to avoid the use of the computation time as such, which depends on the computer speed, in order to enable a comparison of the results achieved by different researchers, the computation time is assumed to be proportional to the number of evalutions of the function plus n times the number of evaluations of the gradient.

In keeping with this practice, we have adopted the same solution, not without remarking that the results strongly depend on the initial point and the other initial parameters, which are not always specified for the published results. If, in general, the initial point is given, the data concerning, for instance, the parameters of the unidimensional optimization subroutine, if is used any, such as the optimum approximation precision, the initial searching step, etc., are not given; yet, experiments carried out by ourselves and by others [2, page 108], prove that the performances considerably depend on these parameters.

In general, both precision and the first searching step in the unidimensional optimization subroutine used by us vary from an iteration to another, being a function of the total step length in the previous iteration; for instance:

$$\varepsilon^k = \varepsilon^0 \lambda^{k-1},\ \lambda_o^k = c\lambda^{k-1} / \| d^k \|$$

where:

ε^0 is the precision with iteration 1;
λ^k is the total step length in the k iteration
d^k is the direction to iteration k;
λ_o^k denotes here the length of the initial searching step to iteration k;
c is a constant, for instance 0.25; 0.5; 1; 1.5 etc.

Experiments with fixed step or precision were also carried out. The tables show the best results yielded. We cannot conclude this preliminary discussion without observing that a study on the computation time would also be useful. For instance, it is certain that the computation time does not depend only on the number of evaluations of the function or gradient, but also in the number of logical operations strictly necessary to the algorithm. However, at such a stage it would be difficult to remove the variation introduced not so much by the computer speed, as the quality of the programme and, consequently, any comparison would be useless. Nevertheless, such a study is important as regards the efficiency in the computation laboratory where it is conducted. Of course, all this should be supplemented by computations regarding the need for storage, the time required for preparing the problem [32], etc. At last, we should specify that our programmes and subroutines, written in FORTRAN IV, are at a stage of "running in" and amenable to improvement in many respects. At the immediatley next stage we have in view to programme again the most promising ones, so that their performances concerning the computation speed, the need for storage and, implicitly, the capability of solving problems of large dimensions shall increase.

b) *Test problems*

The experiments carried out so far were connected with test problems known in the literature for their degree of difficulty:

1. First all the algorithms were tested in view of the programme correctness, for the quadratic problem, which involves no particular difficulty:

$$max\ f(x) = 7x_1 + 4x_2 + x_1x_2 - x_1^2 - x_2^2.$$

Here, the optimum point is $x^* = (6, 5)$, $f(x^*) = 31.0$
The initial point was $x^\circ = (3, 1)$, $f(x^\circ) = 18$

2. Rosenbrock's test function (7):

$$min\ f(x_1, x_2) = 100\ (x_2 - x_1^2)^2 + (1 - x_1)^2.$$

This function has a minimum at $x^* = (1, 1)$ of value $f(x^*) = 0$ and contains a steep valley along the parabola $x_2 = x_1^2$ and an asymmetrical valley along the negative side of the ox_2 axis. The initial point was $x^\circ = (-1.2;\ 1)$, $f(x^\circ) = 24.2$

3. Witte and Holst's "cubic" function [33]:

$$min\ f(x_1, x_2) = 100\ (x_2 - x_1^3)^2 + (1 - x_1)^2.$$

This function has a minimum also at $x^* = (1,1)$ of value $f(x^*) = 0$ and contains a steep valley along $x_2 = x_1^3$.

The initial point was $x^\circ = (1 - 1.2;\ 1)$, $f(x^\circ) = 749.037$

4. "Four-dimensions banana" function [32]:

$$min\ f(x) = 100\ (x_1^2 - x_2)^2 + (1 - x_1)^2 + 90\ (x_3^2 - x_4)^2 + (1 - x_3)^2 +$$
$$+ 10.1\ [(x_2 - 1)^2 + (x_4 - 1)^2] + 19.8\ (x_2 - 1)\ (x_4 - 1).$$

This function has a minimum at $x^* = (1, 1, 1, 1)$ of $f(x^*) = 0$.

The initial point was $x^\circ = (-3, -1, -3, -1)$, $f(x^\circ) = 19192$.

The problem was suggested by C. F. Wood, of the Westinghouse Research Laboratories, as a test problem for a comparative study of the different non-linear optimization programmes initiated by Colville (32] and has been worked out in view of evaluating the capability of algorithms of not ending the procedure in a point which is not optimal. Indeed, around the point (−0.968, 0.947, −0.969, 0.951) the gradient of the function tends towards zero and the majority of the gradient methods come to an end [1] although the value of the function in that point is +7.876.

5. With the same idea in mind, Fiacco and McCormick in [1, p. 168] consider the problem:

$$min\ f(x) = (x_1^4 - 3)^2 + x_2^4.$$

Starting from the point $x_0 = (10^{-3}, 10^2)$ Cauchy's gradient method leads to point (0,0), which, obviously, is not a minimum point, since this lies in $x^* = \pm (\sqrt[4]{3}, 0)$, $f(x^*) = 0$.

6. Beale's function [34]:

$$min\ f(x) = \sum_{i=1}^{3} [c_i - x_1\ (1 - x_2^i)]^2,$$

$$c_1 = 1.5,\ c_2 = 2.25,\ c_3 = 2.625.$$

This function has a minimum at $x^* = (3;\ 0{,}5)$ of $f(x^*) = 0$ and a narrow curving valley near straight line $x_2 = 1$.

7. Box's function [17], that, failing to find the original work, we quote from [2]:

$$f(x) = \sum_y \{(e^{-x_1 y} - e^{-x_2 y}) - x_3\ (e^{-y} - e^{-10y})\}^2$$

where $y = 0.1\ (1)\ 1$. We have interpreted that $y \in \{0{,}1\ ;\ 0.1\ ;\ 1.\}$[1]) The function allows a minimum in $x^* = (1, 10, 1)$ with $f(x^*) = 0$, but it is not the single one, for instance $f(x) = 0$ for any $x = (x_1 = x_2,\ x_3 = 0)$.

The thorough results of the experiments carried out are recorded in the appended tables. Here we try to classify the results in agreement with the efficiency. The number of function evaluations for approximately the same precision is given in brackets.

1. For the quadratic function 1, the best behaviour was connected with the algorithms Hooke and Jeeves (76), Powell (72), Simplex with weighted centroid and unidimensional optimization (75), the Algorithm of cycli-

[1]) During the printing I found out in 38 that the summation is over the values of $y = .1, .2, \ldots, 1.$

cal optimization on the axes and diagonal (94), Rosenbrock's algorithm (97) and the Simplex-Cyco mixed algorithm. (112).

2. For Rosenbrock's function, the test problem 2, the best results were yielded by the weighted centroid Simplex (141), Powell (286), D.S.C. (377).

For Rosenbrock's function I investigated together with R. Şerban the effect that might have the computation of the optimum point on the direction by solving equation $f'_\lambda(x^{k-1} + \lambda d^k) = 0$ by means of a special subroutine called SSTEPR; with the aid of this subroutine we carried out runnings where on the d^k direction the first minimum (ICONT = 0) or the best minimum met was chosen (ICONT = 1). Obviously, in both instances, fewer evaluations function evaluations for all the encircled algorithms were obtained. However, it seems worth pointing out the following: It is not necessary, as it is generally mentioned in the steps description of algorithms, to choose the first optimal point on the direction. The policy of choosing the last point, in the optimal sense, is more efficient. For D.S.C. and Powell's algorithms in case of choosing the first optimum on the direction, the number of iterations was the same as when using subroutine STEP for searching the optimum through interpolation. Hence the conclusion that subroutine STEP, with the parameters recorded in the tables, worked with a good precision. In case of choosing the best optimum on the direction, the number of iterations abruptly fell down to two or three. Nevertheless, this did not occur in the same proportion with the number of function evaluations. For the unidimensional optimization simplex algorithm, the results obtained with the aid of subroutine SSTEPR may generally be considered unsatisfactory, since the number of iterations increases and the obtained point is farther from the optimum; in general, it seems that a certain subrelaxing is suitable with the simplex algorithm.

3. In case of cubic function, the best results were obtained with the algorithm with cyclical optimization on the axe (39), the cyclical optimization on the axes and the diagonal (51), D.S.C. (46), Powell (53) and the Simplex — Cyco mixed Algorithm (66). Our results do not confirm Leon's experiment, according to which Cube's function seems to be difficult of optimization [37].

4. Wood's "four-dimensions banana", the test problem 4, is best dealt with by D.S.C. (385), Powell (603) and the Algorithm of cyclical optimization on the axes and the diagonal (632), as compared to the cyclical optimization algorithm, without optimization on the diagonal, which required, for a less good approximation of the optimum, 2331 function evaluations. This function occasioned a first comparison between the Simplex Cyco mixed algorithm without and with variable dimension. While the latter solved the problem with the given accuracy through 140 iterations and 1905 function evaluations, the former, after 2821 function evaluations and 200 iterations did not attain the required accuracy. For this function, the Simplex algorithm with unidimensional optimization is not convergent and the procedure tends to show as an optimal value the value of false optimum specified by 7.8.

5. Fiacco's function, test problem 5, proves to be a good trap for the most efficient algorithms, Powell and D.S.C., both of them indicating after

32 and 40, respectively, function evaluations, the point (0.0) as a minimum point. The best behaviours belong to the Cyco Simplex mixed algorithm (92), Rosenbrock (150) and the cyclical optimization on the axes and the diagonal (185).

6. Beale's function, test problem 6, did not set a serious problem to *any algorithm. The best results were obtained with Powell (81), D.S.C. (85),* Hooke and Jeeves (107).

7. Box's function, as interpreted by us, has led us to a reasonable number of function evaluations towards a point where the value of the function was zero, each time another but never in the point (1, 10, 1), save for the case that the initial point was very close to it.

From the experiment carried out with these test problems, with subroutines in their first edition and, consequently, liable to improvement as regards the way they had been programmed, it follows, as a personal experience of our own — the subsequent remark on the algorithms under consideration:

Powell and D.S.C. algorithms appear among the most efficient and generally solve the problems with adequate accuracy, notwithstanding their sensibility to traps such as those set by Fiacco and McCormick's function. Hooke and Jeeves algorithm, having the same efficiency in case that the number of variables is small; however, the test problem 4 seems to point out that its efficiency diminishes if the number of variables increases.

We should grant the next place to Rosenbrock's algorithm. The cyclical optimization algorithm on the axes solves the problems through a large number of function evaluations — yet, for Cube's function it is particularly efficient — and the optimization on the diagonal increases generally the convergence speed.

The simplex algorithm appears to be an efficient algorithm but our own experiment does not confirm its comparability to Powell's, as asserted in [2]. However, we acknowledge that the efficiency depends on the parameters and the dimension of the initial simplex. Nevertheless, in our experiment, the simplex algorithm has got the drawback that its vertices can be placed on a level surface and, in that case, the algorithm stops too far from the optimum, which caused us to introduce stop criterion other than the original, $\| \Delta f^k \| / N < \varepsilon$ where Δf^k is a vector of component $\Delta f_i^k = f(x_i^k) - f(\bar{x}^k)$, $\bar{x}^k = \sum_{i=1}^{n+1} x_i^k / N + 1$. If the optimum criterion is stronger and we are in such a situation, iterations shall go on, but the progress will be very slow.

With regard to the reliability in solving the problem, it is worth saying that we achieved good results which we did not enter in the tables when, initially or from m to m interations, the simplex was extended by means of an unidimensional optimization subroutine (the maximal simplex); the number of function evaluations was extremely large, however.

The weighted centroid simplex according to author's formula had a better behaviour in all the cases, save for Fiacco and McCormick's function. When we also used the unidimensional optimization subroutine, the results were still better, but this does not hold true with Rosenbrock's function.

The use of unidimensional optimization subroutine seems to be helpful, although we feel that it should be used in a subrelaxed manner or, and this is the same, with an accuracy parameter, ε, so that the search for the unidimensional optimum shall stop with the first interpolation.

Perhaps in a subjective manner, by avaraging, our preference among the simplex algorithms is directed towards the unidimensional optimization weighted simplex. Considering the accuracy and reliability, we appreciate the Simplex Cyco mixed algorithm with variable dimension out of all the simplex algorithms. The number of function evaluations is relatively large because of the cyclical optimization algorithm.

With regard to the unidimensional optimization subroutine, it programms in broad lines Powell's algorithm and, with a few recent changes introduced by the author, it is the one written by Melanie Lenard and published in [24].

The conclusions of these experiments should be judged as any experimental conclusion. Therefore, we kindly ask the reader to turn also to other works on the subject, one of which is Kowalik and Osborne's excellent book [2], or articles by outstanding researchers pointed to in the enclosed references: Fletcher [9], [20], Davies [8], etc.

I for one think that the study on the efficiency of the different algorithms in proportion to the dimension of the problem or in relation to the different classes of functions is most important, perhaps more important than with regard to their capability of solving problems containing ad-hoc difficulties.

I cannot conclude this paper without saying that to the programs coding and algorithms testing, as well as to programmes computer running have contributed:

Sigrid Moga, for Rosenbrock's, D.S.C. and Simplex algorithm; Florica Luban, for Powell's algorithm, Cyclical optimization on the axes and the Cyco Simplex Mixed Algorithm; Radu Şerban for Hooke and Jeeves' algorithm. All three are research workers of the Economic Informatics Laboratory co-ordinated by the author within the Academy of Economic Studies. The author avails of this opportunity to acknowledge his gratitude for their co-operation.

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Function: $max\ f(x_1, x_2) = 7x_1 + 4x_2 + x_1x_2 - x_1^2 - x_2^2$

Optimum point: $x^* = (6, 5)$; $f(x^*) = 31$

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iterations	Number of evaluations		Remarks
						Function	Gradient	
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i^0 = 1$	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = (6.0;\ 5.0)$ $f(x^k) = 31$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k = 4$	76	—	
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 5.99819 \\ 4.99850 \end{bmatrix}$ $f(x^k) = 31.00000$	$\|\| x^k - x^{k-1} \|\| < 10^{-5}$ or $k > 200$	$k=10$	141	—	
Cyclical optimization on the axes and the diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 6.00061 \\ 5.00066 \end{bmatrix}$ $f(x^k) = 31.00000$	$\|\| x^k - x^{k-1} \|\| < 10^{-5}$ or $k > 200$	$k=6$	137	—	
Cyclical optimization on the axes and the diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.1	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 5.99811 \\ 4.99912 \end{bmatrix}$ $f(x^k) = 31.00000$	$\|\| x^k - x^{k-1} \|\| < 10^{-5}$ or $k > 200$	$k=7$	94	—	
Rosenbrock	RSBR $\alpha = 3$ $\beta = 0{,}5$ NO = 1	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 6.0018 \\ 5.0021 \end{bmatrix}$ $f(x^k) = 31.00000$	$\|\| x^k - x^{k-1} \|\| < \varepsilon$ and $\| f(x^k) - f(x^{k-1}) \| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=9$	97	—	
Davies — Swann — Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001 NO = 2	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 5.9970 \\ 4.9978 \end{bmatrix}$ $f(x^k) = 30.9999$	$\|\| x^k - x^{k-1} \|\| < \varepsilon$ and $\| f(x^k) - f(x^{k-1}) \| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=11$	134	—	

Table 1 continued

1	2	3	4	5	6	7	8	9
Powell	POWELL PSTEP 0.1 EPSLON = 0.001	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 6.00087 \\ 5.00083 \end{bmatrix}$ $f(x^k) = 30.99998$	$\|\| x^k - x^{k-1} \| < 10^{-5}$ or $\frac{\|f(x^k) - f(x^{k-1})\|}{\|x^k - x^{k-1}\|} < 0{,}001$	$k=3$	72	—	
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (3.\ 1)$ $x_2^0 = (1.\ 5)$ $x_3^0 = (4.\ 2)$	$x^k = \begin{bmatrix} 6.00690 \\ 5.00679 \end{bmatrix}$ $f(x^k) = 30.99995$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=19$	116	—	
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (3.\ 1)$ $x_2^0 = (1.\ 5)$ $x_3^0 = (4.\ 2)$	$x^k = \begin{bmatrix} 5.9842 \\ 5.0051 \end{bmatrix}$ $f(x^k) = 30.99965$	"	$k=19$	116	—	
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO2 = 1 NO1 = 2	$x_1^0 = (3.\ 1)$ $x_2^0 = (1.\ 5)$ $x_3^0 = (4.\ 2)$	$x^k = \begin{bmatrix} 6.0264 \\ 4.9958 \end{bmatrix}$ $f(x^k) = 30.9991$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=19$	188	—	New start with point obtained after iteration number 10
Simplex with weighted centroid and unidimensional optimization	SMPYWS PSTEP = 0.5 EPSLON = 0.001 NO2 = 2 NO1 = 2	$x_1^0 = (3.\ 1)$ $x_2^0 = (1.\ 5)$ $x_3^0 = (4.\ 2)$	$x^k = \begin{bmatrix} 6.0064 \\ 5.0110 \end{bmatrix}$ $f(x^k) = 30.9998$	"	k=8	75	—	
Cyco-Simplex mixed algorithm	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 0	$x^\circ = (3.\ 1)$ $f(x^\circ) = 18$	$x^k = \begin{bmatrix} 5.99977 \\ 4.99925 \end{bmatrix}$ $f(x^k) = 31.00000$	In CYCO $\|\| x^k - x^{k-1} \| < 10^{-5}$ or $k > 200$	k=11	112	—	Number of CYCO calls: 4

Function*): $min\ f(x_1, x_2) = 100\ (x_2 - x_1^2)^2 + (1 - x_1)^2$
Optimum point $x^* = (1, 1)$; $f(x^*) = 0$

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iteration	Number of evaluations		Remarks
						Function	Gradient	
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i^0 = 0.2$	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.01253\\1.02489\end{bmatrix}$ $f(x^k) = 0.000$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-3}$	$k=62$	226	—	
Hooke and Jeeves	*	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.0001\\1.0001\end{bmatrix}$ $f(x^k) = 1.25\times10^{-8}$	*	*	250	—	Results reported in [2]
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.73287\\0.53709\end{bmatrix}$ $f(x^k) = 0.07136$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=200$	3299	—	
Cyclical optimization on the axes and diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.95486\\0.91214\end{bmatrix}$ $f(x^k) = 0.00205$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=200$	3202	—	
Rosenbrock	RSBR $\alpha = 5$ $\beta = 0.5$	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99927\\0.99855\end{bmatrix}$ $f(x^k) = 0.5\times10^{-6}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=34$	368	—	
Davies-Swann-Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99997\\0.99994\end{bmatrix}$ $f(x^k) = 0.8\times10^{-9}$	"	$k=25$	377	—	
Davies-Swann-Campey	RSBWS and SSTEPR ICONT = 0	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99999\\0.99999\end{bmatrix}$ $f(x^k) = 0.48\times10^{-9}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=24$	73	—	

*) Rosenbrock's function

Table 2 continued

1	2	3	4	5	6	7	8	9
Davies-Swann-Campey	RSBWS and SSTEPR ICONT = 1	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99999\\0.99999\end{bmatrix}$ $f(x^k) = 0.41 \times 10^{-11}$	$\|\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=2$	11	—	
Rosenbrock	* $\alpha = 5$ $\beta = 0.5$	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.0016\\1.0031\end{bmatrix}$ $f(x^k) = 4.37 \times 10^{-6}$	*	*	200	—	Results reported in [2]
Powell	POWELL PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2;\ 4)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99954\\0.99903\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $\dfrac{\|f(x^k) - f(x^{k-1})\|}{\|x^k - x^{k-1}\|} < \varepsilon$ $\varepsilon = 10^{-3}$	$k=13$	286	—	
Powell	POWELL and SSTEPR ICONT = 0	$x^\circ = (1.2;\ 4)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.00000\\1.00000\end{bmatrix}$ $f(x^k) = 0.00000$	"	$k=12$	37	—	
Powell	POWELL and SSTEPR ICONT = 1	$x^\circ = (-1.2;\ 4)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.00000\\1.00000\end{bmatrix}$ $f(x^k) = 0.00000$	"	$k=2$	13	—	
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}1.0054\\1.0126\end{bmatrix}$ $f(x^k) = 0.00028$	$\dfrac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=90$	570	—	
Simplex (Nelder-Mead)	* $\alpha = 1$ $\beta = 0{,}5$ $\gamma = 2$	*	$x^k = \begin{bmatrix}1.0002\\1.0004\end{bmatrix}$ $f(x^k) = 6.6 \times 10^{-8}$	*	*	200	—	Results reported in [2]

Table 2 continued

1	2	3	4	5	6	7	8	9
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix} 1.001 \\ 1.003 \end{bmatrix}$ $f(x^k) = 0.000082$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=22$	141	—	
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2=1	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix} 1.0285 \\ 1.0542 \end{bmatrix}$ $f(x^k) = 0.0021$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=12$	192	—	
Simplex with unidimensional optimization	SMPXWS NO1 = 2 NO2 = 1 and SSTEPR ICONT = 0	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix} 1.17703 \\ 1.38074 \end{bmatrix}$ $f(x^k) = 0.03350$	"	$k=21$	122	—	
Simplex with unidimensional optimization	ICONT = 1	"	$x^k = \begin{bmatrix} 1.16733 \\ 1.35634 \end{bmatrix}$ $f(x^k) = 0.03199$	$\varepsilon = 10^{-5}$ "	$k=21$	164	—	
Simple with weighied centroid and unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 2	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix} 1.0301 \\ 1.0612 \end{bmatrix}$ $f(x^k) = 0.00096$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=21$	277	—	New start with point obtained after iteration number 11

Table 2 continued

1	2	3	4	5	6	7	8	9
Simplex with weighted centroid and unidimensional optimization	SMPXWS NO1 = 2 NO2 = 2 and SSTEPR ICONT = 0	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}1.17895\\1.39099\end{bmatrix}$ $f(x^k) = 0.03213$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=14$	82	–	
Cyco-Simplex mixed algorithm	SIMIXC PSTEP = 0.1 EPSLON = 0.001 NCENT = 1 NDIAG = 0	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99277\\0.98559\end{bmatrix}$ $f(x^k) = 0.00005$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=35$	550	–	
Cyco-simplex mixed algorithm	SIMIXC NCENT = 1 NDIAG = 0 and SSTEPR ICONT = 0	$x^\circ = (-1.2;\ 1)$ $x(x^\circ) = 24.2$	$x^k = \begin{bmatrix}0.99904\\0.99808\end{bmatrix}$ $f(x^k) = 0.00000$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=58$	125	–	
Cyco-simplex mixed algorithm	SIMIXC NCENT = 1 NDIAG = 0 and SSTEPR ICONT = 1	$x^\circ = (-1.2;\ 1)$ $x(x^\circ) = 24.2$	$x^k = \begin{bmatrix}1.000000\\1.00000\end{bmatrix}$ $f(x^k) = 0.00000$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=1$	13	–	
Cyclical optimization on the axes	CYCO with SSTEPR ICONT = 0	"	$x^k = \begin{bmatrix}0.99999\\0.99998\end{bmatrix}$ $f(x^k) = 0.00000$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=70$	211	–	
Cyclical optimization on the axes	CYCO with SSTEPR ICONT = 1	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 24.2$	$x_q = \begin{bmatrix}1.00000\\1.00000\end{bmatrix}$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=2$	13	–	

Function *): $min\ f(x_1, x_2) = 100\ (x_2 - x_1^3)^2 + (1 - x_1)^2$

Optimum point: $x^* = (1; 1)$; $f(x^*) = 0$

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iteration	Number of evaluations		Remarks
						Function	Gradient	
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_1^0 = 1$	$x^\circ = (-1.2; 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}0.99929\\0.98877\end{bmatrix}$ $f(x^k) = 0.000$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k=49$	233	—	
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2; 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}0.99999\\0.99998\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=2$	39	—	
Cyclical optimization on the axes and the diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2\ .1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}1.00000\\0.99999\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=2$	51	—	
Rosenbrock	RSBR $\alpha = 3$ $\beta = 0.5$ NO = 1	$x^\circ = (-1.2; 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}0.98542\\0.95677\end{bmatrix}$ $f(x^k) = 0.00021$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=22$	234	—	
Rosenbrock	* $\alpha = 3$ $\beta = 0.5$	*	$x^k = \begin{bmatrix}1.0024\\1.0074\end{bmatrix}$ $f(x^k) = 1.27 \times 10^{-5}$	*	*	100	—	Result reported in [2]
Rosenbrock	* $\alpha = 2$ $\beta = 0.3$	$x^\circ = (-1.2; 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}0.7298\\0.3867\end{bmatrix}$ $f(x^k) = 7.34 \times 10^{-2}$	*	*	300	—	„

*)Witte and Holst's Cube function

Table 3 continued

1	2	3	4	5	6	7	8	9
Davies-Swann-Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001 NO = 2	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}1.00003\\1.00001\end{bmatrix}$ $f(x^k) = 0.2 \cdot 10^{-8}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x)^k - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=2$	46	—	
Powell	POWELL PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}1.00000\\0.99999\end{bmatrix}$ $f(x^k) = 0.00000$	$\|\| x^k - x^{k-1} \|\| < 10^{-5}$ or $\frac{\|f(x^k) - f(x^{k-1})\|}{\|\| x^k - x^{k-1}\|\|} < \varepsilon$ $\varepsilon = 10^{-3}$	$k=3$	53	—	
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}0.9989\\0.9987\end{bmatrix}$ $f(x^k) = 0.00038$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=58$	362	—	
Simplex (Nelder-Mead)	*	$x_1^0 = (-1.2;\ 1)$ Simplex regulated with side 0.5	$x^k = \begin{bmatrix}1.0000\\0.9995\end{bmatrix}$ $f(x^k) = 1.3 \times 10^{-7}$	*	*	140	—	Result reported in [2²]
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}0.93864\\0.82254\end{bmatrix}$ $f(x^k) = 0.0057$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=44$	282	—	

Table 3 continued

1	2	3	4	5	6	7	8	9
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 1	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}1.0217\\1.0656\end{bmatrix}$ $f(x^k) = 0.0005$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=8$	164	—	
Simplex with weighted centroid and unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 2	$x_1^0 = (-1.2;\ 1)$ $x_2^0 = (6.\ 5)$ $x_3^0 = (3.\ 2)$	$x^k = \begin{bmatrix}1.008\\1.026\end{bmatrix}$ $f(x^k) = 0.00007$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=15$	257	—	
Cyco-Simplex mixed algorithm	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 1	$x^\circ = (-1.2;\ 1)$ $f(x^\circ) = 749.037$	$x^k = \begin{bmatrix}0.99999\\0.99997\end{bmatrix}$ $f(x^k) = 0.00000$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=1$	66	—	No. of CYCO calls: 3

Table 4

Function*): $min\ f(x_1, x_2, x_3, x_4) = 100\ (x_2 - x_1^2)^2 + (1 - x_1)^2 + 90\,(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10.1\,[(x_2 - x_1^2)^2 + (x_4 - x_1)^2] + 19.8\,(x_2 - x_1)\ (x_4 - x_1)$

Optimum Point $x^* = (1, 1, 1, 1)$; $f(x^*) = 0$.

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iterations	Number of evaluations: Function	Number of evaluations: Gradient	Remarks
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i^0 = 1$	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 1.00031 \\ 1.00061 \\ 0.99976 \\ 0.99951 \end{bmatrix}$ $f(x^k) = 0{,}000$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k =$ 156	1217	–	
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 1.01675 \\ 1.03385 \\ 0.98295 \\ 0.96611 \end{bmatrix}$ $f(x^k) = 0{,}00103$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=82$	2331	–	
Cyclical optimization on the axes and the diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-2, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 1.00669 \\ 1.01346 \\ 0.99325 \\ 0.98652 \end{bmatrix}$ $f(x^k) = 0{,}00016$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=21$	632		
Rosenbrock	RSBR $\alpha = 3$ $\beta = 0.5$ NO = 1	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 1.00048 \\ 1.00103 \\ 0.99953 \\ 0.99900 \end{bmatrix}$ $f(x^k) = 0{,}15 \times 10^{-5}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=48$	887		

*) C.F. Wood's "four-dimensions banana" function

Table 4 continued

1	2	3	4	5	6	7	8	9
Davies — Swann — Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001 NO = 2	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 0.9996 \\ 0.9999 \\ 1.00009 \\ 1.0001 \end{bmatrix}$ $f(x^k) = 0.33 \times 10^{-7}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=14$	385	—	
Powell	POWELL PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 0.99995 \\ 0.99994 \\ 1.00004 \\ 1.00009 \end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $\frac{\|f(x^k) - f(x^{k-1})\|}{\|x^k - x^{k-1}\|} < \varepsilon$ $\varepsilon = 10^{-3}$	$k=18$	603	—	New start after iteration number 7
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 1	$x^k = \begin{bmatrix} 1.00391 \\ 1.01100 \\ 1.00373 \\ 1.00810 \end{bmatrix}$ $f(x^k) = 0.00470$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x^k) - (\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=273$	2107	—	New start with point obtained with iteration no. 200. Simplex of side 0.5
		$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 6	$x^k = \begin{bmatrix} 1.98900 \\ 0.97579 \\ 1.02156 \\ 1.04133 \end{bmatrix}$ $f(x^k) = 0.00495$		$k=206$	1578	—	New start with point obtained with iteration 60. Simplex of side 0.5
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 1	$x^k = \begin{bmatrix} 1.00523 \\ 1.01080 \\ 0.99002 \\ 0.97992 \end{bmatrix}$ $f(x^k) = 0.001095$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=172$	1321	—	New start with point obtained with iteration no. 140. Simplex of side 0.5
		$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 6	$x^k = \begin{bmatrix} 1.00511 \\ 1.00976 \\ 0.99520 \\ 0.98691 \end{bmatrix}$ $f(x^k) = 0.00134$		$k=239$	1861		New start with point obtained with iteration no. 116. Simplex of side 0.5

Table 4 continued

1	2	3	4	5	6	7	8	9
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2=1	$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 1	$x^k = \begin{bmatrix} 0.98135 \\ 0.93607 \\ 1.03105 \\ 1.05577 \end{bmatrix}$ $f(x^k) = 0.08092$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=76$	1604	—	New start with point obtained with iteration no. 61. Simplex of side 0.5
		$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 6	$x^k = \begin{bmatrix} -0.96152 \\ 0.92594 \\ -0.98346 \\ 0.98313 \end{bmatrix}$ $f(x^k) = 7.88775$		$k=30$	728		
Simplex with weighted centroid and with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 2	$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 1	$x^k = \begin{bmatrix} -0.96097 \\ 0.88216 \\ -0.98631 \\ 1.01886 \end{bmatrix}$ $f(x^k) = 8.25218$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=42$	1022	—	
		$x^\circ = (-3, -1, -3, -1)$ Simplex regulated with side 6	$x^k = \begin{bmatrix} -0.97739 \\ 0.96410 \\ -0.98342 \\ 0.95523 \end{bmatrix}$ $f(x^k) = 7.92963$		$k=16$	361	—	
Cyco-Simplex mixed algorithm	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 0 NRED = 1	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 0.95220 \\ 0.90535 \\ 1.04630 \\ 1.09446 \end{bmatrix}$ $f(x^k) = 0.00819$	in CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=200$	2821		Number of Cyco calls: 47

Table 4 continued

1	2	3	4	5	6	7	8	9
Cyco-Simplex mixed algorithm with variable dimension	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 0 NRED = 0	$x^\circ = (-3, -1, -3, -1)$ $f(x^\circ) = 19192$	$x^k = \begin{bmatrix} 0.99649 \\ 0.99303 \\ 1.00336 \\ 1.00673 \end{bmatrix}$ $f(x^k) = 0.00004$	in CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=140$	1905		Number of Cyco calls: 27

Table 5

Function*): $min\ f(x_1, x_2) = (x_1^4 - 3)^2 + x_2^4$
Optimum Point:: $x^* = (\pm\sqrt[4]{3}.0);\ f(x^*) = 0;\ \sqrt[4]{3} = 1.31607$

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iteration	Number of evaluations: Function	Number of evaluations: Gradient	Remarks
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i^0 = 1$	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) = 100000009$	$x^k = \begin{bmatrix}1.31606\\0.0\end{bmatrix}$ $f(x^k) = 0.000$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k=25$	202	–	
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) = 100000009$	$x^k = \begin{bmatrix}1.31607\\0.01595\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=5$	121	–	
Cyclical optimization on the axes and the diagonal	CYCO PSTEP = 0.1 EPSLON = 0.001 NDIAG = 0	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) = 100000009$	$x^k = \begin{bmatrix}1.31607\\0.00004\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=6$	185	–	
Rosenbrock	RSBR $\alpha = 3$ $\beta = 0.5$ NO = 1	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) = 10^8$	$x^k = \begin{bmatrix}-1.31607\\-0.00304\end{bmatrix}$ $f(x^k) = 0.8 \times 10^{-10}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=13$	150	–	
Davies – Swann – Campey	RSBWS PSTEP = 0.5 EPSLON = 10^{-3} NO = 2	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) \approx 10^8$	$x^k = \begin{bmatrix}0.00099\\-0.00066\end{bmatrix}$ $f(x^k) = 9$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=4$	40	–	
		$x^\circ = (10^2, 10^2)$ $f(x^\circ) \approx 10^{15}$	$x^k = \begin{bmatrix}-1.31608\\-0.11281\end{bmatrix}$ $f(x^k) = 0.000161$		$k=6$	100	–	
Powell	POWELL PSTEP = 0.1 EPSLON = 10^{-3}	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) \approx 10^8$	$x^k = \begin{bmatrix}0.00100\\-0.0137\end{bmatrix}$ $f(x^k) = 9.00000$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=2$	32	–	
		$x^\circ = (10^2, 10^2)$ $f(x^\circ) \approx 10^{15}$	$x^k = \begin{bmatrix}1.31607\\0.01022\end{bmatrix}$ $f(x^k) = 0.00000$		$k=5$	108	–	

*) Fiacco and Mc-Cormick's function

1	2	3	4	5	6	7	8	9
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (10^{-3}, 10^2)$ $x_2^0 = (10^{-2}, 10)$ $x_3^0 = (10^{-1}, 1)$ $f(x_1^0) = 10^8$ $f(x_2^0) = 10009$ $f(x_3^0) = 9999$	$x^k = \begin{bmatrix}1.31733\\0.05314\end{bmatrix}$ $f(x^k) = 0.000141$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=29$	176	—	New start with point obtained with iteration number 13, which was (0 ; 0)
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (10^{-3}, 10^2)$ $x_2^0 = (10^{-2}, 1)$ $x_3^0 = (10^{-1}, 1)$ $f(x_1^0) = 10$ $f(x_2^0) = 10009$ $f(x_3^0) = 9999$	$x^k = \begin{bmatrix}1.31726\\0.07524\end{bmatrix}$ $f(x^k) = 0.000149$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=57$	363		New start with point obtained after iteration number 44 (0 ; 0)
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 1	$x_1^0 = (10^{-3}, 10^2)$ $x_2^0 = (10^{-2}, 10)$ $x_3^0 = (10^{-1}, 1)$ $f(x_1^0) = 10^8$ $f(x_2^0) = 10009$ $f(x_3^0) = 9999$	$x^k = \begin{bmatrix}1.31702\\-0.21266\end{bmatrix}$ $f(x^k) = 0.00212$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=15$	209		New start from point, (0 ; 0) after iteration number 4
Simplex with weighted centroid and unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 2	$x_1^0 = (10^{-3}, 10^2)$ $x_2^0 = (10^{-2}, 10)$ $x_3^0 = (10^{-1}, 1)$	$x^k = \begin{bmatrix}1.31853\\-0.04013\end{bmatrix}$ $f(x^k) = 0.000508$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=10$	133	—	New start from point (0, 0) after iteration number 4
Cyco-Simplex mixed algorithm	SIMIXC PSTEP = 0.1 NCENT = 1 NDIAG = 0	$x^\circ = (10^{-3}, 10^2)$ $f(x^\circ) \approx 10^8$	$x^k = \begin{bmatrix}1.31607\\0.00012\end{bmatrix}$ $f(x^k) = 0.00000$	In Cyco $\|x^k - x^{k-1}\| < 10^{-5}$ $k > 200$	1	92	—	Number of Cyco calls: 3

Table 6

Function *): $min\ f(x_i) = \sum_{i=1}^{3} [c_i - x_1(1 - x_i^2)]^2$; $c_1 = 1.5$; $c_2 = 2.25$; $c_3 = 2.625$

Optimum Point $x^* = (3; 0.5)$; $f(x^*) = 0$.

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iteration	Number of evaluations		Remarks
						Function	Gradient	
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i = 1$	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}3.00000\\0.50000\end{bmatrix}$ $f(x^k) = 0.000$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k=9$	107	—	
Hooke and Jeeves	*	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}3.0\\0.5\end{bmatrix}$ $f(x^k) = 0$	*	*	100	—	Result reported in [2]
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}2.99632\\0.49910\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=36$	534	—	
Cyclical optimization on the axes and the diagonal	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}2.99632\\0.49910\end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=13$	278	—	
Rosenbrock	RSBR $\alpha = 3$ $\beta = 0.5$ NO = 1	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}2.99998\\0.50000\end{bmatrix}$ $f(x^k) = 0.35 \times 10^{-8}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=12$	131	—	
Rosenbrock	* $\alpha = 5$ $\beta = 0.5$	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}2.9989\\0.4996\end{bmatrix}$ $f(x^k) = 2.61 \times 10^{-7}$	*	*	130	—	Result reported in [2]

* Beale's function

1	2	3	4	5	6	7	8	9
Davies — Swann — Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001 NO = 2	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}3.00004\\0.50001\end{bmatrix}$ $f(x^k) = 0.3 \times 10^{-9}$	$\|x^k - x^{k-1}\| < \varepsilon$ and $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=7$	85	—	
Powell	POWELL PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0.\ 0)$ $f(x^\circ) = 14.20$	$x^k = \begin{bmatrix}3.00003\\0.50001\end{bmatrix}$ $f(x^k) = 0.00000$	$\frac{\|f(x)^k - f(x^{k-1})\|}{\|x^k - x^{k-1}\|} < \varepsilon$ $\varepsilon = 10^{-3}$	$k=5$	81	—	
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (0,\ 0)$ $x_2^0 = (1,\ 1)$ $x_3^0 = (2,\ 3)$	$x^k = \begin{bmatrix}3.0166\\0.5023\end{bmatrix}$ $f(x^k) = 0.000117$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=37$	250	—	
Simplex (Nelder-Mead)	* $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$	*	$x^k = \begin{bmatrix}2.9998\\0.4999\end{bmatrix}$ $f(x^k) = 2.01 \times 10^{-8}$	*	*	100	—	Result reported in [2]
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (0,0)$ $x_2^0 = (1,\ 1)$ $x_3^0 = (2,\ 3)$	$x^k = \begin{bmatrix}2.9663\\0.4902\end{bmatrix}$ $f(x^k) = 0.00022$	$\frac{\|\Delta f\|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k/N + 1$ $\varepsilon = 10^{-3}$	$k=40$	252	—	
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 1	$x_1^0 = (0,\ 0)$ $x_2^0 = (1,\ 1)$ $x_3^0 = (2,\ 3)$	$x^k = \begin{bmatrix}3.1310\\0.5191\end{bmatrix}$ $f(x^k) = 0.0058$	"	$k=18$	204	—	

Table 6 continued

1	2	3	4	5	6	7	8	9
Simplex with weighted centroid and unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 2	$x_1^0 = (0, 0)$ $x_2^0 = (1, 1)$ $x_3^0 = (2, 3)$	$x^k = \begin{bmatrix} 2.9754 \\ 0.4947 \end{bmatrix}$ $f(x^k) = 0.00012$	$\frac{\| \Delta f \|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=19$	191	—	
Algorithm mixt Simplex-Cyco	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 0	$x^\circ = (0, 0)$ $f(x^\circ) = 14.2$	$x^k = \begin{bmatrix} 2.99993 \\ 0.49998 \end{bmatrix}$ $f(x^k) = 0.00000$	in CYCO $\| x^k - x^{k-1} \| < 10^{-5}$ or $k > 200$	$k=14$	195	—	

Function*: $min\ f(x) = \sum_{y} \{(e^{-x_1 y} - e^{-x_2 y}) - x_3 (e^{-y} - e^{10y})^2\}; j \in \{0.1; 0.1; 1\}$

Optimum Point: $x^* = (1, 10, 1)$ or $\forall\, x^*\ [x_1 = x_2,\ x_3 = 0]$; $f(x^*) = 0$

Method	Programme and/or subroutine	Initial point	Point where procedure ended	Stop test	Number of iteration	Number of evaluations		Remarks
						Function	Gradient	
1	2	3	4	5	6	7	8	9
Hooke and Jeeves	HOOKJV $\Delta x_i^0 = 1$	$x^\circ = (0, 20. 20)$ $f(x^\circ) = 235.419$	$x^k = (24, 24, 0)$ $f(x^k) = 0$	$\Delta x_i < \varepsilon$ $\varepsilon = 10^{-4}$	$k=9$	145	—	
Cyclical optimization on the axes	CYCO NDIAG = 1 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0, 20. 20)$ $f(x^\circ) = 235.419$	$x^k = \begin{bmatrix} 0.42223 \\ 429.49976 \\ 1.78628 \end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=12$	234	—	
Cyclical optimization on the axes	CYCO NDIAG = 0 PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0, 20. 20)$ $f(x^\circ) = 235.419$	$x^k = \begin{bmatrix} 0.42223 \\ 370.41162 \\ 1.78476 \end{bmatrix}$ $f(x^k) = 0.00000$	$\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$	$k=8$	236	—	
Powell	POWELL PSTEP = 0.1 EPSLON = 0.001	$x^\circ = (0, 20, 20)$ $f(x^\circ) = 235.418$	$x^k = \begin{bmatrix} 0.40657 \\ 307.83301 \\ 1.82005 \end{bmatrix}$ $f(x^k) = 0.00062$	$\|x^k - x^{k-1}\| < 10^{-5}$ $\dfrac{\|f(x^k) - f(x^{k-1})\|}{\|x^k - x^{k-1}\|} < \varepsilon$ $\varepsilon = 10^{-3}$	$k=6$	240	—	New start with point obtained with iteration number 5
Rosenbrock	RSBR $\alpha = 2$ $\beta = 0.5$ NO = 1	$x^\circ = (0, 20. 20)$ $f(x^\circ) = 235.418$	$x_1 \to \infty$ $x_2 \to \infty$	$\|x^k - x^{k-1}\| < \varepsilon$ $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$			—	Exponent surpassing with iteration number 4
Davies — Swann — Campey	RSBWS PSTEP = 0.5 EPSLON = 0.001 NO = 2	$x^\circ = (0, 20. 20)$ $f(x^\circ) = 235.418$	$x^k = \begin{bmatrix} 0.42032 \\ 456.8457 \\ 1.78568 \end{bmatrix}$ $f(x^k) = 0.89 \times 10^{-11}$	$\|x^k - x^{k-1}\| < \varepsilon$ $\|f(x^k) - f(x^{k-1})\| < \varepsilon$ $\varepsilon = 10^{-3}$	$k=8$	146	—	

* M. J. Box's function

Table 7 continued

1	2	3	4	5	6	7	8	9
Simplex with weighted centroid	SMPX $\alpha = 1$ $\beta = 0.5$ $\beta = 2$ NO1 = 1 NO2 = 2	$x_1^0 = (0, 20, 20)$ $x_2^0 = (5, 10, 5)$ $x_3^0 = (10, 5, 10)$ $x_4^0 = (0, 0, 10)$	$x^k = \begin{bmatrix} 6.41273 \\ 1.45754 \\ -0.61290 \end{bmatrix}$ $f(x^k) = 0.00018$	$\frac{\| \Delta f \|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=43$	309	—	New start with point obtained with iteration number 28
Simplex (Nelder-Mead)	SMPX $\alpha = 1$ $\beta = 0.5$ $\gamma = 2$ NO1 = 1 NO2 = 1	$x_1^0 = (0, 20, 20)$ $x_2^0 = (5, 10, 5)$ $x_3^0 = (10, 5, 10)$ $x_4^0 = (0, 0, 10)$	$x^k = \begin{bmatrix} 13.6296 \\ 15.2252 \\ 0.0565 \end{bmatrix}$ $f(x^k) = 0.00054$	$\frac{\| \Delta f \|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=29$	223	—	
Simplex with unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 1	$x_1^0 = (0, 20, 20)$ $x_2^0 = (5, 10, 5)$ $x_3^0 = (10, 5, 10)$ $x_4^0 = (0, 0, 10)$	$x^k = \begin{bmatrix} 4.63 \\ 4.33 \\ -0.03 \end{bmatrix}$ $f(x^k) = 0.00008$	$\frac{\| \Delta f \|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=15$	218	—	New start with point obtained with iteration number 11
Simplex with weighted centroid and unidimensional optimization	SMPXWS PSTEP = 0.5 EPSLON = 0.001 NO1 = 2 NO2 = 1	$x_1^0 = (0, 20, 20)$ $x_2^0 = (5, 10, 5)$ $x_3^0 = (10, 5, 10)$ $x_0^0 = (0, 0, 10)$	$x^k = \begin{bmatrix} 4.99 \\ 5.21 \\ 0.02 \end{bmatrix}$ $f(x^k) = 0.000049$	$\frac{\| \Delta f \|}{N} < \varepsilon$ $\Delta f_i = f(x_i^k) - f(\bar{x}^k)$ $\bar{x}^k = \sum_{i=1}^{N+1} x_i^k / N + 1$ $\varepsilon = 10^{-3}$	$k=6$	83	—	
Cyco-Simplex mixted algorithm	SIMIXC PSTEP = 0.1 EPSLON = 0.1 NCENT = 1 NDIAG = 0	$x^\circ = (0, 20, 20)$	$x_2 \to \infty$	In CYCO $\|x^k - x^{k-1}\| < 10^{-5}$ or $k > 200$			—	Over flow exponent

METHODE D'OPTIMISER LES FONCTIONS SANS EVALUATION DE LA DERIVEE (II)

On donne les résultats numériques concernant l'efficience de certains algorithmes des plus connus pour l'optimisation des fonctions sans évaluation de la dérivée, efficience déterminée par le nombre des évaluations des fonctions et par la précision offerte dans la résolution de sept problèmes test. Par le grand nombre des tests effectués, les résultats présentés constituent un ouvrage complet et unitaire dans le domaine.

Au cours de ces recherches une attention toute particulière fut attachée à l'algorithme simplex de Himsworth, Spendley et Hext, développé par la suite par Nedler et Mead, algorithme pour lequel on propose différentes variantes et modifications, dont l'une a trait au calcul pondéré du centroïde. Ces modifications semblent être justifiées par les expérimentations numériques effectuées. Regardant ce même algorithme on propose un algorithme mixte- simplex à optimisation cyclique sur les axes — qui, selon l'avis de l'auteur, serait à même de faire réduire l'effort de calcul par rapport à l'algorithme d'optimisation cyclique sur les axes, tout en augmentant la certitude d'obtenir une solution pour chacun des algorithmes qui le constituent.

L'auteur regrette de ne pas pouvoir, faute d'espace, publier aussi, comme il en avait fait la promesse dans la première partie, les schémas logiques et les programmes qui auraient été utiles au moins en ce qui concerne l'enseignement; il exprime l'espoir de remplir, tout de même, plus tard cette promesse par la publication d'un livre en ce domaine.

МЕТОДЫ ОПТИМИЗАЦИИ ФУНКЦИЙ БЕЗ ОЦЕНКИ ПРОИЗВОДНОЙ (ЧАСТЬ II)

Приводятся числовые результаты эффективности некоторых из наиболее известных алгоритмов оптимизации функций без оценки производной, определяемой числом оценок функций и точностью решения 7 задач проверки. Вследствие большого числа произведенных проверок полученные результаты представляют собой один из наиболее полных трудов, опубликованных в этой области.

В процессе указанных исследований особое внимание уделялось алгоритму симплекс Химсуорса, Спендлея и Хекста, развитому впоследствии Недлером и Мидом, по которому предлагаются различные варианты и изменения, из которых одно касается взвешенного расчета центроида, причем эти изменения кажутся обоснованными проведенными числовыми экспериментами. В связи с этим же алгоритмом предлагается смешанный алгоритм симплекс с цикличной оптимизацией по осям, что, по мнению автора, сократило бы трудоемкость расчета по сравнению с алгоритмом цикличной оптимизации по осям и повысило бы уверенность при получении решения по сравнению с каждым из составляющих его алгоритмов.

Автор сожалеет, что из-за недостатка места он не может опубликовать, как он обещал в части I, логические схемы и программы, что было бы полезно, по меньшей мере, с точки зрения обучения, но надеется сделать это позднее путем издания книги в этой области.

24.XII.1973