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The Matrix of Growth

"... in a relatively simplified way, with the aid of the matrix of growth, it is possible to set up the relation between the two states of the quantities in the periods t and t — 1."

We will consider theoretical and practical aspects of the system of productions and the related differential equations [1]. We shall start with the system of difference equations based on the matrix of growth; and we will set up a model which can be used for programming of future developments. By simplifying the system, through direct and indirect rate of growth, it is possible to include and to follow the movement of basic categories of economic development.

The theoretical considerations of the indirect relations do not emphasize especially the assumption about the existence of dependence between the observed quantities, for the fact that the direct and the indirect rates of growth can exist in the two states whether between the observed quantities exists a corresponding dependence or that the form of that dependence according to its nature is the same for all the quantities. Here we shall assume, as in the other similar cases, that between the quantities we are observing exists a certain degree of dependence, although, as we have early stated, this assumption will not have a priori measuring when determining the elements of the matrix of growth. In the other cases, when setting up these systems and their use for extrapolative aims the assumption about the interdependence may be of importance, depending on the concrete empirical research.

The relations which express the direct and the indirect changes can serve as basis for setting up specific systems which link the quantities under observation into two different periods. The specifity of this dependence is expressed by the facts that it is limited to the period being observed which covers the elements of stationary state in the ex post conditions. In other

words, such systems are based on elements which result from real relations and they actually express the observed quantities in the periods t and $t - 1$. In that way there exist relatively more possibilities regarding the coverage of the different categories and the area of the corresponding systems.

The needs for the study of the economic systems which express the connection of the stationary states, especially in conditions when there is no possibility of linking the quantities by the classical form of the mathematical interdependence, are relatively more. In the first place they appear when observing the economic categories in the various spheres, such as the movement of production according to the branches and activities, movement of prices, circulations of finance, imports and exports, or in the more complicated cases, where at one moment we observe a collection of more important categories which characterise the state of the economy in the given period. In such cases it is possible to come to the direct and the indirect relations which in the form of the matrix of growth show the basic parameters of the development of economy and they make it more possible to see the movement which exists in the ex post conditions of the economy as a whole. In the case of enterprises, where it is necessary to observe certain quantities, such as production, income, personal income, accumulation, capital, number of the workers, etc., the relations can be observed through the matrix of growth and on their basis can be followed the corresponding movements, with the possibility of programming certain relations and states in the following periods. As well as that, by setting up the theoretical bases, there exists the chance of linking the various quantities and categories for which it can not explicitly be established the existence of dependence in the classical sense, that is the dependence between the quantities in the mathematical or statistical form, with the deterministic or stohastic aspects of the expression. Apart from these forms of interdependence, in economics, especially in the economy, there are noticeable more complicated forms of links which in the form of the systems express the direct and the indirect relations. With these considerations there is a general tendency of consistence in the movement of the quantities being observed which can be expressed by the direct and the indirect rates of growth in the ex post conditions. The matrix of growth in such conditions can in a satisfactory way express the consistent dependence between the two states of the quantities.

Let us now consider the mathematical-methodological aspects of the system of the corresponding relation which helps to set up a model based on the direct and the indirect rates of growth.

Let us assume that there are two known stationary states of quantities Y_i, $i = 1, 2, \ldots n$, in the periods of $t - 1$ and t, i.e. :

$$Y_{1,\,t-1},\ Y_{2,\,t-1},\ \ldots,\ Y_{n,\,t-1},\ Y_{1,\,t},\ Y_{2,\,t},\ \ldots,\ Y_{n,\,t}, \tag{1}$$

and that they are mutually linked relations:

$$Y_{i,\,t} = Y_{i,\,t-1} + \Delta Y_i \quad \text{i.e.} \quad Y_{i,\,t} - \Delta Y_i = Y_{i,\,t-1}, \quad i = 1, 2, \ldots, n. \tag{2}$$

By the stationary states of quantities $Y_{i,\,t}(i = 1, 2, \ldots n)$ is understood to be the set realised values of the quantities observed in the period t.

Let us assume, also, that the increment ΔY_i, is, generally, a function of the variables $Y_{i,t}$, i.e.

$$\Delta Y_i = F_i(Y_{1,t}, Y_{2,t}, ..., Y_{n,t}), \ (i = 1, 2, ..., n) \tag{3}$$

where F_i means the function of the corresponding variables. Through the system (3), which expresses the link between the increments and the corresponding variables, we determine the system (2) by which are linked the two stationary states in the ex post conditions, realized in the periods $t-1$ and t, i.e.

$$Y_{i,t} - F_i(Y_{1,t}, Y_{2,t}, ..., Y_{n,t}) = Y_{i,t-1}, \ (i = 1, 2, ..., n), \tag{4}$$

or in the developed form:

$$\begin{aligned}
Y_{1,t} - F_1(Y_{1,t}, Y_{2,t}, ..., Y_{n,t}) &= Y_{1,t-1} \\
Y_{2,t} - F_2(Y_{1,t}, Y_{2,t}, ..., Y_{n,t}) &= Y_{2,t-1} \\
... \qquad\qquad ... \qquad\qquad &... \\
... \qquad\qquad ... \qquad\qquad &... \\
Y_{n,t} - F_n(Y_{1,t}, Y_{2,t}, ..., Y_{n,t}) &= Y_{n,t-1}.
\end{aligned} \tag{5}$$

In the study we shall limit ourselves to the consideration the linear systems [2] i.e. we shall consider the system (3) in the form

$$\Delta Y_i = \sum_{j=1}^{n} a_{ij} Y_{jt}. \tag{6}$$

After substitution in (4) we get:

$$Y_{it} - \sum_{j=1}^{n} a_{ij} Y_{jt} = Y_{i,t-1}, \tag{7}$$

or in the developed form

$$\begin{aligned}
(1 - a_{11})\, Y_{1t} - a_{12} Y_{2t} - ... - a_{1n} Y_{nt} &= Y_{1,t-1} \\
- a_{21} Y_{1t} + (1 - a_{22})\, Y_{2t} - ... - a_{2n} Y_{nt} &= Y_{2,t-1} \\
... \qquad\qquad ... \qquad\qquad &... \\
... \qquad\qquad ... \qquad\qquad &... \\
- a_{n1} Y_{1t} - a_{n2} Y_{2t} + ... + (1 - a_{nn}) Y_{nt} &= Y_{n,t-1},
\end{aligned} \tag{8}$$

that is:

$$\begin{Bmatrix} 1 - a_{11} & a_{12} & ... & - a_{1n} \\ - a_{21} & 1 - a_{22} & ... & - a_{2n} \\ ... & ... & & ... \\ ... & ... & & ... \\ - a_{n1} & - a_{n2} & ... & 1 - a_{nn} \end{Bmatrix} \begin{Bmatrix} Y_{1t} \\ Y_{2t} \\ ... \\ ... \\ Y_{nt} \end{Bmatrix} = \begin{Bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ ... \\ ... \\ Y_{n,t-1} \end{Bmatrix}. \tag{9}$$

If with A we denote the matrix of the system

$$A = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix} \tag{10}$$

but with $Y_t = \{Y_{1t}, Y_{2t}, \dots, Y_{nt}\}$ and $Y_{t-1} = \{Y_{1,t-1}, Y_{2,t-1}, \dots, Y_{n,t-1}\}$ vectors of quantities which express the stationary states in the periods $t-1$ and t, then the equation (9) can be written:

$$(I - A)Y_t = Y_{t-1}. \tag{11}$$

The solution of the system (11) gives:

$$Y_t = (I - A)^{-1}Y_{t-1}, \tag{12}$$

where $(I - A)^{-1}$ is the inverse matrix of the system. The solution (12) expresses the stationary states of the quantities Y_i in the period t by the matrix of the system and the stationary state in the considered quantities in the period $t-1$.

The linearization and the determination of the coefficients of the systems deserve special attention because in this particular, but important, case can be introduced the assumption which lead to linking the stationary states in the ex post conditions, realised in the periods t and $t-1$. With this aim we start from the following consideration [3].

If as before with $Y_{i,t-1}$ and $Y_{i,t}(i = 1, 2, \dots n)$ we denote the quantity Y_i in the periods $t-1$ and t, respectively, the indirect or the crossed rate of growth we define as the relation between the increment $\Delta Y_i = Y_{i,t} - Y_{i,t-1}$ in the periods $(t-1, t)$ and the absolute value Y_j in the t period, i.e.:

$$r_{ij} = \frac{\Delta Y_i}{Y_{jt}} = \frac{Y_{it} - Y_{i,t-1}}{Y_{jt}}, \qquad (i, j = 1, 2, \dots, n), \tag{13}$$

in the developed form the direct and the indirect rates of growth can be shown in the form of the following matrix of growth:

$$R = \begin{Vmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \dots & \dots & & \dots \\ \dots & \dots & & \dots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{Vmatrix}, \tag{14}$$

where the elements in the main diagonals represent the direct but the rest of the elements represent the indirect or the crossed rate of growth. According to the relation (13) the rate of growth r_{ij} is defined in relation to the

value Y_j in the t — moment. However, it is possible to find the crossed rate of growth in relation to the value Y_j from the period $t-1$, so that it can be expressed in the form:

$$r'_{ij} = \frac{\Delta Y_i}{Y_{j,\,t-1}}, \qquad (i, j = 1, 2, \ldots, n). \tag{15}$$

That is in the form of the matrix

$$R' = \begin{Vmatrix} r'_{11} & r'_{12} & \ldots & r'_{1n} \\ r'_{21} & r'_{22} & \ldots & r'_{2n} \\ \ldots & \ldots & & \ldots \\ \ldots & \ldots & & \ldots \\ r'_{n1} & r'_{n2} & \ldots & r'_{nn} \end{Vmatrix}. \tag{16}$$

The link between the rate of growth in (13) and (15) and also in (14) and (16) exists in the form:

$$r_{ij} = \frac{r'_{ij}}{1 + r'_{jj}}, \qquad \text{or} \qquad r'_{ij} = \frac{r_{ij}}{1 - r_{jj}}, \tag{17}$$

so that through the given relations it can relatively easily be crossed from the one to the other form. For instance, it is possible to write the matrix of growth (14) in the form:

$$R = \begin{Vmatrix} \dfrac{r'_{11}}{1 + r'_{11}} & \ldots & \dfrac{r'_{1n}}{1 + r'_{nn}} \\ \ldots & & \ldots \\ \ldots & & \ldots \\ \dfrac{r'_{n1}}{1 + r'_{11}} & & \dfrac{r'_{nn}}{1 + r'_{nn}} \end{Vmatrix}, \tag{18}$$

and the matrix of growth (16) in the form:

$$R' = \begin{Vmatrix} \dfrac{r_{11}}{1 - r_{11}} & \ldots & \dfrac{r_{1n}}{1 - r_{nn}} \\ \ldots & & \ldots \\ \ldots & & \ldots \\ \dfrac{r_{n1}}{1 - r_{11}} & & \dfrac{r_{nn}}{1 - r_{nn}} \end{Vmatrix}. \tag{19}$$

As we can see the matrix of growth expresses the direct and the indirect rates of growth between the observed quantities. The matrix embraces all the possible indirect relations and links the stationary positions of the observed quantities in two different periods. The matrix of growth, as it is defined, does not imply the assumption about the existence of dependence between the observed quantities, as such the matrix exists in the relation which exist in the ex post conditions, that is in those relations which are realised in the periods t and $t - 1$. The matrix of growth is singular because the vectors of the columns and rows are proportional.

From the economics point of view the matrix can be significant in the cases where there is need to link the quantities (categories) in the dynamic sense and by following them up in the definite periods, especially in those cases when explicitly the relations can not be put into the functional expression. When following up the dynamics of the economic categories, very frequently the defined relations can not mutually be linked and expressed in the form of the system because of the circumstance that among the various economic categories there is not usually that form of dependence which mathematically can be expressed by the corresponding relations. Most frequently the forms of the economic dependence are expressed by the relations which are the results of the action of a very large number of factors which in the direct and in the indirect forms express the general conditions, but without the possibility of the precise mathematical formulations through the corresponding functions. For this reason, when considering the stationary states, that is their links from period to period, we wish to put into one final form, in the ex post conditions, the corresponding effects and that on their basis we should estimate their actual relations. In that case the matrix in the general expression of the corresponding relations which help link the various quantities whether or not among the observed quantities exist dependence.

Let us now come back to the examination of the system, especially the linear system (7), in which the matrix of elements expresses the relationship between the observed states in the periods t and $t - 1$. We shall observe one special case of the system (7), with the matrix of the coefficient in which the elements of the systems are expressed by the direct and by the indirect rates of growth. Hence, if in the place of the relationship (6) we take into consideration the form:

$$\Delta Y_i = \frac{1}{n} \sum_{j=1}^{n} r_{ij} Y_{jt}, \qquad (i = 1, 2, ..., n), \tag{20}$$

where are r_{ij} the indirect rates of growth, and n the number of variables, then system (7) becomes:

$$Y_{it} - \frac{1}{n} \sum_{j=1}^{n} r_{ij} Y_{jt} = Y_{i,t-1}, \qquad (i = 1, 2, ..., n). \tag{21}$$

In the developed form it can be written *as*:

$$
\begin{aligned}
Y_{1t} - \frac{1}{n}(r_{11}Y_{1t} + r_{12}Y_{2t} + \ldots + r_{1n}Y_{nt}) &= Y_{1,t-1} \\
Y_{2t} - \frac{1}{n}(r_{21}Y_{1t} + r_{22}Y_{2t} + \ldots + r_{2n}Y_{nt}) &= Y_{2,t-1} \qquad (22) \\
\ldots \qquad\qquad \ldots \qquad\qquad \ldots & \\
\ldots \qquad\qquad \ldots \qquad\qquad \ldots & \\
Y_{nt} - \frac{1}{n}(r_{n1}Y_{1t} + r_{n2}Y_{2t} + \ldots + r_{nn}Y_{nt}) &= Y_{n,t-1}
\end{aligned}
$$

or

$$
\left[\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & & \ldots & \\ \ldots & & \ldots & \\ 0 & 0 & \ldots & 1 \end{pmatrix} - \frac{1}{n}\begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \ldots & \ldots & & \\ \ldots & \ldots & & \\ r_{n1} & r_{n2} & \ldots & r_{nn} \end{pmatrix}\right] \begin{pmatrix} Y_{1t} \\ Y_{2t} \\ \ldots \\ \ldots \\ Y_{nt} \end{pmatrix} = \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ \ldots \\ \ldots \\ Y_{n,t-1} \end{pmatrix}, \qquad (23)
$$

that is:

$$
\left(I - \frac{1}{n}R\right)Y_t = Y_{t-1}, \qquad \text{or} \qquad (N - R)Y_t = nY_{t-1}, \qquad (24)
$$

where R is the matrix of growth, defined by the equation (14) whose elements are the direct and the indirect rates of growth.

The solution of the system (24) gives:

$$
Y_t = \left(I - \frac{1}{n}R\right)^{-1} Y_{t-1}, \qquad \text{or} \qquad Y_t = n(N - R)^{-1}Y_{t-1} \qquad (25)
$$

the link between the two stationary states in the periods t and $t - 1$, through the matrix of growth R.

The relation (20) which is the basis of the assumption of the system (21) comes from the simultaneous incirclement of all the changes between the corresponding stationary states. In the case of the partial consideration different assumptions are possible, for instance, if we start from the assumption that all the quantities depend on one fixed k — th quantities, i.e. if the changes of the individual quantities are a function of $Y_{k,t}$, then the system can be expressed in the form:

$$
\Delta Y_i = F_i(Y_{k,t}), \qquad (i = 1, 2, \ldots, n) \qquad (26)
$$

in that case the relation (20) can be expressed as:

$$
\Delta Y_i = r_{ik}Y_{kt}, \qquad (27)
$$

whence after substituion in (7) we have:

$$Y_{it} - r_{ik}Y_{k,t} = Y_{i,t-1}, \qquad (i = 1, 2, ..., n), \tag{28}$$

or in the developed form:

$$\begin{aligned} Y_{1t} - r_{1k}Y_{1t} &= Y_{1,t-1} \\ Y_{2t} - r_{2k}Y_{2t} &= Y_{2,t-1} \\ \cdots \quad &\cdots \quad \cdots \\ \cdots \quad &\cdots \quad \cdots \\ Y_{nt} - r_{nk}Y_{nt} &= Y_{n,t-1} \end{aligned} \tag{29}$$

that is:

$$\begin{Bmatrix} 1 & 0 & \ldots & r_{1k} & \ldots & 0 \\ 0 & 1 & \ldots & r_{2k} & \ldots & 0 \\ \cdots & & & \cdots & & \\ \cdots & & & \cdots & & \\ 0 & 0 & \ldots & r_{nk} & \ldots & 1 \end{Bmatrix} \begin{Bmatrix} Y_{1t} \\ Y_{2t} \\ \cdots \\ \cdots \\ Y_{nt} \end{Bmatrix} = \begin{Bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ \cdots \\ \cdots \\ Y_{n,t-1} \end{Bmatrix}. \tag{30}$$

this not difficult to see that the relation (27), after suming up to K, gives:

$$\Delta Y = \frac{1}{n} \sum_{k=1}^{n} r_{ik}Y_{kt},$$

which as (20) expresses the simultaneous effect of all the quantities. Of course, the general connection does not have to be complete in the sense of the absolute incirclement of the quantifies, but it can be limited to two or more factors.

At last, if two stationary states expressed by the quantities $Y_{i,t}$ and $Y_{i,t-1}$ and realized in the periods t and $t - 1$ were observed we could ask the question about their link with the states from the preceding periods. The linking of the stationary states in the period t does not always have to be completed with the corresponding states from the period $t - 1$, but by the matrix of growth it could be extended to the periods which cover more stationary states in the past. This kind of expression can be useful because the large number of periods which it covers demand the matrices of growth which suit the covered periods. That helps to come to the corresponding basis for comparative analysis of the direct and the indirect rates according to the periods. This approach can be useful also for purposes of planning when by formulate new matrices of growth which by the assumption be realised in the future periods, it is possible to estimate the new states of the observed quantities.

To show this we start from the matrix of growth R_t which joins the stationary states Y_t and Y_{t-1} in the periods t and $t - 1$. Then the relation (24) becomes:

$$\left(I - \frac{1}{n} R_t\right) Y_t = Y_{t-1}, \tag{31}$$

$$Y_t = \left(I - \frac{1}{n} R_t\right)^{-1} Y_{t-1}. \tag{32}$$

For $t = 1, 2, \ldots m$, we have:

$$\begin{aligned}
\left(I - \frac{1}{n} R_1\right) Y_1 &= Y_0 \\
\left(I - \frac{1}{n} R_2\right) Y_2 &= Y_1 \\
\ldots \quad &\quad \ldots \\
\ldots \quad &\quad \ldots \\
\left(I - \frac{1}{n} R_m\right) Y_m &= Y_{m-1}
\end{aligned} \tag{33}$$

that is:

$$\begin{aligned}
Y_1 &= \left(I - \frac{1}{n} R_1\right)^{-1} Y_0 \\
Y_2 &= \left(I - \frac{1}{n} R_2\right)^{-1} Y_1 \\
\ldots \quad &\quad \ldots \\
\ldots \quad &\quad \ldots \\
Y_m &= \left(I - \frac{1}{n} R_m\right)^{-1} Y_{m-1}.
\end{aligned} \tag{34}$$

Taken into successives series it can be written:

$$\left(I - \frac{1}{n} R_1\right)\left(I - \frac{1}{n} R_2\right) \ldots \left(I - \frac{1}{n} R_m\right) Y_m = Y_0, \tag{35}$$

whence we come to the solution in the form:

$$Y_m = \left\{\left(I - \frac{1}{n} R_1\right)\left(I - \frac{1}{n} R_2\right) \ldots \left(I - \frac{1}{n} R_m\right)\right\}^{-1} Y_0. \tag{36}$$

The relation we have got shows that the stationary states in the m-th period can be expressed by the corresponding matrices of growth in the m preceeding periods and the initial stationary state Y_0.

In the same way from the system (34) and (36), we can determine the solution through the successive series of the matrices of growth in the form:

$$Y_m = \left(I - \frac{1}{n} R_m\right)^{-1} \left(I - \frac{1}{n} R_{m-1}\right)^{-1} \ldots \left(I - \frac{1}{n} R_1\right)^{-1} Y_0. \tag{37}$$

If the matrices of growth are equal in the successive periods, i.e., $R_t = \bar{R}$, for $t = 1, 2, \ldots, m$, where $\bar{R}$ denotes the average matrix of growth, in the period $(0, m)$, the solution (35) can be written in the form:

$$\left(I - \frac{1}{n}\bar{R}\right)^m Y_m = Y_0, \tag{38}$$

with the solution

$$Y_m = \left\{\left(I - \frac{1}{n}\bar{R}\right)^m\right\}^{-1} Y_0. \tag{39}$$

When considering the stationary system we start from the assumption that between the observed quantities exists a definite degree of interdependence which approximately can be expressed in the mathematical formula in the same way as does express the indirect rate of growth. If this mathematical interpretation of interdependence is inadequate for bigger intervals, i.e. if the deviation of the real values is larger than that which can be accepted, the system can have meaning only for the stated stationary state from which we start when formulating the system. In other words, the system will always be satisfied for the initial states in the periods t and $t - 1$. In the same way the values of the corresponding indirect rates of growth expressed by the matrix of growth will completely represent the actual relations for the stated stationary states. As we have earlier on stated, in the cases when between the observed quantities does not exist a corresponding interdependence this kind of system can be formed and it has corresponding periods. In other words, when in the ex post conditions we formulate the matrix of growth and at the same time the system which links the corresponding quantities of two successive periods then that is the real picture of the relations. In these cases the systems completely satisfy the initial assumptions regarding the values of the quantities in the stationary states, whether it is about the interdependence of the quantities or not. However, when we make some changes in the system, the changes of the rate of growth or of the other elements, the question is asked about the reality of the expression of such systems and the possibilities of using the values in the sense of extrapolation, that is, the estimation of new stationary states. As long as among the observed quantities there is a large degree of interdependence the estimation will be better, that is there exist real chances that the estimation will be nearer the expected changes and vice-versa if the quantities are to a larger degree independent the deviation will be greater and less reliable will be the estimation. However, in both cases, by approximation it can be improved if the assumption about the changes of the individual elements of the system are small, or relatively small so that the system stays nearly within the limits of the existing relations. We expect the system in the se cases to point out to the changes of the individual quantities when some of the elements change.

Let us now consider the example of production according to the economic actitivies of the Yugoslav economy from the year 1966 to 1971. The matrices of growth R_1, R_2, R_3, R_4, R_5 express the relations in the successive

years from the year 1966 to 1971, respectively. The matrix of growth R_1 gives the direct and the indirect rates of growth for the state of production from the year 1966 and 1967, the matrix R_2 from the state in the year 1967/68, etc. By comparing the corresponding elements of the matrices of growth according to the years we come to the conclusion that the individual relations show certain tendencies which can be significant for the movement of the economy. As we can see the matrix of growth helps to explicitly examine the various relations in the direct and in the indirect forms. For instance, the indirect rate of growth of communication against industry in the period 1966/67, $r_{51} = 0.0165$ (the matrix of growth R_1) show the relative increase of communication against the industry, and the rate of growth $r_{15} = 0.2940$ is the relative increase of the industry against communication. Although this information does not speak much for themselves about the tempo of the individual activities as against others, they are important when they are compared against themselves according to the years in the matrices of growth R_2, R_3, ... R_n etc., namely, from the comparative rates of growth of communication against industry according to years we got the relative mild increases viz.:

Year	1966/1967	1967/1968	1968/1969	1969/1970	1970/1971
r_{51}	0.0165	0.0157	0.0223	0.0303	0.0355.

However, by observing the indirect rate of growth of industry against communication we see that these relations are considerably more intensive, i.e.:

Year	1966/1967	1967/1968	1968/1969	1969/1970	1970/1971
r_{15}	0.2940	0.5060	0.8989	1.0669	1.5677

In these results we consider summarily we see that from the period 1966/1971 the indirect rate of growth of communication against industry was increased by 2.15 times while the indirect rate of growth of industry against communication was increased by 5.33 times. From this, we can see that the tempo of growth of industry against communication was about 2.48 times more than the corresponding tempo of communication against industry.

Similary we can analyse the relationship between industry and trade. We also start from the period 1966/1971 with the rate of growth of trade against industry 1.8 and the rate of growth of industry against trade in the same period 4.79. The relationship of these two rates shows that the rate of growth of industry against trade was 2.63 times more than the rate of growth of trade against industry.

In the other similar cases the matrix of growth helps to analyse the indirect rate of growth of the observed activities. Indicatively the matrix explicitly expressed the indirect relations and offers the possibility for the complex analytical understanding.

Martrices of growth of production

	Industry	Agricul-ture	Forestry	Con-struction	Communi-cation	Trade	Crafts	Kom-munal
				1966/1967				
$R_1 =$	0.0474	0.0910	2.1330	0.2285	0.2940	0.1723	0.3737	0.3724
	0.0024	0.0047	0.1096	0.0117	0.0151	0.0089	0.0192	0.1734
	0.0010	0.0018	0.0432	0.0046	0.0060	0.0035	0.0076	0.0682
	0.0371	0.0711	1.6660	0.1785	0.2296	0.1346	0.2919	2.6340
	0.0165	0.0316	0.7412	0.0794	0.1021	0.0599	0.1299	1.1718
	0.0405	0.0776	1.8198	0.1949	0.2508	0.1470	0.3189	2.8772
	0.0169	0.0324	0.7604	0.0815	0.1048	0.0614	0.1332	1.2023
	0.0020	0.0039	4.0919	0.0098	0.0127	0.0074	0.0161	0.1453
				1967/1968				
$R_2 =$	0.0828	0.1800	3.8760	0.3763	0.5061	0.2906	0.6567	5.4385
	0.0179	0.0389	0.8387	0.0814	0.1095	0.0629	0.1421	1.1768
	0.0010	0.0021	0.0455	0.0044	0.0060	0.0034	0.0077	0.0639
	0.0297	0.0645	1.3895	0.1349	0.1814	0.1042	0.2354	1.9497
	0.0157	0.0340	0.7330	0.0712	0.0957	0.0550	0.1242	1.0286
	0.0325	0.0707	1.5223	0.1478	0.1988	0.1141	0.2579	2.1360
	0.0097	0.0211	0.9549	0.0442	0.0594	0.0341	0.0771	0.6383
	0.0023	0.0051	0.1090	0.0106	0.0142	0.0082	0.0185	0.1530
				1968/1969				
$R_3 =$	0.1458	0.3104	6.6417	0.6384	0.8989	0.4925	1.1578	9.8025
	0.0765	0.1628	3.4845	0.3349	0.4716	0.2584	0.6074	5.1428
	0.0037	0.0079	0.1681	0.0162	0.0228	0.0125	0.0293	0.2481
	0.0405	0.0817	1.8343	0.1763	0.2483	0.1360	0.3198	2.7073
	0.0223	0.0475	1.0173	0.0978	0.1377	0.0754	0.1774	1.5015
	0.0524	0.1116	2.3890	0.2296	0.3233	0.1772	0.4145	3.5259
	0.0181	0.0386	0.8263	0.0794	0.1118	0.0613	0.1440	0.2195
	0.0019	0.0040	0.0846	0.0081	0.0115	0.0063	0.0148	0.1249
				1969/1970				
$R_4 =$	0.1751	0.4031	7.2659	0.7302	1.0693	0.5779	1.4011	11.8949
	0.0469	0.1080	1.9464	0.1956	0.2858	0.1548	0.3753	3.1864
	0.0060	0.0138	0.2486	0.0250	0.0365	0.0198	0.0479	0.4070
	0.0514	0.1184	2.1339	0.2144	0.3133	0.1697	0.4115	3.4933
	0.0303	0.0698	1.2586	0.1265	0.1848	0.1001	0.2427	2.0605
	0.0588	0.1354	2.4400	0.2452	0.3583	0.1941	0.4705	3.9946
	0.0211	0.0486	0.8752	0.0880	0.1235	0 0696	0.1688	1.4328
	0.0025	0.0056	0.1018	0.0102	0.0149	0.0081	0.0196	0.1666

Matrices of growth of production

	Industry	Agriculture	Forestry	Construction	Communication	Trade	Crafts	Kommunal
				1970/1971				
$R_5 =$	0.2489	0.5976	11.9610	1.0675	1.5677	0.8258	2.1908	18.2126
	0.0902	0.2166	4.3348	0.3860	0.5682	0.2993	0.7939	6.6005
	0.0027	0.0065	0.1301	0.0116	0.0171	0.0090	0.0238	0.1982
	0.0530	0.1273	2.5487	0.2275	0.3341	0.1760	0.4668	3.8809
	0.0355	0.0852	1.7057	0.1522	0.2236	0.1178	0.3124	2.5973
	0.0738	0.1773	3.5487	0.3167	0.4651	0.2450	0.6500	5.4035
	0.0197	0.0474	0.9486	0.0847	0.1243	0.0655	0.1738	1.4444
	0.0026	0.0063	0.1254	0.0112	0.0164	0.0087	0.0230	0.1909
				1966/1971				
$R_6 =$	0.5376	1.2909	25.8367	2.3059	3.3864	1.7837	4.7323	39.3406
	0.1621	0.3893	7.7924	0.6955	1.0213	0.5379	1.4273	11.8652
	0.0105	0.0252	0.5034	0.0449	0.0660	0.0348	0.0922	0.7666
	0.1503	0.3609	7.2239	0.6447	0.9468	0.4987	1.3231	10.9996
	0.0884	0.2123	4.2487	0.3792	0.5569	0.2933	0.7782	6.4693
	0.1874	0.4499	0 0047	0.8037	1.1802	0.6217	1.6493	13.7112
	0.0602	0.1445	0.8925	0.2582	0.3791	0.1997	0.5298	4.9043
	0.0078	0.0188	0.3762	0.0336	0.0493	0.0260	0.0689	0.5728

The average (constant) matrix of growth in the period 1966/1971.

$\bar{R} =$	0.1397	0.3012	6.3283	0.6288	0.8701	0.4839	1.1504	9.8342
	0.0421	0.0908	1.9086	0.1897	0.2624	0.1460	0.3470	2.9660
	0 0027	0.0059	0.1233	0.0123	0.0170	0.0094	0.0224	0.1916
	0.0391	0.0842	1.7694	0.1758	0.2433	0.1353	0.3217	2.7496
	0.0230	0.0495	1.0407	0.1034	0.1431	0.0796	0.1892	1.6172
	0.0487	0.1050	2.2056	0.2192	0.3033	0.1687	0.4010	3.4275
	0.0156	0.0337	0.7085	0.0704	0.0974	0.0542	0.1288	1.1010
	0.0020	0.0044	0 0921	0.0092	0.0127	0.0070	0.0168	0.1432

The formation of the system on basis of given matrices of growth is accomplished according to the relation (24):

$$\left(I - \frac{1}{n} R_t\right) Y_t = Y_{t-1},$$

with the solution

$$Y_t = \left(I - \frac{1}{n} R_t\right)^{-1} Y_{t-1},$$

that is:

$$Y_t = n(N - R_t)^{-1} Y_{t-1},$$

where n is the series of the matrix R_t, and N the scalar matrix with the elements equal to n and the main diagonal, but the rest being equal to zero.

In the concrete case for the given vector of production Y_{66}, which expresses the value of production according to activities in the year 1966, i.e.

$$Y_{66} = \{84373,\ 46379,\ 1885,\ 15110,\ 12834,\ 20802,\ 9745,\ 1065\},$$

and the matrix of growth R_1 it is possible to give the relation

$$\left(I - \frac{1}{n} R_1\right) Y_{67} = Y_{66}$$

and to determine the production in the year 1967 in the form:

$$Y_{67} = n(N - R_1)^{-1} Y_{66}$$

where $(N - R)^{-1}$ is the inverse matrix. Hence, the vector of production for the year 1967 is:

$$Y_{67} = \{88575,\ 46181,\ 1970,\ 18392,\ 14294,\ 24387,\ 11243,\ 1246\}.$$

We can also establish the link between the vectors of production in the other periods according to the relations:

$$Y_{68} = n(N - R_2)^{-1} Y_{67}$$
$$Y_{69} = n(N - R_3)^{-1} Y_{68}$$
$$Y_{70} = n(N - R_4)^{-1} Y_{69}$$
$$Y_{71} = n(N - R_5)^{-1} Y_{70}.$$

In this way there are uniform relations between the various states of production according to the years through the corresponding matrix of growth. On the basis of given relations there is also the relation between the stationary states of production in the period 1966 and 1971 in the form:

$$\left(I - \frac{1}{n} R_1\right)\left(I - \frac{1}{n} R_2\right)\left(I - \frac{1}{n} R_3\right) \ldots \left(I - \frac{1}{n} R_5\right) Y_{71} = Y_{66}.$$

that is:

$$(N - R_1)(N - R_2)(N - R_3)(N - R_4)(N - R_5) Y_{71} = n^5 Y_{66},$$

with the solution:

$$Y_{71} = n^5 (N - R_5)^{-1} (N - R_4)^{-1} (N - R_3)^{-1} (N - R_2)^{-1} (N - R_1)^{-1} Y_{66}.$$

If we put:

$$B = (N - R_1)(N - R_2)(N - R_3)(N - R_4)(N - R_5),$$

we have:

$$B Y_{71} = n^5 Y_{66}.$$

with the solution:

$$Y_{71} = n^5 B^{-1} Y_{66}.$$

On the other side, independent of the above results, it is possible directly to link the relationship between the states of production between the year 1966 and 1971 through the matrix of growth R_6 in which the elements express the relative increase in the course of the whole period and cumulatively cover the rates of growth of the corresponding period. Such we can write as:

$$\left(I - \frac{1}{n} R_6\right) Y_{71} = Y_{66}$$

that is:

$$Y_{71} = n(N - R_6)^{-1} Y_{66}$$

where $(N - R_6)^{-1}$ is the inverse matrix of the system.

At last it is possible to estimate the values of production according to activities for the following years 1972. That we do in two ways: in the first place directly by copying the existing relations through the matrix of growth for the year 1770/71. In the second place by estimating from the preceding results the direct and the indirect relations r_{ij} in the matrix of growth which is expected in the following period.

In the first case we have:

$$\left(I - \frac{1}{n} R_5\right) Y_{72} = Y_{71}$$

$$Y_{72} = n(N - R_5)^{-1} Y_{71}.$$

Numerically that can be expressed thus:

$$\frac{1}{8} \begin{pmatrix} 7.7511 & -0.5976 & -11.9610 & -1.0675 & -1.5677 & -0.8258 & -2.1908 & -18.2126 \\ -0.0902 & 7.7834 & -1.3348 & -0.3868 & -0.5682 & -0.2993 & -0.7940 & -6.6005 \\ -0.0027 & -0.0065 & 7.8699 & -0.0116 & -0.0171 & -0.0090 & -0.0238 & -0.1982 \\ -0.0530 & -0.1273 & -2.5487 & 7.7725 & -0.3341 & -0.1760 & -0.4668 & -3.8809 \\ -0.0355 & -0.0852 & -1.7057 & -0.1522 & 7.7764 & -0.1178 & -0.3124 & -2.5793 \\ -0\,0738 & -0.1773 & -3.5487 & -0.3167 & -0.4651 & 7.7500 & -0.6500 & -5.4035 \\ -0.0197 & -0.0474 & -0.9486 & -0.0847 & -0.1243 & -0.0655 & 7.8263 & -1.4444 \\ -0.0026 & -0.0063 & -0.1254 & -0.0112 & -0.0164 & -0.0087 & -0.0230 & 7.8091 \end{pmatrix} \begin{pmatrix} Y_{1.72} \\ Y_{2.72} \\ Y_{3.72} \\ Y_{4.72} \\ Y_{5.72} \\ Y_{6.72} \\ Y_{7.72} \\ Y_{8.72} \end{pmatrix} = \begin{pmatrix} 182\,449 \\ 75\,977 \\ 3\,796 \\ 42\,532 \\ 28\,962 \\ 54\,984 \\ 20\,725 \\ 2\,493 \end{pmatrix}.$$

The solution is in the from:

$$\begin{pmatrix} Y_{1.72} \\ Y_{2.72} \\ Y_{3.72} \\ Y_{4.72} \\ Y_{5.72} \\ Y_{6.72} \\ Y_{7.72} \\ Y_{8.72} \end{pmatrix} = \begin{pmatrix} 1.0392 & 0.0942 & 1.8855 & 0.1683 & 0.2471 & 0.1302 & 0,3454 & 2.8710 \\ 0.0142 & 1.0341 & 0.6833 & 0.0610 & 0,0896 & 0\,0472 & 0.1252 & 1.0405 \\ 0.0004 & 0,0010 & 1.0205 & 0.0018 & 0.0027 & 0.0014 & 0.0038 & 0.0312 \\ 0.0084 & 0.0201 & 0.4018 & 1.0359 & 0.0527 & 0.0277 & 0.0736 & 0.6118 \\ 0.0056 & 0.0134 & 0.2689 & 0.0240 & 1.0352 & 0.0186 & 0.0492 & 0.4094 \\ 0.0116 & 0.0279 & 0.5594 & 0.0499 & 0.0733 & 1.3086 & 0.1024 & 0.8518 \\ 0.0031 & 0.0075 & 0.1495 & 0.0133 & 0.0196 & 0.0103 & 1.0274 & 0,2277 \\ 0.0004 & 0.0010 & 0.0198 & 0.0018 & 0.0026 & 0.0014 & 0.0036 & 1.0301 \end{pmatrix} \begin{pmatrix} 182\,449 \\ 75\,977 \\ 3\,796 \\ 42\,532 \\ 28\,962 \\ 54\,984 \\ 20\,725 \\ 2\,493 \end{pmatrix},$$

where after multiplying we estimate the vector of production for the year 1972.

$$Y_{72} = \{239708, 96728, 4419, 54733, 37128, 71972, 25266, 3093\}.$$

In the second case, in the same way, we have:

$$\left(I - \frac{1}{n} R_c\right) Y_{72} = Y_{71},$$

the solution

$$Y_{72} = n(N - R_c)^{-1} Y_{71},$$

where R_c is the estimated matrix of growth for the period 1971/72.

Thus, after substitution, we have:

$$\frac{1}{8}\begin{Bmatrix} 7.700 & -0.700 & -15.000 & -1.200 & -1.700 & -0.900 & -3.000 & -22.000 \\ -0.100 & 7.750 & -5.000 & -0.500 & -0.700 & -0.350 & -1.000 & -9.000 \\ -0.005 & -0.010 & 7.800 & -0.020 & -0.200 & -0.020 & -0.050 & -0.300 \\ -0.050 & -0.140 & -3.000 & 7.750 & -0.350 & -0.180 & -0.500 & -4.000 \\ -0.040 & -0.090 & -2.000 & -0.180 & 7.750 & -0.130 & -0.350 & -3.000 \\ -0.080 & -0.200 & -4.000 & -0.400 & -0.600 & 7.700 & -0.700 & -6.000 \\ -0.020 & -0.050 & -1.000 & -0.080 & -0.120 & -0.070 & 7.820 & -1.500 \\ -0.003 & -0.007 & -0.150 & -0.010 & -0.018 & -0.009 & -0.025 & 7.700 \end{Bmatrix} \begin{Bmatrix} Y_{1.72} \\ Y_{2.72} \\ Y_{3.72} \\ Y_{4.72} \\ Y_{5.72} \\ Y_{6.72} \\ Y_{7.72} \\ Y_{8.72} \end{Bmatrix} = \begin{Bmatrix} 182\,449 \\ 75\,977 \\ 3\,796 \\ 42\,532 \\ 28\,962 \\ 54\,984 \\ 20\,725 \\ 2\,493 \end{Bmatrix},$$

with the solution:

$$\begin{Bmatrix} Y_{1.72} \\ Y_{2.72} \\ Y_{3.72} \\ Y_{4.72} \\ Y_{5.72} \\ Y_{6.72} \\ Y_{7.72} \\ Y_{8.72} \end{Bmatrix} = \begin{Bmatrix} 1.049 & 0.116 & 2.468 & 0.202 & 0.341 & 0.153 & 0.484 & 3.634 \\ 0.017 & 1.042 & 0.845 & 0.081 & 0.134 & 0.059 & 0.165 & 1.453 \\ 0.001 & 0.002 & 1.042 & 0.004 & 0.028 & 0.004 & 0.008 & 0.062 \\ 0.008 & 0.023 & 0.488 & 1.041 & 0.069 & 0.030 & 0.083 & 0.669 \\ 0.007 & 0.015 & 0.331 & 0.030 & 1.049 & 0.029 & 0.059 & 0.495 \\ 0.013 & 0.033 & 0.674 & 0.065 & 0.112 & 1.049 & 0.119 & 1.009 \\ 0.003 & 0.008 & 0.166 & 0.014 & 0.023 & 0.012 & 1.030 & 0.248 \\ 0.001 & 0.001 & 0.024 & 0.002 & 0.004 & 0.002 & 0.004 & 1.033 \end{Bmatrix} \begin{Bmatrix} 182\,449 \\ 75\,977 \\ 3\,796 \\ 42\,532 \\ 28\,962 \\ 54\,984 \\ 20\,725 \\ 2\,493 \end{Bmatrix},$$

which gives the estimated vector of production for the year 1972, with the matrix of growth R_c, i.e.

$$Y_{72} = \{255634, 103083, 5782, 56488, 38893, 76253, 25724, 3191\}.$$

Thus the estimated production according to the branches of activities in the two different cases — in the first case when the matrix of growth is taken from the preceeding period 1970/71, and in the second case, when the matrix is estimated through the matrix R_c — is:

Estimated production in the year 1972 in mill. dinars with the matrix of growth

Branch of activity	From 1970/71.	Acc. to estimate R_c
1. Industry	239 708	255 634
2. Agriculture	96 728	103 083
3. Forestry	4 419	5 782
4. Construction	54 733	56 488
5. Communication	37 128	38 893
6. Trade	71 972	76 253
7. Crafts	25 266	25 724
8. Communal	3 093	3 191

In the end we can conclude that by introducing new mathematical methodological relations for expressing the dependence of the economic quantities, it enables to follow certain structural movements in the dynamic sence. The matrix of growth which expresses the direct and the indirect relations enables us to set up the dynamic systems which cover the relations in the various periods. Particularly it is necessary to emphasize the flexibility of the methodological approach which ensures that the corresponding approach can be applied to conditions when there are not sufficient classical forms of mathematical interdependence. In that sense it is not only important that through the matrix of growth and the system based on it can be linked and followed the relations between the observed quantities but that on the basis of the general tendencies can be estimated for the next period. That is why the matrix of growth offers more possibilities for linking the systems and discovering the important dynamic relations for following and programming the economic and business relations.

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LA MATRICE DE CROISSANCE

On y fait l'analyse des problèmes concernant les aspects méthodologiques du taux direct et indirect de croissance de certaines grandeurs données. L'étude traite de la forme générale du taux de croissance, c'est-à-dire de la forme renfermant aussi bien les modifications directes, que les modifications indirectes, exprimant de la sorte pleinement les rapports réciproques des

grandeurs étudiées, pour certaines périodes données. L'expression explicite de ces rapports conduit à des systèmes exprimant des modifications spécifiques. La matrice de croissance, qui met en corrélation les grandeurs de deux périodes différentes, sert à exprimer et à examiner les soi-disant états stationnaires des grandeurs soumises à la recherche. De cette manière, relativement sans difficulté, il devient possible, au moyen de la matrice de croissance, d'établir les relations qui existent entre les deux états des grandeurs aux périodes t et $t - 1$.

МАТРИЦА РОСТА

В настоящей статье рассматриваются вопросы в связи с методологическими аспектами прямой и косвенной нормы роста некоторых данных величин. В статье рассматривается общая форма нормы роста, то-есть форма, которая охватывает как непосредственные, так и косвенные изменения и, таким образом, полностью выражает взаимоотношения исследуемых величин за определенные данные периоды. Развернутое выражение этих отношений ведет к определенным системам, выражающим специфические изменения. Матрица роста, увязывающая величины двух различных периодов, служит для выражения и рассмотрения так-называемых стационарных состояний исследуемых величин. Таким образом, сравнительно просто при помощи матрицы роста становится возможным установление существующих соотношений между двумя состояниями величин в периодах t и $t - 1$.

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