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An Economic Growth Model with Discrete Variables and with Linear Production Function

"We consider that the models of economic growth with discrete variables where all the phenomena connected to time factor are mirrored, better corresponds to the concrete requirements for a long-term forecasting and planning".

Manea Mănescu

In the countries with a planned economy, planning represents the vital function of the economic system and the state plan represents the most complex social scheduling. The continuous improvement of the methods of prognosis, management and planning of the national economy is one of the constant and most important preoccupations of the decision-making forums.

A new theory allowing a better substantiation of the prognosis on the directions of development of the national economy has developed during the past decade. We refer to the theory of economic growth, a domain in which a huge volume of works has appeared during the last 10—15 years.

By using a modern mathematical apparatus, the models of economic growth are elaborated for the substantiation of the major economic decisions such as the choice of the optimal relation between accumulation and consumption, the optimal relation between the sectors of the national economy, a.s.o.

In connection with a prognosis model of economic growth or a hierarchical system of models, simulation with the aid of the electronic computer of the different trajectories is playing a special part.

By formulating various hypotheses on the economic level one wishes to attain, the future policy in the field of investments, in the field of consumption, in that of foreign trade and international cooperation, various

trajectories of economic growth can be generated and on the basis of pre-established indices (the output sizes) various admissible trajectories can be analysed by way of comparison.

This causes simulation to be an extremely useful instrument for the elaboration of prognoses and decisions on economic growth.

Nevertheless, we consider that this must not exclude or diminish the role of optimization.

On the contrary, by attaching an objective-function to the model and by applying an adequate mathematical technique for the optimization of the dynamic systems (for example, algorithms and techniques based on Pontreaghin's maximum principle) extremely precious data on the optimum behaviour of the modelled system can be obtained. Even if the optimal trajectories are sometimes "capricious", since they are vertex-solutions and not always applicable in practice, they can supply the simulation process with valuable data on the intervals in which we must vary various parameters of the model in the simulation process.

Moreover, an important advantage of the optimization models is the fact that extremely important data on the behaviour of the trajectories of development can be obtained on the basis of the qualitative study of its mathematical properties. In fact, a number of "Theorems of prognosis and planning" of the type of the turnpike theorems are obtained.

In the literature of speciality, models of economic growth with continuous variables and less with discrete variables have been analysed.

In our opinion, the models with discrete variables can better model the economic phenomena, especially the macroeconomic ones, because the decisions at this level are made at discrete periods of time. On the other hand, the statistical data necessary for the elaboration of models can also be obtained only at discrete time intervals.

Although most of the qualitative properties are common to the models with continuous variables and to those with discrete variables, yet there are certain properties specific of each class of models.

In this paper a particular unisectorial model of economic growth is studied namely, a model in which the production function is linear and the utility function of consumption is of Bernoulli type.

An analogous model with discrete variables has been studied by S. Chakravarty [6] and other authors, but only for the case where the controls are interior to the admissible domain of controls.

In the first part of the paper the model to be studied is described and in the second part the optimum conditions are deduced and the qualitative study of the optimum trajectories is made. The qualitative study refers to the behaviour of optim controls with respect to several relations satisfied by data of the model. Thus, the optimal control is monotonously decreasing or monotonously increasing depending on the data of the model and it is possible to have at most two switch points.

In the last part of the paper an algorithm for calcullating the optim trajectory is presented. This algorithm has been programmed for the electronic computer IBM 360.

Let Y_n be total output at moment n, K_n be capital stock.
We shall suppose that:

$$Y_n = F_n(K_n) = e_n \cdot K_n$$

where e_n is the efficiency coefficient of the capital.
We shall suppose that:

$$e_n \geqslant e_{n-1} \text{ and } 0 < e_n < 1 \text{ for any } n.$$

The evolution equation of the capital is:

$$K_{n-1} = (1 - \mu_n) K_n + I_n, \qquad K_0 - \text{given}$$

where μ_n is the average coefficient of physical and moral wear of the capital and I_n — the volume of gross investments in the $n - th$ period. We define the performance of the system:

$$max\ J = max \sum_{n=0}^{N-1} P_n \cdot U_n(c_n)$$

where $c_n = \dfrac{C_n}{P_n}$, C_n being the total consumption alloted in stage n and P_n the volume of the population in this period.
We shall suppose the evolution equation of the population is:

$$P_{n-1} = p \cdot P_n, \qquad P_0 = 1$$

where $\bar{p} = p - 1$ is the growth rate of the population.
We shall suppose that the utility function is of Bernoulli type:

$$U_n(c) = \delta^n \frac{c^{1-\nu}}{1-\nu}, \quad \nu > 0, \quad 0 < \delta \leqslant 1$$

where $r = \dfrac{1-\delta}{\delta}$ is the discount rate of the consumption.

In other words, $U(c)$ is a strictly concave function and $U'(c)$ has a constant elasticity, namely ν.
We define the control parameter of the system:

$$V_n = \frac{I_n}{Y_n}$$

and we shall suppose that $V_n \in [\alpha, \beta]$ where $\alpha \geqslant 0$ and $\beta \leqslant 1$, $\beta > \alpha$.
The optimum problem we shall further analyse is:

$$K_{n+1} = a_n \cdot K_n + V_n \cdot e_n K_n, \qquad K_0 - \text{given},$$

$$n = 0, 1, 2, \ldots, N-1, \qquad K_N \geqslant \omega$$

$$\max J(v) = \max \sum_{n=0}^{N-1} \frac{\overline{\delta}^n}{1-\nu} [(1 - V_n)\, \lambda_n K_n]^{1-\nu}$$

$$0 \leqslant \alpha \leqslant V_n \leqslant \beta \leqslant 1, \qquad K_0 > 0,\ 0 < e_n < 1$$

where we denoted:

$$a_n = 1 - \mu_n; \quad \lambda_n = \frac{e_n}{p_n}, \quad \overline{\delta} = \delta \cdot p.$$

2. OPTIMUM CONDITIONS.

To find the necessary optimum conditions we shall apply the *maximum principle of Pontryagin* for discrete systems.

The Hamiltonian of the system is:

$$H_n(K_n, V_n, \psi_n) = \psi_n[a_n K_n + V_n \cdot e_n K_n] + S_0 \frac{\overline{\delta}^n}{1-\nu} [(1 - V_n) \cdot \lambda_n K_n]^{1-\nu}$$

where S_0 is a non-negative multiplier and the dual variable results from the dual equation

$$\psi_{n-1} = \psi_n[a_n + V_n e_n] + S_0 \overline{\delta}^n [(1 - V_n)\, \lambda_n K_n]^{-\nu} (1 - V_n)\, \lambda_n$$

or:

$$\psi_{n-1} = \psi_n[a_n + V_n e_n] + S_0 \delta^n [(1 - V_n)\, \lambda_n K_n]^{-\nu} (1 - V_n) e_n$$

$$\text{with } \widetilde{\psi}_{N-1} = -S_1 \geqslant 0 \text{ and } \psi_n \equiv 0 \text{ for } n \geqslant N.$$

Further, we shall note with $\widetilde{V}_n$ the optimum value of the control variable with $\widetilde{K}_n$ the corresponding state and with $\widetilde{\psi}_n$ the solution to the dual equation which results after V_n and K_n are replaced by the optimal values.

Maximal principle of Pontryagin's provides the following necessary optimum condition:

$$\frac{\partial H_n(\widetilde{K}_n, \widetilde{V}_n, \widetilde{\psi}_n)}{\partial V} (V - \widetilde{V}_n) \leqslant 0$$

whatever the admissible control:

$$V \in [\alpha, \beta]$$

$$\text{and} \qquad n = 0, 1, \ldots, N-1.$$

Moreover, we have the following information about the two multipliers:

$$S_0 \geqslant 0; \qquad -S_1 \geqslant 0 \qquad \text{and } S_0 - S_1 > 0$$

and the *transversality* condition is

$$\widetilde{\psi}_{N-1}[\widetilde{K}_N - \omega] = 0.$$

Proposition 1

If $\omega < \check{K}_N$, where $\check{K}_N$ results from the recurrence relation:

$$\check{K}_{n+1} = a_n \check{K}_n + \beta \cdot e_n \check{K}_n, \qquad \check{K}_0 = K_0$$

then

$$S_0 > 0.$$

Proof

Let us suppose that $S_0 = 0$. In this case $S_1 \neq 0$ and thus

$$\tilde{\psi}_{N-1} = -S_1 > 0.$$

The dual equation becomes:

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n [a_n + \tilde{V}_n e_n].$$

As $a_n + V_n e_n > 0$, all $\tilde{\psi}_n$ have the same sign, namely they are positive. We have:

$$\frac{\partial H_n(\tilde{K}_n, \tilde{V}_n, \tilde{\psi}_n)}{\partial V} = \tilde{\psi}_n \cdot e_n \tilde{K}_n$$

and, thus the optimum condition is:

$$\tilde{\psi}_n e_n \tilde{K}_n (V - \tilde{V}_n) \leqslant 0$$
$$n = 0,\ 1 \ldots, N-1.$$

But $K_0 > 0$ and thus $\tilde{K}_n > 0$. It results thereof that the optimal control is $\tilde{V}_n = \beta$, $n = 0, 1 \ldots, N-1$ and thus $\tilde{K}_N = \check{K}_N$. But, in this case the transversality condition: $\tilde{\psi}_{N-1}[\tilde{K}_N - \omega] = 0$ is no longer fulfilled.

Considering $S_0 = 1$ the optimum condition is:

$$[\tilde{\psi}_n e_n - \bar{\delta}^n \lambda_n [(1 - \tilde{V}_n)\lambda_n \tilde{K}_n]^{-\nu}] (V - \tilde{V}_n) \leqslant 0$$

for any $V \in [\alpha, \beta]$ and $n = 0, 1, \ldots, N-1$

or

$$[\tilde{\psi}_n - \delta^n [(1 - \tilde{V}_n)\, \lambda_n \tilde{K}_n]^{-\nu}] (V - \tilde{V}_n) \leqslant 0.$$

Denoting:

$$\tilde{q}_n = \delta^{-n} \tilde{\psi}_n$$

the *optimum conditions* are:

a. if	$\alpha < \tilde{V}_n < \beta$	then $\tilde{q}_n = [(1 - \tilde{V}_n)\lambda_n \tilde{K}_n]^{-\nu}$
b. if	$\tilde{V}_n = \alpha$	then $\tilde{q}_n \leqslant [(1 - \alpha)\lambda_n \tilde{K}_n]^{-\nu}$
c. if	$\tilde{V}_n = \beta$	then $\tilde{q}_n \geqslant [(1 - \beta)\lambda_n \tilde{K}_n]^{-\nu}$

The *equation for* $\tilde{q}_n$ is:

$$\tilde{q}_{n-1} = \tilde{q}_n \delta[a_n + \tilde{V}_n e_n] + \delta[(1 - \tilde{V}_n)\lambda_n \tilde{K}_n]^{-\nu} (1 - \tilde{V}_n) e_n$$

which in the case $\tilde{V}_n \in (\alpha, \beta)$ becomes:

$$\tilde{q}_{n-1} = \tilde{q}_n \delta[a_n + e_n].$$

Proposition 2

The necessary optimum conditions provided by the maximum principle are sufficient optimum conditions. The optimal trajectory is unique.

For the proof see [1]

REMARKS:

1. By introducing a new control variable:

$$U_n = \frac{V_n - \alpha}{\beta - \alpha}$$

the field of the admissible controls becomes the interval [0, 1].

2. Due to proposition 2, the implications in the optimum conditions a), b) and c) are valid in both directions.

The following propositions supply information on the behaviour of the optimal trajectory:

Proposition 3

If the data of the problem satisfy conditions:

$$\delta[a_n + e_n] \left[\frac{p \dfrac{e_{n-1}}{e_n}}{a_{n-1} + \beta e_{n-1}} \right]^{\nu} \geqslant 1, \quad n = 1, 2, \ldots, N-1,$$

then, the optimal control is monotonously decreasing as a time function and may have at most two switch points, from $\tilde{V}_{n_1-1} = \beta$

to $\tilde{V}_{n_1} \in (\alpha, \beta)$ and from $\tilde{V}_{n_2} \in (\alpha, \beta)$ to $\tilde{V}_{n_2+1} = \alpha$.

If $\tilde{V}_{n-1} \in (\alpha, \beta)$ then $\tilde{V}_n < \tilde{V}_{n-1}$.

Proof

Consider $\tilde{V}_n = \beta$; we should prove that $\tilde{V}_{n-1} = \beta$

We have

$$\tilde{q}_n \geqslant [(1 - \beta)\lambda_n \tilde{K}_n]^{-\nu}$$

and

$$\tilde{q}_{n-1} = \tilde{q}_n \delta[a_n + \beta e_n] + \delta[(1 - \beta)\lambda_n \tilde{K}_n]^{-\nu} (1 - \beta) e_n$$

whereof:

$$\tilde{q}_{n-1} \geqslant [(1-\beta)\,\lambda_n \tilde{K}_n]^{-\nu}\,\delta[a_n+e_n] = [(1-\beta)\,\lambda_n(a_{n-1}+$$

$$+\,\tilde{V}_{n-1}e_{n-1})\,\tilde{K}_{n-1}]^{-\nu}\delta[a_n+e_n] = [(1-\beta)\,\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}\,\frac{\delta[a_n+e_n]}{(a_{n-1}+\tilde{V}_{n-1}e_{n-1})^{\nu}}\times$$

$$\times\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\nu} = [(1-\beta)\,\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}\frac{\delta[a_n+e_n]}{[a_{n-1}+\tilde{V}_{n-1}e_{n-1}]^{\nu}}\cdot\left[p\,\frac{e_{n-1}}{e_n}\right]^{\nu} \geqslant$$

$$\geqslant [(1-\beta)\,\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}$$

whereof results that $\tilde{V}_{n-1} = \beta$

Consider now $\tilde{V}_{n-1} = \alpha$; we should prove that $\tilde{V}_n = \alpha$.

Obviously, according to those shown above $\tilde{V}_n \neq \beta$.

Let us suppose that $\tilde{V}_n \in (\alpha, \beta)$.

We have:

$$\tilde{q}_n = \frac{\tilde{q}_{n-1}}{\delta[a_n+e_n]} \leqslant \frac{[(1-\alpha)\,\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}}{\delta[a_n+e_n]} =$$

$$= \frac{\left[(1-\alpha)\,\lambda_{n-1}\dfrac{\tilde{K}_n}{a_{n-1}+\alpha e_{n-1}}\right]}{\delta[a_n+e_n]} =$$

$$= [(1-\alpha)\lambda_n\tilde{K}_n]^{-\nu}\,\frac{[a_{n-1}+\alpha e_{n-1}]}{\delta[a_n+e_n]}\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\nu} < [(1-\alpha)\,\lambda_n\tilde{K}_n]^{-\nu}\times$$

$$\times\,\frac{(a_{n-1}+\beta e_{n-1})^{\nu}}{\delta[a_n+e_n]}\left(\frac{1}{p}\,\frac{e_n}{e_{n-1}}\right)^{\nu} \leqslant [(1-\alpha)\,\lambda_n\tilde{K}_n]^{-\nu}$$

which contradicts the hypothesis $\tilde{V}_n > \alpha$.

Consider now $\tilde{V}_{n-1} \in (\alpha, \beta)$. According to those shown, we cannot have $\tilde{V}_n = \beta$.

If $\tilde{V}_n = \alpha$, then relation $\tilde{V}_n < \tilde{V}_{n-1}$ is proved. But, let us suppose that also $\tilde{V}_n \in (\alpha, \beta)$.

We have:

$$\tilde{q}_n = \frac{\tilde{q}_{n-1}}{\delta[a_n+e_n]} = \frac{[(1-\tilde{V}_{n-1})\,\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}}{\delta[a_n+e_n]} =$$

$$= [(1-\tilde{V}_{n-1})\,\lambda_n\tilde{K}_n]^{-\nu}\,\frac{[a_{n-1}+\tilde{V}_{n-1}e_{n-1}]^{\nu}}{\delta[a_n+e_n]}\cdot\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\nu} <$$

$$< [(1-\tilde{V}_{n-1})\,\lambda_n\tilde{K}_n]^{-\nu}\,\frac{[a_{n-1}+\beta e_{n-1}]^{\nu}}{\delta[a_n+e_n]}\left[\frac{e_n}{pe_{n-1}}\right]^{\nu} \leqslant [(1-\tilde{V}_{n-1})\,\lambda_n\tilde{K}_n]^{-\nu}.$$

But, on the other hand

$$\tilde{q}_n = [(1 - \tilde{V}_n)\,\lambda_n \tilde{K}_n]^{-\nu}$$

and thus

$$[(1 - \tilde{V}_n)\,\lambda_n \tilde{K}_n]^{-\nu} < [(1 - \tilde{V}_{n-1})\,\lambda_n \tilde{K}_n]$$

thus

$$\tilde{V}_n < \tilde{V}_{n-1}.$$

REMARK

If $a_n = a$ and $e_n = e$ then, the condition in proposition 3 becomes:

$$\frac{\delta(a + e) \cdot p^{\nu}}{(a + \beta e)^{\nu}} \geqslant 1.$$

Corollary 1

$$\text{If } e_n = e, \quad a_n = a, \quad \beta = 1, \quad a + e > p,$$

and the rate of growth of the population is lower to the discount rate, in this case the condition in proposition 3 can be satisfied only if $\nu < 1$.

Proof

We have:

$$\nu \leqslant \frac{\ln(a + e) + \ln \delta}{\ln(a + e) - \ln(p)} = \frac{\ln(a + e) + \ln \dfrac{1}{1 + r}}{\ln(a + e) - \ln(1 + \bar{p})}$$

where r is the discount rate, and $\bar{p} = p - 1$ is the growth rate of the population. Thus

$$\nu \leqslant \frac{\ln(a + e) - \ln(1 + r)}{\ln(a + e) - \ln(1 + \bar{p})} < 1.$$

Proposition 4

If the data of the problem satisfy conditions:

$$\delta[a_n + e_n]\left[\frac{p \cdot \dfrac{e_{n-1}}{e_n}}{a_{n-1} + \alpha e_{n-1}}\right]^{\nu} \leqslant 1 \qquad n = 1, 2, \ldots, N - 1$$

then the optimal control is monotonously increasing as a time function and may have at most two switch points from $\tilde{V}_{n_1 - 1} = \alpha$ to $\tilde{V}_{n_1} \in (\alpha, \beta)$ and from $\tilde{V}_{n_2} \in (\alpha, \beta)$ to $\tilde{V}_{n_2 + 1} = \beta$. If $\tilde{V}_{n-1} \in (\alpha, \beta)$ then $\tilde{V}_n > \tilde{V}_{n-1}$

Proof

Consider $\widetilde{V}_n = \alpha$; in this case we shall show that also $\widetilde{V}_{n-1} = \alpha$. We have the optimum condition:

$$\widetilde{\widetilde{q}}_n \leqslant [(1-\alpha)\,\lambda_n \widetilde{K}_n]^{-\nu}$$

and

$$\widetilde{\widetilde{q}}_{n-1} = \widetilde{\widetilde{q}}_n \delta[a_n + \alpha e_n] + \delta[(1-\alpha)\,\lambda_n \widetilde{K}_n]^{-\nu}(1-\alpha)\,e_n \leqslant$$

$$\leqslant [(1-\alpha)\,\lambda_n \widetilde{K}_n]^{-\nu}\delta[a_n + e_n] =$$

$$= [(1-\alpha)\,\lambda_n \widetilde{K}_n]^{-\nu}\,\delta[a_n + e_n]\,\frac{\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\nu}}{(a_{n-1} + \alpha e_{n-1})^{\nu}} <$$

$$< [(1-\alpha)\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu}.$$

and thus $\widetilde{V}_{n-1} = \alpha$.

Consider now $\widetilde{V}_{n-1} = \beta$; we should prove that also $\widetilde{V}_n = \beta$. According to those proved, $\widetilde{V}_n \neq \alpha$. Let us suppose that $\widetilde{V}_n \in (\alpha, \beta)$

We have:

$$\widetilde{\widetilde{q}}_n = \frac{\widetilde{\widetilde{q}}_{n-1}}{\delta[a_n + e_n]} \leqslant \frac{[(1-\beta)\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu}}{\delta[a_n + e_n]} =$$

$$= [(1-\beta)\,\lambda_n \widetilde{K}_n]^{-\nu}\,\frac{(a_{n-1} + \beta e_{n-1})^{\nu}}{\delta[a_n + e_n]} \cdot \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\nu} >$$

$$> [(1-\beta)\,\lambda_n \widetilde{K}_n]^{-\nu}.$$

which contradicts $\widetilde{V}_n < \beta$.

Consider now $\widetilde{V}_{n-1} \in (\alpha, \beta)$. According to those proved $\widetilde{V}_n > \alpha$; if $\widetilde{V}_n = \beta$. the property is proved. Let us suppose that $\widetilde{V}_n \in (\alpha, \beta)$. We have:

$$\widetilde{\widetilde{q}}_n = \frac{\widetilde{\widetilde{q}}_{n-1}}{\delta[a_n + e_n]} = \frac{[(1-\widetilde{V}_{n-1})\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu}}{\delta[a_n + e_n]} =$$

$$= [(1-\widetilde{V}_{n-1})\,\lambda_n \widetilde{K}_n]^{-\nu}\,\frac{(a_{n-1} + \widetilde{V}_{n-1}e_{n-1})^{\nu}}{\delta[a_n + e_n)}\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\nu} >$$

$$> [(1-\widetilde{V}_{n-1})\,\lambda_n \widetilde{K}_n]^{-\nu}$$

thus:

$$\widetilde{\widetilde{q}}_n = [(1-\widetilde{V}_n)\,\lambda_n \widetilde{K}_n]^{-\nu} > (1-\widetilde{V}_{n-1})\,\lambda_n \widetilde{K}_n]^{-\nu}$$

whereof

$$\widetilde{V}_n > \widetilde{V}_{n-1}.$$

Corollary 2

If $e_n = e$, $a_n = a$, then the conditions in proposition 4 may take place only if:

$$\alpha > \frac{p-a}{e} = \frac{\bar{p}+\mu}{e}$$

where $\bar{p} = p - 1$ is the growth rate of the population.

Proof

If $\alpha < \frac{p-a}{e}$, then from the relation required by proposition 4 results

$$\nu < -\frac{\ln(a+e) + \ln\delta}{\ln p - \ln(a+\alpha e)} < 0$$

which is impossible.

Example:

If $p = 1.01$; $a = 0.95$ and $e = 0.33$ then to have the conditions in proposition 4 satisfied, it is necessary that $\alpha > 0.18$.

Proposition 5

If the data of the problem satisfy condition:

$$\delta[a_n + e_n]\left[\frac{p \cdot \frac{e_{n-1}}{e_n}}{a_{n-1} + \beta e_{n-1}}\right]^{\nu} < 1$$

then $\tilde{V}_{n-1} = \beta$ implies $\tilde{V}_n = \beta$
and if:

$$\delta[a_n + e_n]\left[\frac{p\,\frac{e_{n-1}}{e_n}}{a_{n-1} + \alpha e_{n-1}}\right]^{\nu} > 1$$

then $\tilde{V}_{n-1} = \alpha$ implies $\tilde{V}_n = \alpha$

Proof

Consider the first relation satisfied and let us suppose that $\tilde{V}_{n-1} = \beta$. Let us suppose that $\tilde{V}_n = \alpha$. We have:

$$\tilde{q}_{n-1} = \delta\tilde{q}_n[a_n + \alpha e_n] + \delta[(1-\alpha)\lambda_n\tilde{K}_n]^{-\nu}(1-\alpha)e_n \leqslant$$

$$\leqslant \delta[(1-\alpha)\lambda_n\tilde{K}_n]^{-\nu}[a_n + e_n] =$$

$$= [(1-\alpha)\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}\,\delta[a_n + e_n] \cdot \frac{\left[\frac{\lambda_{n-1}}{\lambda_n}\right]^{\nu}}{[a_{n-1} + \beta e_{n-1}]^{\nu}} <$$

$$< [(1-\alpha)\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu} < [(1-\beta)\lambda_{n-1}\tilde{K}_{n-1}]^{-\nu}$$

which contradicts the hypothesis that $\tilde{V}_{n-1} = \beta$.

If $\widetilde{V}_{n-1} = \beta$ and $\widetilde{V}_n \in (\alpha, \beta)$ we have:

$$\widetilde{q}_{n-1} = \delta[a_n + e_n]\,\widetilde{q}_n = \delta[a_n + e_n]\,[(1 - \widetilde{V}_n)\,\lambda_n \widetilde{K}_n]^{-\nu} =$$

$$= [(1 - \widetilde{V}_n)\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu}\,\delta[a_n + e_n]\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\nu} / [a_{n-1} + \beta e_{n-1}]^{-\nu} <$$

$$< [(1 - \widetilde{V}_n)\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu} < [(1 - \beta)\,\lambda_{n-1}\widetilde{K}_{n-1}]^{-\nu}$$

which also contradicts the hypothesis $\widetilde{V}_{n-1} = \beta$.
The other implication of the proposition is similarly proved.

Corollary 3

If the data of the problem satisfy conditions:

$$\delta[a_n + e_n]\left[\frac{p \cdot \dfrac{e_{n-1}}{e_n}}{a_{n-1} + \beta e_{n-1}}\right]^{\nu} < 1$$

and

$$\delta[a_n + e_n]\left[\frac{p\,\dfrac{e_{n-1}}{e_n}}{a_{n-1} + \alpha e_{n-1}}\right]^{\nu} > 1, \qquad n = 1, 2, \ldots, N-1$$

then the optimal control can have at most one switch point, namely from $\widetilde{V}_{n_1} \in (\alpha, \beta)$ to $\widetilde{V}_{n_1+1} = \alpha$ or $\widetilde{V}_{n_1+1} = \beta$.

3. OPTIMAL PATHS FOR THE AUTONOMOUS CASE

In case the system is autonomous and for a period of time $[n_1, n_2] \subseteq [0, N-1]$ the optimal control is found inside the domain of the admissible controls, the next proposition supplies the explicite formulas for the optimal control and for the optimal state.

It can be seen from the respective formulas that for the interval in which $\widetilde{V}_n \in (\alpha, \beta)$ the consumption per capita increases exponentially having the rate of growth: $[\delta(a + e)]^{1/\nu}$.

The formulas given by proposition 6 can be extended immediately also for the noautonomous case.

Proposition 6

Consider $a_n = a, \quad e_n = e \quad (n = 0, 1, \ldots, N)$
and $\widetilde{V}_n \in (\alpha, \beta)$ for $n = n_1, \; n_1 + 1, \ldots, n_2$
$(0 \leqslant n_1 < n_2 \leqslant N - 1)$.

We have:

$$\widetilde{K}_{n_1+n} = \gamma^n \omega_1 - [\gamma^{n_2-n_1}\omega_1 - \omega_2]\,\frac{\varepsilon^{n_1+n} - \gamma^{n_1+n}}{\varepsilon^{n_2} - \gamma^{n_2}}$$

$$\tilde{V}_{n_1+n} = 1 - \frac{\tilde{C}_{n_1+n}}{e\tilde{K}_{n_1+n}}.$$

$$n = 0, 1, \ldots, n_2 - n_1$$

where we denoted:

$$\gamma = a + e; \qquad \varepsilon = [\delta(a+e)]^{1/\nu} \cdot p$$

and

$$\omega_1 = \tilde{K}_{n_1}; \qquad \omega_2 = \tilde{K}_{n_2}$$

$$\tilde{C}_{n_1+n} = \frac{\varepsilon^{n_1+n}[\varepsilon - \gamma]}{\varepsilon^{n_2} - \gamma^{n_2}} [\gamma^{n_2-n_1}\omega_1 - \omega_2].$$

Proof

For

$$n = n_1,\ n_1 + 1, \ldots, n_2$$

we have:

$$\tilde{K}_n = [a + e\tilde{V}_{n-1}]\, \tilde{K}_{n-1}$$

$$\tilde{q}_n = [(1 - \tilde{V}_n)\, \lambda_n \tilde{K}_n]^{-\nu} = \frac{\tilde{q}_{n-1}}{\delta(a+e)}.$$

According to the introduced notation, we have:

$$\tilde{C}_n = (1 - \tilde{V}_n)\, e\tilde{K}_n = p^n(1 - \tilde{V}_n)\, \lambda_n \tilde{K}_n,$$

thus

$$\tilde{K}_{n+1} = (a + e)\, \tilde{K}_n - \tilde{C}_n$$

and

$$\tilde{C}_n = p^n \tilde{q}_n{}^{-1/\nu}$$

thus:

$$\tilde{q}_n = p^{n \cdot \nu} \tilde{C}_n^{-\nu}$$

whereof:

$$p^{n\nu}\tilde{C}_n^{-\nu} = \frac{p^{(n-1)\nu}\tilde{C}_{n-1}^{-\nu}}{\delta(a+e)}$$

and thus, the evolution equation for $\tilde{C}_n$ is:

$$\tilde{C}_n = p\,[\delta(a+e)]^{1/\nu}\, \tilde{C}_{n-1}, \quad n = n_1, \ldots, n_2$$

or

$$\widetilde{C}_n = p^{n-n_1}[\delta(a+e)]^{\frac{n-n_1}{\nu}}\widetilde{C}_{n_1},\ n = n_1, ..., n_2.$$

We denote:

$$b = p^{-n_1}[\delta(a+e)]^{-\frac{n_1}{\nu}}\widetilde{C}_{n_1}$$

and thus

$$\widetilde{C}_n = b\varepsilon^n$$

whereof:

$$\widetilde{K}_{n+1} = \gamma\widetilde{K}_n - b\varepsilon^n, \qquad n = n_1, ..., n_2$$

or:

$$\widetilde{K}_{n_1+n+1} = \gamma\widetilde{K}_{n_1+n} - b\varepsilon^{n_1+n}, \qquad n = 0, 1, .., n - n_2,$$

Supposing $\gamma \neq \varepsilon$, the solution of the above equation is:

$$\widetilde{K}_{n_1+n} = \gamma^n\widetilde{K}_{n_1} - b\frac{\varepsilon^{n_1+n} - \gamma^{n_1+n}}{\varepsilon - \gamma},\ n = 0, 1, ..., n_2 - n_1.$$

According to the introduced notation we have:

$$\widetilde{K}_{n_1} = \omega_1; \qquad \widetilde{K}_{n_2} = \omega_2$$

whereof:

$$\omega_2 = \gamma^{n_2-n}\widetilde{K}_{n_1} - b\frac{\varepsilon^{n_2} - \gamma^{n_2}}{\varepsilon - \gamma}$$

and thus:

$$b = [\gamma^{n_2-n_1}\widetilde{K}_{n_1} - \omega_2]\frac{\varepsilon - \gamma}{\varepsilon^{n_2} - \gamma^{n_2}}$$

and thus:

$$\widetilde{K}_{n_1+n} = \gamma^n\omega_1 - [\gamma^{n_2-n_1}\omega_1 - \omega_2]\frac{\varepsilon^{n_1+n} - \gamma^{n_1+n}}{\varepsilon^{n_2} - \gamma^{n_2}}$$

$$\widetilde{C}_{n+n_1} = [\gamma^{n_2-n_1}\omega_1 - \omega_2]\frac{\varepsilon - \gamma}{\varepsilon^{n_2} - \gamma^{n_2}}\varepsilon^{n_1+n}\ n = 0, 1, ..., n_2 - n_1.$$

The following proposition gives the elasticity of consumption with respect to the imposed final condition, when:

$$\widetilde{V}_n \neq \alpha \text{ and } \widetilde{V}_n \neq \beta.$$

Proposition 7

If $\tilde{V}_n \in (\alpha, \beta)$ for $n = 0, 1, ..., N-1$

then

$$E_\omega \tilde{C}_n = \frac{1}{\left(\frac{a+e}{1+\bar{f}}\right)^N - 1}$$

where $E_\omega \tilde{C}_n$ is the elasticity of $\tilde{C}_n$ with respect to ω, and $\bar{f}$ is the average growth rate of the capital. Thus:

$$\omega = (1 + \bar{f})^N \cdot K_0.$$

The proof easily results from the direct calculus of $E_\omega \tilde{C}_n$.

Proposition 8

If the data of the problem satisfy condition: $\delta(a+e)\left(\frac{p}{a+\beta e}\right)^\nu \geqslant 1$ and n_1, n_2 are the switch points of the optimal control then, they satisfy relations:

$$n_1 \ln \frac{\varepsilon}{a+\beta e} - n_2 \ln \frac{\varepsilon}{a+\alpha e} \leqslant \ln \frac{1-\alpha}{1+\beta} \frac{\omega}{(a+\alpha e)^N K_0} -$$

$$- n_1 \ln \frac{\gamma}{a+\beta e} + n_2 \ln \frac{\gamma}{a+\alpha e} \geqslant \ln \frac{\omega}{(a+\alpha e)^N K_0}$$

and

$$n_2 \geqslant \frac{\ln \frac{\omega}{(a+\alpha e)^N K_0}}{\ln \frac{\gamma}{a+\alpha e}}.$$

where $\gamma = a + e$ and $\varepsilon = [\delta(a+e)]^{1/\nu} \cdot p$.

Proof

For $n = 0, 1, ..., n_2 - n_1$ we have: $\tilde{V}_{n_1+n} = 1 - \frac{\tilde{C}_{n_1+n}}{e\tilde{K}_{n_1+n}}$.

From $\alpha \leqslant \tilde{V}_{n_1+n} \leqslant \beta$ results:

$$(1-\alpha)\, e\tilde{K}_{n_1+n} \geqslant \tilde{C}_{n_1+n} \geqslant (1-\beta)\, e\tilde{K}_{n_1+n}.$$

According to proposition 3 we have:

$$\tilde{V}_n < \tilde{V}_{n-1}$$

thus, to have the above inequalities fulfilled, it suffices to have:

$$\widetilde{C}_{n_1} \geqslant (1-\beta)\, e\widetilde{K}_{n_1} = (1-\beta)\, e\omega_1$$

$$\widetilde{C}_{n_2} \leqslant (1-\alpha)\, e\widetilde{K}_{n_2} = (1-\alpha)\, e\omega_2.$$

According to proposition 6, results:

$$\frac{\varepsilon^{n_1}[\varepsilon-\gamma]}{\varepsilon^{n_2}-\gamma^{n_2}}[\gamma^{n_2-n_1}\omega_1-\omega_2] \geqslant (1-\beta)\, e\omega_1$$

$$\frac{\varepsilon^{n_2}(\varepsilon-\gamma)}{\varepsilon^{n_2}-\gamma^{n_2}}[\gamma^{n_2-n_1}\omega_1-\omega_1] \leqslant (1-\alpha)\, e\omega_2$$

whereof by division we have:

$$\varepsilon^{n_2-n_1} \leqslant \frac{1-\alpha}{1-\beta}\frac{\omega_2}{\omega_1}.$$

But

$$\widetilde{V}_n = \beta \quad \text{for} \quad n = 0,\ 1,\ \ldots,\ n_1 - 1$$

thus

$$\omega_1 = (\alpha+\beta e)^{n_1} K_0$$

and $\widetilde{V}_n = \alpha$ for $n = n_2+1,\ n_2+2,\ \ldots,\ N-1$

thus

$$\omega_2 = \frac{\omega}{(a+\alpha e)^{N-n_2}}$$

thus

$$(n_2-n_1)\ln\varepsilon \leqslant \ln\frac{1-\alpha}{1-\beta}\cdot\frac{\omega}{(a+\alpha e)^N K_0} - n_1\ln(a+\beta e) + n_2\ln(a+\alpha e),$$

$$n_1\ln\frac{a+\beta e}{\varepsilon} - n_2\ln\frac{a+\alpha e}{\varepsilon} \geqslant \ln\frac{1-\beta}{1-\alpha}\frac{(a+\alpha e)^N K_0}{\omega}.$$

From condition:

$$\frac{\delta(a+e)}{(a+\beta e)\,\nu}\cdot p^{\nu} \geqslant 1 \text{ results: } \frac{\varepsilon}{a+\beta e} \geqslant 1.$$

Supposing $\dfrac{\varepsilon}{a+\beta e} > 1$ we have:

$$n_1\ln\frac{\varepsilon}{a+\beta e} - n_2\ln\frac{\varepsilon}{a+\alpha e} \leqslant \ln\frac{1-\alpha}{1-\beta}\frac{\omega}{(a+\alpha e)^N K_0}.$$

On the other hand, from $\widetilde{C}_n \geqslant 0$ results: $\gamma^{n_2-n_1}\omega_1 - \omega_2 \geqslant 0$ or:

$$(n_2 - n_1)\ln\gamma + n_1\ln(a + \beta e) + \ln K_0 -$$

$$- n_2\ln(a + \alpha e) - \ln\frac{\omega}{(a + \alpha e)^N} \geqslant 0$$

or:

$$n_1\ln\frac{a + \beta e}{\gamma} - n_2\ln\frac{a + \alpha e}{\gamma} \geqslant \ln\frac{\omega}{(a + \alpha e)^N \widetilde{K}_0}$$

or:

$$-n_1\ln\frac{\gamma}{a + \beta e} + n_2\ln\frac{\gamma}{a + \alpha e} \geqslant \ln\frac{\omega}{(a + \alpha e)^N K_0}.$$

As $n_1 \geqslant 0$ it results from the above relation:

$$n_2 \geqslant \frac{\ln\dfrac{\omega}{(a + \alpha e)^N K_0}}{\ln\dfrac{\gamma}{a + \alpha e}}.$$

REMARK

In case the data of the problem satisfy conditions:

$$\delta[a + e]\cdot\left[\frac{p}{a + \alpha e}\right]^{\nu} \leqslant 1$$

a number of properties similar to those given in proposition 7 may be inferred.

4. ALGORITHMS FOR CALCULATING THE OPTIMAL TRAJECTORY

We shall present the calculus algorithm only for the case when the data of the problem satisfy condition:

$$\delta(a + e)\left(\frac{p}{a + \beta e}\right)^{\nu} \geqslant 1.$$

For the other cases, the algorithms are described in similar ways. In this case, according to proposition 3 we know that the optimal control may have at most two switch points and is monotonously decreasing. Proposition 8 supplies data on the switch points.

We shall notice that function:

$$\widetilde{C}_{n_2} = \frac{\varepsilon^{n_2}[\varepsilon - \gamma]}{\varepsilon^{n_2} - \gamma^{n_2}}[\gamma^{n_2-n_1}\cdot\omega_1 - \omega_2]$$

is decreasing with respect to $n_1(\gamma > 1)$.

The algorithm of calcullating the optimal trajectory in this case is the following:

Step 1

Consider

$$S_2 = \frac{\ln \dfrac{\omega}{(a+\alpha e)^N K_0}}{\ln \dfrac{\gamma}{a+\alpha e}}$$

and $S_2 \leqslant n_2 \leqslant N-1$.
For n_2 fixed, consider $n_1^*(n_2)$ the smallest positive integer number satisfying inequalities:

$$n_1 \ln \frac{\varepsilon}{a+\beta e} - n_2 \ln \frac{\varepsilon}{a+ae} \leqslant \ln \frac{1-\alpha}{1+\beta} \frac{\omega}{(a+\alpha e)^N K_0},$$

$$-n_1 \ln \frac{\gamma}{a+\beta e} + n_2 \ln \frac{\gamma}{a+\alpha e} \geqslant \ln \frac{\omega}{(a+\alpha e)^N K_0},$$

$$\frac{\varepsilon^{n_2}(\varepsilon-\gamma)}{\varepsilon^{n_2}-\gamma^{n_2}} [\gamma^{n_2-n_1^*(n_2)} \omega_1 - \omega_2] \leqslant (1-\alpha) e\omega_2,$$

$$\frac{\varepsilon^{n_1^*(n_2)}(\varepsilon-\gamma)}{\varepsilon^{n_2}-\gamma^{n_2}} [\gamma^{n_2-n_1^*(n_2)} \omega_1 - \omega_2] \geqslant (1-\beta)\omega_1.$$

If $n_1^*(n_2) \geqslant 0$ exists and $n_1^*(n_2) < n_2$ then $n_1^*(n_2)$ and n_2 are the sought switch points and in this case we pass to step 2. If for the chosen n_2 there is no $n_1^*(n_2)$ non-negative, smaller or at least equal to n_2 which should satisfy the four above inequalities, then another integer number $n_2 \in [S_2, N-1]$ is chosen. The seeking process begins with $n_2 = S_2$.

Step 2

The optimal control is:

$$\begin{cases} \widetilde{K}_n = (a+\beta e)^n K_0, & n = 0, 1, \ldots, n_1^*(n_2) \\ \widetilde{K}_{n_1^*(n_2)+n} = \gamma^n \omega_1 - [\gamma^{n_2-n_1^*(n_2)} \cdot \omega_1 - \omega_2] \times \dfrac{\varepsilon^{n_1^*(n_2)+n} - \gamma^{n_1^*(n_2)+n}}{\varepsilon^{n_2} - \gamma^{n_2}} \\ \qquad\qquad n = 1, 2, \ldots, n_2 - n_1^*(n_2) \\ \widetilde{K}_{n_2+n} = \dfrac{\omega}{(a+\alpha e)^{N-n-n_2}}, \quad n = 1, 2, \ldots, N-n_2 \end{cases}$$

and the optimal control is:

$$\widetilde{V}_n = \beta, \quad n = 0, 1, ..., n_1^*(n_2) - 1$$

$$V_{n_1^*(n_2)+n} = 1 - \frac{\widetilde{C}_{n_1^*(n_2)+n}}{e\widetilde{K}_{n_1^*(n_2)+n}}, \quad n = 0, 1, ..., n_2 - n_1^*(n_2)$$

and

$$\widetilde{V}_{n_2+n} = \alpha, \quad n = 1, 2, ..., N - n_2$$

where:

$$\widetilde{C}_{n_1^*(n_2)+n} = \frac{\varepsilon^{n_1^*(n_2)+n}(\varepsilon - \gamma)}{\varepsilon^{n_2} - \gamma^{n_2}} [\gamma^{n_2 - n_1^*(n_2)} \cdot \omega_1 - \omega_2]$$

$$\gamma = a + e \qquad \varepsilon = [\delta(a + e)]^{1/\nu} \cdot p$$

$$\omega_1 = \widetilde{K}_{n_1}; \qquad \omega_2 = \widetilde{K}_{n_2}.$$

CONCLUSIONS

The algorithm presented in this paper can be easily extended also for the nonautonomous case where the macroeconomic production function is of the form:

$$Y_n = F_n(K_n) = e_n \cdot K_n$$

It results that the algorithm can be applied to a wide class of models since if the unit of measuring time is small enough, any concave function can be approximated with linear functions.

If the production function has the form $Y_n = F_n(L_n, K_n)$, where F_n (1)is homogenous of the first degree, we can write $F_n(L_n, K_n) = L_n f_n(R_n)$, where $f_n(R_n) = F_n(1, R_n)$ and $R_n = K_n/L_n$. If $f_n(R_n) \approx e_n K_n$, then $F_n(L_n, K_n) \approx e_n K_n$ and in this case, we can also apply our algorithm.

A remark we want to make refers to the sensitivity calculations of the optimal trajectory made by several authors [7], [8].

Most part of the authors have also calcullated the sensitivity of optimal trajectories depending on the modifications of some parameters of the model considering only the case where the control is interior to the admissible order field, case where the optimal trajectories can also be obtained using the method of classical, variational calculus.

But, in the general case, the optimal controls can take values also on the frontier of the admissible field. In this case, thc modifications of some parameters of the models also modify the switch points and in this way the calculation ot optimal trajectory sensitivity is much more complex.

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UN MODELE DE CROISSANCE ECONOMIQUE A VARIABLES DISCRETES AVEC UN FONCTION LINÉAIRE DE PRODUCTION

L'auteur étudie un modèle unisectoriel particulier de croissance économique, à savoir un modèle où la fonction de production est linéaire et la fonction d'utilité de la consommation est de type Bernoulli. Un modèle similaire à variables discrètes fut étudié par S. Chakravarty et par d'autres auteurs, mais uniquement pour le cas où les commandes sont à l'intérieur du domaine des commandes admissibles.

La première partie de l'article comprend la description du modèle à étudier et dans la seconde partie on déduit les conditions d'optimum et on fait l'étude qualitative des trajectoires optimales.

L'étude qualitative traite notamment du comportement de la commande optimale en fonction de différentes relations satisfaites par les données du modèle. Ainsi, en fonction des données du modèle la commande optimale est monotonement décroissante ou croissante, pouvant avoir deux points de commutation tout au plus et au moins un seul point.

La dernière partie contient un algorithme pour le calcul de la trajectoire optimale. Cet algorithme a été mis en programme sur un ordinateur éléctronique IBM 360.

МОДЕЛЬ ЭКОНОМИЧЕСКОГО РОСТА С ДИСКРЕТНЫМИ ПЕРЕМЕННЫМИ И ЛИНЕЙНОЙ ПРОИЗВОДСТВЕННОЙ ФУНКЦИЕЙ

В настоящей статье изучается частная односекторная модель экономического роста а именно модель, в которой производственная функция является линейной, а функция полезности потребления представляет собой функцию типа Бернулли.

Аналогичная модель с дискретными переменными была изучена С. Чакраварти [6] и другими авторами, но только в случае, когда управление находится в пределах допустимого.

В первой части статьи описывается подлежащая изучению модель, после чего во второй части выводятся условия оптимума и производится качественное изучение оптимальных траекторий.

Качественное изучение касается, в основном, поведения оптимального управления в зависимости от различных соотношений, удовлетворяемых данными модели. Так, в зависимости от данных модели оптимальное управление является монотонно уменьшающимся или монотонно возрастающим и может иметь не более двух точек коммутации или не более одной единственной точки коммутации.

В последней части статьи представляется алгоритм для расчета оптимальной траектории. Этот алгоритм программирова для электронной вычислительной машины I.B.M. 360.

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