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Application of Mathematical Programming in Production Planning

The authors suggest the building of a mathematical model for optimal planning of the production process in a tinned goods enterprise.

The proportions and many-sided links within our national economy, the planned management of its development dictate a higher accuracy of economic computation, the broader use of quantitative analysis in measurement and comparison between resources and the efficiency of their utilization.

Mathematical programming is one of the most powerful intrumeans of mathematical modelling of economic processes, of computation of the indices and prices, of analysis of the variants of solutions to economic problems at the level of national economy, of branches and enterprises.

This survey is aimed at providing the mathematical model of mechanized production of a tinned goods factory conducive to an optimal planning variant of this unit's production.

The tinned goods factory near Constantza is the chief section of the "Munca" food combine. It turns out a large number of products out of tomatoes and fruits — quinces, apricots, peaches and morello cherries. Its chief output — some two-thirds — is obtained from tomatoes, under the form of double and triple concentrated tomato paste, tomato juice, etc. — in a varying range of packings with regard to material and size — and also peeled tomato preserves destined mainly for export. Fruits are processed into nectar — with the juice residue as an auxiliary product, jams and compotes.

Production is fully mechanized — excepting the manufacture of jams and compotes which have not been included in the calculus — and proceeds on five technological lines: two for concentrated paste, one for peeled tomatoes, one for tomato juice and one for fruit juice, which may be used for syrups as well.

The activity of the tinned goods factory has an uneven course in the year.

In July, August, September, and sometimes part of October, work is carried on 23 hours out of 24 (one hour being required for the cleaning and disinfection of the technological line), and the amount of man-power grows 20 times as against the winter period. The factory is supplied under contracts concluded with the socialist agricultural units in its vicinity, as it is sited in a zone enjoying broad possibilities for ensuring the raw material needed for a smooth production process. Since the raw material consists almost only of perishables products (which cannot be stocked), and the technological lines are not adapted to the production of other preserves that can be manufactured all the year long, work in the factory is much reduced in winter.

We propose to elaborate a mathematical model on the basis of which we would determine an optimum planning variant for minimizing production costs.

There are four groups of restrictions in the formulation of conditions:

a) *restrictions referring to capacities and their utilization*:
b) *restrictions referring to costs*;
c) *restrictions referring to packings*;
d) *restrictions referring to the raw material.*

Output is obtained following the processing of the raw material on four technological lines. Let us mark by:

x_j — the unknown quantities of the product j ($j = 1, ..., 20$) or, more exactly, the unknown number of preserves of type j to be produced *;

μ_j — the quantities of finished product which is packed in boxes of type j;

c_i — the maximum capacities of the technological lines (the capacities have been taken in a differentiated manner from the viewpoint of the various technological lines for the interval of 2—3 months covered by the production campaign), in terms of kilograms of finished product ($i = 1, ..., 5$).

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* The problem will thus be of programming whole numbers. Seeing that the quantity of x_j is of tens of thousands, the neglecting of fractions leads to very small errors, utterly negligible practically. On the other hand, considering the dimensions of the problem, the solving in whole numbers would be very difficult, expensive and of long duration, if not impossible.

Let $j_i = 1, ..., j_i$ be the number of products j turned out on line i. The restrictions imposed by the capacities of the technological lines are then in the form:

$$\sum_{j=j_{i-1}}^{j_i} \mu_j x_j \leqslant c_i \ (i = 1, ..., 5)$$

Yet, we have practically only four restrictions because the first two lines are identical.

It is also under the name of restrictions of capacities that we have marked the conditions of inferior limiting of the utilization of the capacities of technological lines, which are imposed by the fact that some of the inferior limits are plan figures and their non-observance would be prejudicial to the enterprise or even to the national economy (non-delivery of products to be exported under contract, lack of an assortment).

These restrictions are expressed for the utilization of the technological lines:

$$\sum_{j=j_i-1}^{j_i} \mu_j x_j \geqslant d_i$$

where d_i is the inferior utilization limit (in the finished product) or, for the products to be exported under contracts:

$$x_j > d'_j$$

where d'_j is the quantity of product j required (number of boxes).

b) The second group of restrictions refers to production costs. These restrictions show that the expenses for raw material, supply, etc., must not exceed certain limits, set by the plan.

So, with regard to raw material, if we mark by:

M — the sum-total for the permissible expenses related to raw material under the plan;

φ_j — the cost in lei of the raw material for a j-type container, the condition will be:

$$\sum_{j=1}^{20} \varphi_j x_j \leqslant M.$$

The next restrictions refers to direct materials. These include the containers, which differ as to capacity and material, and the caps and labels.

Let E_j be the expenses for direct materials per j-type container. These costs include the possible unavoidable losses resulted from the breaking up of the glass containers, the tearing out of labels, and wrong capping of bottles.

R — the permissible sum-total of expenses with direct materials.

We must have:

$$\sum_{j=1}^{20} E_j x_j \leqslant R.$$

Another restriction refers to the supply costs of the enterprise (especially for raw materials). In connection with this restriction, and taking into account the enterprise's experience, the conclusion has been reached that there is no need to estimate higher costs than the planned ones, as there are no cases of complete loss of the tomato crop to entail the supply of this raw material from other vegetable-growing areas of this country.

If we mark by α the supply expenses per container, and by *S.E.* the sum-total of the supply expenses allowed under plan, the restriction will read:

$$\sum_{j=1}^{20} \alpha_j x_j \leqslant S.E. \ (j = 1, ..., 20).$$

The wage expenses for the productive workers will be the object of other restrictions, namely: taking into account the fact that these expenses are directly proportional to the amount of finished products resulted, the specific costs per container may be appreciated as quite invariable.

Let:

β_j = expenses with direct wages per unit of container j, and

W.F. = the total fund of direct wages allowed under plan, we shall have the following restriction:

$$\sum_{j=1}^{20} \beta_j x_j < W.F.$$

The restrictions further analyzed refer to the expenses of the main fabrication sections, the expenses concerning amortization, the general expenses of the enterprises, the expenses required by the introduction of new techniques, and the sale expenses. For all these expenses, restrictions similar to those mentioned above (referring to costs) will be formulated, as the coefficients related to those costs per container are approximately invariable, and have the character of average. By marking by:

γ_j — the unitary costs required by the expenses of the chief manufacturing sections for a container type j, and *M.E.* the sum-total of these expenses under plan;

δ_j — the unitary costs required by the amortization of the equipment for a container type j, and by *E.E.* the sum-total of these expenses allowed by plan;

φ_j — the unitary costs required by the general expenses of the enterprises for a container type j, and by *G.E.* the sum-total allowed by the plan;

ρ_j — the unitary costs required by the expenses for the introduction of new technique for a container type j, and by *N.T.* the sum-total allowed by plan;

ξ_j — the unitary costs required by the sale expenses for a container type j, and by *S.E.* the sum-total allowed by plan.

The restrictions will synthesize in a general form in the following relations:

$$\sum_{j=1}^{20} \gamma_j x_j \leqslant M.E.$$

$$\sum_{j=1}^{20} \delta_j x_j \leqslant E.E.$$

$$\sum_{j=1}^{20} \varphi_j x_j \leqslant G.E.$$

$$\sum_{j=1}^{20} \rho_j x_j \leqslant N.T.$$

$$\sum_{j=1}^{20} \xi_j x_j \leqslant S.E.$$

The possibility exists that several of these restrictions be amalgamated in a single one for lessening the total number of equations of the system, in accordance with the capacity of the electronic computer by means of which these problem will be solved.

c) The group of restrictions concerning the packing-cases (containers) occurs in the problem only for those products that are packed in metal boxes, to the purpose of indicating the superior limit of the production of the assortments. This limiting is due to the fact that the enterprise is being supplied, under plan, with a certain quantity of such containers, which cannot for the time being be supplemented. A great many of these metal packings are imported, mainly from the countries to which the finished products will be delivered, while the others are of Romanian make.

In the framework of the surveyed problem, the restrictions refer to the following products:

— peeled tomatoes in 1/1 kg. metal boxes;

— tomato juice in 1/2 kg. metal boxes;

— tomato paste of 38—40 per cent concentration, in 1/10 kg. metal boxes;

— type B tomato paste in 5 kg. metal boxes.

Taking into account the fact that we marked by x_j the quantity of containers type j, that there are no several assortments packed in the same kind of boxes, and that we marked by C_j the total amount of containers the enterprise can dispose of, the restriction will be formulated, in a general (form, as follows:

$$x_j < C_j$$

(j indicating the products making use of the metal containers mentioned above).

d) The restrictions related to the raw material are determined at least by two causes, namely:

1. The necessary of taking into consideration the possibility of a variation of the raw material availabilities, a variation which is limited bilaterally.

2. The limiting of the possibilities of the raw material's damaging, in the sense of the use of a high-grade material, purchased at a higher price, for the fabrication of products requiring a less-pretentious raw material.

Bearing in mind the fact that the raw material used in the enterprise comes directly from agriculture and consequently is directly influenced by natural conditions of the respective agricultural year, it emerges that in shaping the optimal production plan we must consider possible modifications in the envisaged agricultural output.

Following the discussions held in enterprises on supply and the conclusion of contracts with purveyors, it has been noted that the maximal variations in the production under contract did not exceed the limits $\pm$ 10 per cent. This means — transposed to the daily output capacity of the factory — the prolongation or shortening of the production campaign by 6 days at most.

The formulation of these restrictions is necessary for in case of a good crop, the enterprise has no interest in halting production before comsuming the whole quantity of raw material deriving from the respective vegetable-growing area. The extra output might be sold if this additional raw material is destined to the fabrication of the same products, in demand on the market.

Contrarily, when owing to unfavourable conditions (natural calamities) part of the tomato crop is demaged, the enterprises will have to end the production campaign several days earlier. In this case, it will try to produce the minimal quantities allowed for various assortments, with special emphasis on the products destined to export.

Hence, this restriction will be formulated as follows:

$$\sum_{j=1}^{20} x_j c_j - \theta T \leqslant 0$$

$$\sum_{j=1}^{20} x_j c_j - \Phi T > 0$$

where x_j are the quantities of products in containers;
c_j — the specific consumption in quantitative term of container;
θ — the maximal percentage of excess of the quantity of raw material under contract;
Φ — the minimal percentage of reduction of the quantity of raw material under contract;
T — the quantity of raw material purchased under contract.

In order to define this restriction it has been necessary to express the whole finished output in the light of the quantity of raw material used, a transformation we have performed bearing in mind the specific consumption levels set under state standards. The transformations are in keeping with the homogeneity of the raw material which is processed into various products. For example, at peeled tomatoes, the raw material is of particular quality and, hence, the computation must be done separately; as tomato juice and paste require the same raw material, they formed another group, while fruit nectar implied another series of computation. We mention that under state standards, specific consumptions are calculated per kilogram; hence the

transformation of these specific consumptions per container was necessary. Exception to this rule are the products packed in barrels, where x_j represents the quantity in kilograms.

Besides these restrictions related to the quantity of raw material under contract, there is need for the formulation of a further restriction concerning the deterioration of the raw material. This is the case of the San-Marzanna tomatoes which are high-grade raw material and must not be used in the manufacture of juice or concentrated paste, for instance.

The restriction might be formulated only by resorting to the expression of value, namely:

$$x_j m_j \geqslant V$$

where x_j — is the quantity of tinned peeled tomatoes;
m_j — the fraction of the total value expression of the high-quality raw material used in the production of a unit of product
V — the value in lei calculated as the result between the quantity of containers and the unitary expenses, generated exclusively by the raw materials.

As is known, optimization may mean the maximization of a function (at profits) or its minimization, as it occurs in case of cost prices. In the light of the experience and conditions specific to the enterprise, the chosen optimization criterion was the cost price.

In this case, the expression of the objective function will be:

$$f(x) = \sum_{j=1}^{20} x_j p_j \rightarrow \text{minimum}$$

where x_j — is the quantity of finished products per container
p_j — the single cost price per container.

The full formulation of the problem with the function to be minimized and the whole system of restrictions — on the basis of the data in the plan of the enterprise in the year 1964 and of those calculated with their help — may be followed in the appendix.

The solving of the problem, considering its dimensions, can proceed only with the help of electronic computers. The solution found will indicate the optimum quantities from the point of view of minimizing cost prices the enterprise has to produce from the various assortments obtained on mechanized technological lines. The rest of output has to be achieved in the quantities demanded by the recipient.

As we already said, we intend in this survey to formulate a mathematical model and the possibility is not excluded that, in the course of solving the problem, some non-essential transformations may occur, some amalgamations (restrictions) or exclusion of restrictions, as well as reduction of the number of unknown quantities may be necessary.

A comparison between the results achieved following the solving of the problem and the real data of the year 1964, permit conclusions both with regard to cost price and profit, and the mathematical model will in the future be adaptable, with small alterations, to the optimum, planning of the assortment in other canneries with a similar profile.

To minimize:

$$I.\quad f(x) = 4.680\,x_1 + 3.212\,x_2 + 4.339\,x_3 + 0.775\,x_4 + 1.561\,x_5 + 0.483\,x_6 + 0.563\,x_7 + 4.359\,x_8 + 28.312\,x_9 + 3.213\,x_{10} + 1.048\,x_{11} + 6.381\,x_{12} + 0.974\,x_{13} + 1.018\,x_{14} + 0.800\,x_{15} + 1.683\,x_{16} + 1.506\,x_{17} + 17.929\,x_{18} + 1.111\,x_{19} + 1.170\,x_{20} \rightarrow \rightarrow \textit{minimum}$$

under conditions:

II.

1. $0.440\,x_2 + x_3 + x_8 + 5x_9 + 0.440\,x_{10} + 0.100\,x_{11} + x_{12} \leqslant 3{,}220{,}000$ kg.
2. $x_1 \leqslant 650{,}000$ kg.
3. $0.500\,x_4 + 0.500\,x_5 + 0.250\,x_6 + 0.333\,x \leqslant 2{,}166{,}000$ kg.
4. $0.333\,x_{13} + 0.333\,x_{14} + 0.333\,x_{15} + 0.333\,x_{16} + 0.500\,x_{17} \leqslant \leqslant 600{,}000$ kg
5. $x_{18} \leqslant 50{,}000$ kg.
6. $0.400\,x_2 + x_3 + x_8 + 5x_9 + 0.440\,x_{10} + 0.100\,x_{11} + x_{12} \geqslant \geqslant 3{,}145{,}000$ kg
7. $0.100\,x_{11} + x_{12} \geqslant 452{,}000$ kg.
8. $x_1 \geqslant 300{,}000$ kg.
9. $0.500\,x_4 + 0.500\,x_5 + 0.250\,x_6 + 0.333\,x_7 \geqslant 2{,}000{,}000$ kg.
10. $0.250\,x_4 + 0.500\,x_5 \geqslant 1{,}042{,}000$ kg.
11. $0.333\,x_{13} + 0.333\,x_{14} + 0.333\,x_{15} + 0.333\,x_{16} + 0.500\,x_{17} \geqslant 500{,}000$kg

12. $x_{18} \geqslant 30{,}000$ kg.

13. $1.726\ x_1 + 1.281\ x_2 + 2.405\ x_3 + 0.347\ x_4 + 0.268\ x_5 +$
$+ 0.166\ x_6 + 0.186\ x_7 + 3.009\ x_8 + 15.996\ x_9 + 1.486\ x_{10} +$
$+ 0.265\ x_{11} + 4.051\ x_{12} + 0.504\ x_{13} + 0.480\ x_{14} + 0.328\ x_{15} +$
$+ 0.898\ x_{16} + 0.727\ x_{17} + 12.810\ x_{18} + 0.675\ x_{19} + 0.586\ x_{20} \leqslant$
$\leqslant 12{,}828{,}886$ lei.

14. $0.230\ x_1 + 0.296\ x_2 + 0.379\ x_3 + 0.043\ x_4 + 0.023\ x_5 +$
$+ 0.021\ x_6 + 0.015\ x_7 + 0.383\ x_8 + 2.789\ x_9 + 0.242\ x_{10} +$
$+ 0.037\ x_{11} + 0.496\ x_{12} + 0.070\ x_{13} + 0.082\ x_{14} + 0.047\ x_{15} +$
$+ 0.124\ x_{16} + 0.124\ x_{17} + 1.750\ x_{18} + 0.111\ x_{19} + 0.249\ x_{20} \leqslant$
$\leqslant 1{,}840{,}397$ lei

15. $1.444\ x_1 + 0.758\ x_2 + 0.003\ x_3 + 0.055\ x_4 + 0.0889\ x_5 +$
$+ 0.047\ x_6 + 0.041\ x_7 + 0.008\ x_8 + 3.346\ x_9 + 0.702\ x_{10} +$
$+ 0.500\ x_{11} + 0.011 x_{12} + 0.064\ x_{13} + 0.096\ x_{14} + 0.050\ x_{15} +$
$+ 0.049\ x_{16} + 0.154\ x_{17} + 0.015\ x_{19} \leqslant 3{,}468{,}527$ lei.

16. $0.582\ x_1 + 0.093\ x_2 + 0.097\ x_3 + 0.076\ x_4 + 0.073\ x_5 +$
$+ 0.043\ x_6 + 0.070\ x_7 + 0.090\ x_8 + 0.596\ x_9 + 0.093\ x_{10} +$
$+ 0.034\ x_{11} + 1.442\ x_{12} + 0.096\ x_{13} + 0.097\ x_{14} + 0.098\ x_{15} +$
$+ 0.099\ x_{16} + 0.131\ x_{17} + 0.041\ x_{18} + 0.030\ x_{19} + 0.029\ x_{20} \leqslant$
$\leqslant 1{,}099{,}464$ lei

17. $0.596\ x_1 + 0.482\ x_2 + 0.827\ x_3 + 0.153\ x_4 + 0.192\ x_5 +$
$+ 0.128\ x_6 + 0.161\ x_7 + 0.495\ x_8 + 3.615\ x_9 + 0.388\ x_{10} +$
$+ 0.133\ x_{11} + 1.045\ x_{12} + 0.143\ x_{13} + 0.162\ x_{14} + 0.159\ x_{15} +$
$+ 0.266\ x_{16} + 0.227\ x_{17} + 1.912\ x_{18} + 0.163\ x_{19} + 0.082\ x_{20} \leqslant$
$\leqslant 3{,}628{,}551$ lei

18. $0.244\ x_1 + 0.145\ x_2 + 0.340\ x_3 + 0.064\ x_4 + 0.072\ x_5 +$
$+ 0.050\ x_6 + 0.058\ x_7 + 0.185\ x_8 + 1.204\ x_9 + 0.192\ x_{10} +$
$+ 0.052\ sx_{11} + 0.392\ x_{12} + 0.064\ x_{13} + 0.071\ x_{14} + 0.065\ x_{15} +$
$+ 0.133\ x_{16} + 0.100\ x_{17} + 0.782\ x_{18} + 0.050\ x_{19} + 0.053\ x_{20} \leqslant$
$\leqslant 1{,}422{,}127$ lei.

19. $0.150\ x_1 + 0.132\ x_2 + 0.270\ x_3 + 0.036\ x_4 + 0.042\ x_5 + + 0.027\ x_6 + 0.031\ x_7 + 0.130\ x_8 + 0.741\ x_9 + 0.077\ x_{10} + + 0.026\ x_{11} + 0.236\ x_{12} + 0.032\ x_{13} + 0.029\ x_{14} + 0.052\ x_{15} + + 0.112\ x_{16}\ x + 0.041\ x_{17} + 0.623\ x_{18} + 0.050\ x_{19} + 0.050\ x_{20} \leqslant \leqslant 906.436$ lei.

20. $0.005\ x_1 + 0.003\ (x_2 + x_3) + 0.001\ (x_4 + x_6 + x_7 + x_{11} + + x_{13} + x_{14} + x_{15}) + 0.002\ (x_5 + x_{16} + x_{17}) + 0.004\ (x_8 + + x_{10}) + 0.025\ x_9 + 0.008\ x_{12} + 0.011\ x_{18} \leqslant 27{,}388$ lei.

21. $0.003\ x_1 + 0.022\ x_2 + 0.015\ x_3 + 0.055\ x_8 + 0.028\ x_{10} + + 0.017\ x_{19} + 0.121\ x_{20} + \leqslant 121{,}750$ lei.

22. $x_1 \leqslant 323{,}000$ pieces

23. $x_5 \leqslant 739{,}200$ pieces

24. $x_9 \leqslant 52{,}400$ pieces

25. $x_{11} \leqslant 2{,}060{,}500$ pieces

26. $4.700\ (x_2 + x_3) + 0.625\ (x_4 + x_5) + 0.312\ x_6 + 0.416\ x_7 + + 6.800\ x_8 + 34.000\ x_9 + 3.133\ x_{10} + 0.859\ x_{11} + 8.950\ x_{12} — — 20{,}817{,}061 \geqslant 0.$

27. $4.700\ (x_2 + x_3) + 0.625\ (x_4 + x_5) + 0.312\ x_6 + 0.416\ x_7 + + 6.800\ x_8 + 34.000\ x_9 + 3.133\ x_{10} + 0.859\ x_{11} + 8.590\ x_{12} — — 25{,}441{,}975 \leqslant 0.$

28. $1.450\ x_1 — 515{,}249 \leqslant 0.$

29. $1.450\ x_1 — 421{,}669 \geqslant 0.$

30. $0.217\ x_{13} — 250{,}955 \leqslant 0.$

31. $0.217\ x_{13} — 195{,}327 \geqslant 0.$

32. $0.220\ x_{14} + 0.330\ x_{17} — 89{,}057 \leqslant 0.$

33. $0.220\ x_{14} + 0.330\ x_{17} — 72{,}865 \geqslant 0.$

34. $0.284\ x_{16} - 57{,}686 \leqslant 0.$

35. $0.284\ x_{16} - 47{,}198 \geqslant 0.$

36. $0.200\ x_{15} - 35{,}455 \leqslant 0.$

37. $0.200\ x_{15} - 29{,}009 \geqslant 0.$

38. $1.726\ x_1 \geqslant 557{,}458$ lei.

III. $x_1 \geqslant 0$

REFERENCES

1. JONES, W. G. and ROPE C. M., *Linear Programming Applied to Production Planning — — A Case Study*, "Operations Research Quarterly", Vol. 15, No. 4, Dec. 1964.
2. MIRESCU, P., MIHĂILĂ N. and others, *Determinarea planului optim de producție al unei fabrici de produse lactate.*
3. IONESCU, H., MIHĂILĂ, P., NĂDEJDE, I., *Programarea liniară*, Editura Științifică, București, 1964
4. MIHOC, GH., URSEANU, V., *Matematici aplicate în statistică*, Editura Academiei Republicii Socialiste România, București, 1962
5. *Calculul economic*, under the editorship of Professor M. Mănescu, Corresponding Member of the Academy of the Socialist Republic of Romania, București, 1964.

APPLICATION DE LA PROGRAMMATION MATHEMATIQUE A LA PLANIFICATION DE LA PRODUCTION

L'intention des auteurs est de construire un modèle mathématique, applicable à la planification optimale du processus de production dans le cadre d'une entreprise de conserves avec activité saisonière.

On a choisi pour critère d'optimisation la minimisation du prix de revient, tandis que les restrictions représentant la transposition mathématique des conditions économiques concrètes ont été groupées comme suit:

a) les restrictions concernant la capacité de production des lignes automatiques, en tenant compte des capacités, maximale et minimale;

b) les restrictions concernant les coûts, dans le cadre desquelles le total des dépenses doit être moindre qu'une valeur donnée;

c) les restrictions concernant les emballages de nature à constituer une limite supérieure pour le processus de certains produits;

d) les restrictions concernant la matière première en tenant compte:

— de la quantité de matière première contractée et des déviations possibles en plus ou en moins par rapport à la quantité contractée;

— de l'utilisation de la matière de qualité supérieure seulement pour la réalisation de produits supérieurs.

ПРИМЕНЕНИЕ МАТЕМАТИЧЕСКОГО ПРОГРАММИРОВАНИЯ ПРИ ПЛАНИРОВАНИИ ПРОИЗВОДСТВА

В статье авторы ставят перед собой цель сформулировать математическую модель, применимую при оптимальном планировании производственного процесса на консервном предприятии с сезонной деятельностью.

Критерием оптимизации является минимизация себестоимости, а ограничения, представляющие собой математическое выражение конкретных экономических условий, сгруппированы следующим образом:

а) ограничения в отношении производственной мощности автоматических линий с учетом наибольшей и наименьшей мощности;

б) ограничения в отношении себестоимости, при которых общие расходы должны быть меньше заданной стоимости;

в) ограничения в отношении тары, направленные на установление верхнего предела производства определенных изделий;

г) ограничения в отношении сырья с учетом:

— законтрактованного количества сырья и возможных отклонений в плюс или в минус по сравнению с законтрактованным количеством;

— использования высококачественного сырья только для получения высококачественных изделий.