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Optimization Algorithm and Qualitative Properties of One-Sector Economic Growth Models with Discrete Variables

In the paper there is discussed a general one-sector economic growth model with discrete variables and finite time horizon. In order to infer the optimality conditions, the Maximum Principle for discrete systems is used. Some qualitative properties of the optimal trajectories are obtained, as well as an algorithm for calculating these trajectories.

In the last part of the paper, the case of autonomous model is considered.

1. INTRODUCTION

In this paper we deal with the following general discrete model of economic growth:

$$\text{I}\begin{cases} K_{n+1} = g_n(K_n) + v_n h_n(K_n),\ n = 0, 1, \ldots, N-1 \\ 0 < k_0 \text{ given},\ k_N \geqslant \omega \\ max\ J(v) = max \sum_{n=0}^{N-1} U_n[\lambda_n(1 - v_n)h_n(K_n)] \\ \lambda_n \text{ given},\ \alpha \leqslant v_n \leqslant \beta,\ n = 0, 1, \ldots, N-1 \\ 0 \leqslant \alpha < \beta \leqslant 1 \text{ fixed}. \end{cases}$$

In the above model K_n stands for the state variable and v_n for the control parameter.

We suppose that the functions $g_n(\cdot)$, $h_n(\cdot)$, $U_n(\cdot)$ belong to class C^2 and that they are increasing and concave.

We suppose also that

$$g_n(0) = h_n(0) = U_n(0) = 0.$$

We note with ω the minimal level imposed for the final value of the state variable.

In order to show the economic relevance of such a model, we suppose that the national income or the social product, Y_n, in period n, is given by the value of a function $F_n(K_n, L_n)$, where L_n stands for the labour force at moment n in the productive sphere, and K_n for the productive capital at the same moment.

We suppose that the functions:

$$F_n : R_+^2 \to R_+, \quad n = 0, 1, ..., N-1$$

satisfy the usual properties of the neoclassical production functions. Taking into account the impact of technical progress, we suppose that:

$$F_{n-1}(L, K) \leqslant F_n(L, K), \quad n = 0, 1, ..., N-1.$$

Examples

1) $F_n(L_n, K_n) = F(a_n L_n, K_n)$ (Harrod)

2) $F_n(L_n, K_n) = F(L_n, b_n K_n)$ (Solow)

3) $F_n(L_n, K_n) = F(a_n L, b_n K_n)$.

It is to be mentioned that in the general frame we work in, we may suppose that the form of the production function, as well as the type of technical progress may change from one period to another.

As concerns the labour force, we suppose that its growth is exogenous to the model we are dealing with. The law of labour force growth may be, for instance, of the following type:

$$L_{n+1} = l_n L_n, \quad L_0 \text{ given}$$

and $l_0, l_1, ..., l_n, ...,$ given exogenously ($l_0 = 1$).

Hence

$$L_n = l_0 l_1 ... l_{n-1} L_0 = \prod_{i=0}^{n-1} l_i L_0.$$

We shall note

$$h_n(K_n) = F_n\left(\prod_{i=0}^{n-1} l_i L_0, K_n\right).$$

According to the manner of its repartition, the national income can be written as $Y_n = C_n + Z_n$, where C_n stands for the amount of total resources allotted to consumption at moment n, and Z_n for the amount of gross investment allotted to the productive sphere at moment n.

The equation describing the evolution of the capital is:

$$K_{n+1} = g_n(K_n) + Z_n.$$

For a fixed K_n, $g_n(K_n)$ is that part of capital K_n, which remains at the end of period n, i.e. after subtracting from the value of K_n the value of its physical and moral wear and tear.

The function $g_n : R_+ \to R_+$ may be, for instance, of the form:

$$g_n(K) = (1 - \mu_n)K.$$

We suppose that $g_n(\cdot)$ is a concave function which satisfies also the constraint $g_n(K) \leqslant K$.

The control parameter will be $v_n = \dfrac{Z_n}{Y_n}$, i.e. the *rate of productive accumulation.*

We shall suppose that

$$v_n \in [\alpha, \beta],\ 0 \leqslant \alpha < \beta \leqslant 1.$$

Therefore, the evolution of capital will be described by the equation:

$$K_{n+1} = g_n(K_n) + v_n h_n(K_n);\ K_0 \text{ given.}$$

The functions:

$$U_n : R_+ \to R_+$$

are utility functions; they give a quantitative measure of the utility of consumption at each of the n stages.

The expression $(1 - v_n)h_n(K_n)$ represents the consumption C_n in period n.

The most usually met utility functions in economic growth theory are:

a) $U_n(C_n) = U(C_n)$

or

$$U_n(C_n) = \delta^n U(C_n)$$

where δ is a discount rate;

b) $U_n(\lambda_n C_n) = \delta^n U\left(\dfrac{1}{L_n} C_n\right)$

c) $U_n(\lambda_n C_n) = \delta^n P_n U\left(\dfrac{1}{P_n} C_n\right)$

where L_n and P_n are the labour force in the productive sphere and the total population, respectively.

The degree of generality of model I is such that it allows for the form of the utility function to change from one period to another.

In order to infer the optimality conditions we use Pontryagin's Maximum principle for discrete systems.

The Hamiltonian function of the system is:

$$H_n(K_n, v_n, \Psi_n) =$$
$$= \Psi_n[g_n(K_n) + v_n h_n(K_n)] + \mu_0 U_n[\lambda_n(1 - v_n)h_n(K_n)]$$

where the adjoint variable Ψ_n is the solution of the dual equation:

$$\Psi_{n-1} = \Psi_n[g_n'(K_n) + v_n h_n'(K_n)] +$$
$$+ \mu_0 U_n'[\lambda_n(1 - v_n)h_n(K_n)] \cdot \lambda_n(1 - v_n)h_n'(K_n)$$

with $\tilde{\Psi}_{N-1} = -\mu_1$ and $\Psi_n = 0$, for $n \geqslant N$ where μ_0 and μ_1 are two multipliers.

Further, we note with $\tilde{v}_n$ the optimal control and with $\tilde{K}_n$ and $\tilde{\Psi}_n$ the corresponding values of the state variable and the dual variable, respectively.

The Maximum principle provides the following necessary optimality conditions:

$$\frac{\partial H_n(\tilde{K}_n, \tilde{v}_n, \tilde{\Psi}_n)}{\partial v}(v - \tilde{v}_n) \leqslant 0$$

for any admissible control $v \in [\alpha, \beta]$ and $n = 0,1, \ldots, N-1$.

Moreover, it provides the following information about the multipliers μ_0 and μ_1;

$$\mu_0 \geqslant 0, \quad -\mu_1 \geqslant 0, \quad \mu_0 - \mu_1 > 0.$$

The transversality condition is:

$$\tilde{\Psi}_{N-1}[\tilde{K}_N - \omega] = 0.$$

Proposition 1. Let $\check{K}_n$ be the solution of the equations

$$\check{K}_{n+1} = g_n(\check{K}_n) + \beta h_n(\check{K}_n).$$
$$\check{K}_0 = K_0$$

If $\omega < \check{K}_N$, then $\mu_0 > 0$.

Proof

Let us suppose that $\mu_0 = 0$. Then $\mu_1 \neq 0$ and hence

$$\tilde{\Psi}_{N-1} = -\mu_1 > 0.$$

The adjoint equation becomes:

$$\tilde{\Psi}_{n-1} = \tilde{\Psi}_n[g_n'(\tilde{K}_n) + \tilde{v}_n h_n'(\tilde{K}_n)].$$

It follows obviously that all $\tilde{\Psi}_n$ have the same sign and, since

$$\tilde{\Psi}_{N-1} = -\mu_1 > 0,$$

it results that $\tilde{\Psi}_n > 0$ for all n.

When $\mu_0 = 0$ the first optimality condition becomes:

$$\tilde{\Psi}_n h_n(\tilde{K}_n)(v - \tilde{v}_n) \leqslant 0, \; n = 0, 1, \ldots, N-1,; \; K_0 > 0$$

and hence $\tilde{K}_n > 0$, whence $h_n(\tilde{K}_n) > 0$. Therefore:

$$\tilde{v}_n = \beta, \; n = 0, 1,., N-1.$$

But this means

$$\tilde{K}_N = \check{K}_N > \omega$$

and the transversality condition:

$$\tilde{\Psi}_{N-1}[\tilde{K}_N - \omega] = 0$$

is no longer satisfied. Q.E.D.

Based on Proposition 1, we may consider, without loss of generality, that $\mu_0 = 1$. Then, the optimality conditions become:

$$h_n(\tilde{K}_n)[\tilde{\Psi}_n - \lambda_n U_n'(\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n))](v - \tilde{v}_n) \leqslant 0.$$

$$h_n(\tilde{K}_n) > 0,$$

Therefore:

a) if $\tilde{v}_n \in (\alpha, \beta)$, then $\tilde{\Psi}_n = \lambda_n U_n'[\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)]$

b) if $\tilde{v}_n = \alpha$, then $\tilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1 - \alpha)h_n(\tilde{K}_n)]$

c) if $\tilde{v}_n = \beta$ then $\tilde{\Psi}_n \geqslant \lambda_n U_n'[\lambda_n(1 - \beta)h_n(\tilde{K}_n)]$.

The evolution equations for $\tilde{\Psi}_n$ are:

a) if $\tilde{v}_n \in (\alpha, \beta)$ then

$$\tilde{\Psi}_{n-1} = \varphi_n'(\tilde{K}_n)\tilde{\Psi}_n$$

where

$$\varphi_n(K) = g_n(K) + h_n(K)$$

b) if $\tilde{v}_n \notin (\alpha, \beta)$, then

$$\tilde{\Psi}_{n-1} = \tilde{\Psi}_n[g_n'(\tilde{K}_n) + \tilde{v}_n h_n'(\tilde{K}_n)] + \lambda_n U_n'[\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)](1 - \tilde{v}_n)h_n'(\tilde{K}_n)$$

where $\tilde{v}_n = \alpha$ or $\tilde{v}_n = \beta$, respectively. In the case $\tilde{v}_n \in (\alpha, \beta)$ from the optimality condition:

$$\tilde{\Psi}_n = \lambda_n U_n'[\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)]$$

it results:

$$\tilde{v}_n h_n(\tilde{K}_n) = h_n(\tilde{K}_n) - \frac{1}{\lambda_n} W_n\left(\frac{\tilde{\Psi}_n}{\lambda_n}\right)$$

where $W_n(\cdot)$ is the function inverse to $U_n'(\cdot)$ The above relation can be rewritten as follows:

$$\tilde{v}_n = 1 - \frac{W_n(\tilde{\Psi}_n \lambda_n^{-1})}{\lambda_n h_n(\tilde{K}_n)}.$$

Therefore, when $\tilde{v}_n \in (\alpha, \beta)$, the Hamiltonian system takes the form

$$\begin{cases} \tilde{K}_{n+1} = \varphi_n(\tilde{K}_n) - \dfrac{W_n(\tilde{\Psi}_n \lambda_n^{-1})}{\lambda_n} \cdot \\ \tilde{\Psi}_{n-1} = \varphi_n'(\tilde{K}_n)\tilde{\Psi}_n. \end{cases}$$

Proposition 2

The necessary optimality conditions provided by the Maximum principle are also sufficient optimality conditions.
If, for some n, $U_n''(\cdot) < 0$ or $\varphi_n''(\cdot) < 0$, where $\varphi_n(K) = g_n(K) + h_n(K)$, then the optimal trajectory is unique.

For the proof see [2].

Remark 1.

Choosing a new control

$$u_n = \frac{v_n - \alpha}{\beta - \alpha},$$

the range of admissible controls becomes the interval [0, 1].

From $u_n = \dfrac{v_n - \alpha}{\beta - \alpha}$ it follows $v_n = (\beta - \alpha)u_n + \alpha$. In this case, the dynamic equation becomes:

$$K_{n+1} = \bar{g}_n(K_n) + u_n \bar{h}_n(K_n)$$

where

$$\bar{g}_n(K_n) = g_n(K_n) + \alpha h_n(K_n)$$

$$\bar{h}_n(K_n) = (\beta - \alpha)h_n(K_n).$$

The objective — function becomes

$$\sum_{n=0}^{N-1} U_n[\lambda_n(\gamma - u_n)\bar{h}_n(K_n)]$$

where

$$\gamma = \frac{1 - \alpha}{\beta - \alpha}$$

Hence, model I is equivalent to the following model:

$$\text{I}' \quad \begin{cases} K_{n+1} = \bar{g}_n(K_n) + u_n \bar{h}_n(K_n), \ n = 0, 1, \ldots, N-1 \\ 0 < K_0, \text{ given}, \ K_N \geqslant \omega \\ max \ \bar{j}(u) = max \sum_{n=0}^{N-1} U_n[\lambda_n(\gamma - u_n)\bar{h}_n(K_n)] \\ \lambda_n \text{ given}, \gamma = \dfrac{1 - \alpha}{\beta - \alpha}; \ 0 \leqslant u_n \leqslant 1, \ n = 0, 1, \ldots, N-1. \end{cases}$$

3. QUALITATIVE PROPERTIES OF THE OPTIMAL TRAJECTORIES OF THE GENERAL MODEL

In this section we shall establish some qualitative properties of the general model described in section 1. Further, these properties will be used in order to infer some peculiar qualitative properties of the autonomous model.

Proposition 3 gives the maximal attainable level of fixed assets (capital) on the optimal trajectory.

In proposition 4 there is described the behaviour of the optimal trajectories of capital $\tilde{K}_n$ and of the adjoint variable $\tilde{\Psi}_n$ with respect to their position against the "golden-rule points" K_n^* (the solution of equations $\varphi_n'(K) - 1 = 0$) and with respect to their position against the simple reproduction curves, whose equations are $\Psi_n^0(K) = \lambda_n U_n'[\lambda_n(\varphi_n(K) - K)]$ (they correspond to the case $K_{n+1} = K_n$).

Propositions 5 and 6 give sufficient conditions in order to obtain monotonously increasing, respectively monotonously decreasing trajectories of per capita consumption $\tilde{c}_n = \lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)$.

Proposition 3

Let us consider

$$\overline{\varphi}_n(K) = g_n(K) + \beta h_n(K)$$

If

$$\lim_{K \downarrow 0} \overline{\varphi}_n'(K) > 1 \text{ and } \lim_{K \to \infty} \overline{\varphi}_n'(K) < 1, \quad n = 0,1, \ldots, N-1$$

then

$$\tilde{K}_n \leqslant max \ (\bar{K}, \ K_0)$$

for each $n = 0,1, \ldots, N$; where

$$\bar{K} = \max_{n \in \{0,1,\ldots,N-1\}} \bar{K}_n.$$

and $\bar{K}_n$ is the non-zero solution of equation:

$$\overline{\varphi}_n(K) - K = 0$$

Proof

Let $\hat{K}_n$ be the solution of equation $\overline{\varphi}_n'(K) - 1 = 0$ from the hypotheses, this solution exists and is unique and we have $\hat{K}_n < \bar{K}_n$.

The function $\overline{\varphi}_n(K) - K$ is increasing for $K \leqslant \hat{K}_n$ and decreasing for $K \geqslant \hat{K}_n$. Let us suppose $\widetilde{K}_n \leqslant \overline{K}_n$. If $\widetilde{v}_n \in (\alpha, \beta)$, then

$$\widetilde{K}_{n+1} = \varphi_n(\widetilde{K}_n) - \frac{1}{\lambda_n} W_n(\widetilde{\Psi}_n \lambda_n^{-1}) \leqslant \overline{\varphi}_n(\widetilde{K}_n) \leqslant \overline{\varphi}_n(\overline{K}_n) = \overline{K}_n \leqslant \overline{K};$$

if $\widetilde{v}_n = \beta$, then

$$\widetilde{K}_{n+1} = \overline{\varphi}_n(\widetilde{K}_n) \leqslant \overline{\varphi}_n(\overline{K}_n) = \overline{K}_n \leqslant \overline{K};$$

if $\widetilde{v}_n = \alpha$, then

$$\widetilde{K}_{n+1} = g_n(\widetilde{K}_n) + \alpha h_n(\widetilde{K}_n) \leqslant \overline{\varphi}_n(\widetilde{K}_n) \leqslant \overline{\varphi}_n(\overline{K}_n) \leqslant \overline{K}.$$

Suppose there exists an n such that $\widetilde{K}_n > \overline{K}_n$. If $\widetilde{v}_n \in (\alpha, \beta)$, then

$$\widetilde{K}_{n+1} = \varphi_n(\widetilde{K}_n) - \frac{1}{\lambda_n} W_n(\widetilde{\Psi}_n \lambda_n^{-1}) \leqslant \overline{\varphi}_n(\widetilde{K}_n) \leqslant \widetilde{K}_n.$$

If $\widetilde{v}_n = \beta$, then

$$\widetilde{K}_{n+1} = \overline{\varphi}_n(\widetilde{K}_n) \leqslant \widetilde{K}_n.$$

If $\widetilde{v}_n = \alpha$, then

$$\widetilde{K}_{n+1} = g_n(\widetilde{K}_n) + \alpha h_n(\widetilde{K}_n) \leqslant \overline{\varphi}(\widetilde{K}_n) \leqslant \widetilde{K}_n.$$

Hence, if $K_0 > \overline{K}$, then $\widetilde{K}_n < K_0$ for every n.

Proposition 4

Let K_n^* be the solution of equation $\varphi_n'(K) - 1 = 0$;

$\varphi_n(K_n, \varrho) = g_n(K_n) + \varrho h_n(K_n)$ and $\Psi_n^0(K) = \lambda_n U_n'[\lambda_n(\varphi_n(K) - K)]$.

Then the following implications hold:

a) If $\widetilde{K}_n \neq K_n^*$ and $\lambda_n U_n'(\lambda_n(1-\beta)h_n(\widetilde{K}_n)) > \widetilde{\Psi}_n > \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)]$ then $(\widetilde{K}_n - K_n^*)(\widetilde{\Psi}_n - \widetilde{\Psi}_{n-1}) > 0$; if $\widetilde{K}_n \leqslant K_n^*$ and

$$\widetilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)], \text{ then } \widetilde{\Psi}_{n-1} \geqslant \widetilde{\Psi}_n$$

If $\widetilde{K}_n \geqslant K_n^*$ and $\widetilde{\Psi}_n \geqslant \lambda_n U_n'(\lambda_n(1-\beta)h_n(\widetilde{K}_n))$ then $\widetilde{\Psi}_{n-1} \leqslant \widetilde{\Psi}_n$

b) if $\widetilde{K}_n = K_n^*$ and $\lambda_n U_n'(\lambda_n(1-\beta)h_n(\widetilde{K})) > \widetilde{\Psi}_n > \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)]$,

$$\text{then } \widetilde{\Psi}_n = \widetilde{\Psi}_{n-1};$$

c) if $\widetilde{K}_n = K_n^*$ and $\widetilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)]$, then $\widetilde{\Psi}_n \leqslant \widetilde{\Psi}_{n-1}$

and if $\widetilde{\Psi} \geqslant \lambda_n U_n'(\lambda_n(1-\beta)h_n(\widetilde{K}_n))$ then $\widetilde{\Psi}_n \geqslant \widetilde{\Psi}_{n-1}$

d) if $\widetilde{\Psi}_n \neq \Psi_n^0(\widetilde{K}_n)$, $U_n''(\cdot) < 0$, and $\varphi_n(\widetilde{K}_n, \alpha) \leqslant \widetilde{K}_n \leqslant \varphi_n(\widetilde{K}_n, \beta)$

$$\text{then } [\widetilde{\Psi}_n - \Psi_n^0(\widetilde{K}_n)][\widetilde{K}_{n+1} - \widetilde{K}_n] > 0;$$

e) if $\widetilde{\Psi}_n = \Psi_n^0(\widetilde{K}_n)$ and $\varphi_n(\widetilde{K}_n, \alpha) < \widetilde{K}_n < \varphi_n(\widetilde{K}_n, \beta)$, then $\widetilde{K}_{n+1} = \widetilde{K}_n$.

Proof

a) If $\widetilde{K}_n \neq K_n^*$ then, $\varphi_n'(\cdot)$ being a decreasing function, it results $(\widetilde{K}_n - K_n^*)[1 - \varphi_n'(\widetilde{K}_n)] > 0$.

From the hypotheses it follows $\widetilde{v}_n \in (\alpha, \beta)$, $\widetilde{\Psi}_{n-1} = \varphi_n'(\widetilde{K}_n)\widetilde{\Psi}_n$ which implies

$$(\widetilde{K}_n - K_n^*)(\widetilde{\Psi}_n - \widetilde{\Psi}_{n-1}) > 0.$$

If $\widetilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)]$, then $\widetilde{\Psi}_{n-1} \geqslant \varphi_n'(\widetilde{K}_n)\widetilde{\Psi}_n$; therefore, $K_n \leqslant K_n^*$ implies $\widetilde{\Psi}_{n-1} \geqslant \widetilde{\Psi}_n$. The other case can be analogously proved.

b) Fram the hypotheses it follows $\widetilde{v}_n \in (\alpha, \beta)$ and

$$\varphi_n'(\widetilde{K}_n) = 1 \text{ which implies } \widetilde{\Psi}_{n-1} = \varphi_n'(\widetilde{K}_n)\widetilde{\Psi}_n = \widetilde{\Psi}_n.$$

c) If $\widetilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)]$

then $\widetilde{\Psi}_{n-1} \geqslant \widetilde{\Psi}_n \varphi_n'(\widetilde{K}_n) = \widetilde{\Psi}_n$ if $\widetilde{\Psi}_n \geqslant \lambda_n U_n'(\lambda_n(1-\beta)h_n(\widetilde{K}_n))$ then

$$\widetilde{\Psi}_{n-1} \leqslant \widetilde{\Psi}_n \varphi_n'(\widetilde{K}_n) \leqslant \widetilde{\Psi}_n.$$

d) If $\widetilde{v}_n \in (\alpha, \beta)$ then the optimum condition is:

$$\widetilde{\Psi}_n = \lambda_n U_n'[\lambda_n(1-\widetilde{v}_n)h_n(\widetilde{K}_n)].$$

We have

$$\widetilde{\Psi}_n - \Psi_n^0(\widetilde{K}_n) = \lambda_n U_n'[\lambda_n(1-\widetilde{v}_n)h_n(\widetilde{K}_n)] -$$

$$- \lambda_n U_n'[\lambda_n(\varphi_n(\widetilde{K}_n) - \widetilde{K}_n)] = \lambda_n^2 U''(\xi_n)[(1-\widetilde{v}_n)h_n(\widetilde{K}_n) - \varphi_n(\widetilde{K}_n) + \widetilde{K}_n] =$$

$$= \lambda_n^2 U''(\xi_n)[\widetilde{K}_n - \widetilde{K}_{n+1}]$$

Since $U_n''(\cdot) < 0$, it results:

$$[\widetilde{\Psi}_n - \Psi_n^0(\widetilde{K}_n)][\widetilde{K}_{n+1} - \widetilde{K}_n] = -\lambda_n^2 U_n''(\xi_n)[\widetilde{K}_{n+1} - \widetilde{K}_n]^2 > 0.$$

If $\widetilde{v}_n = \beta$, then $\widetilde{K}_{n+1} = \overline{\varphi}_n(\widetilde{K}_n) \geqslant \widetilde{K}_n$ and

$$\widetilde{\Psi}_n \geqslant \lambda_n U_n'[(1-\beta)h_n(\widetilde{K}_n)] \geqslant \lambda_n U_n'[\lambda_n(\varphi_n(\widetilde{K}_n) - \widetilde{K}_n)] = \Psi_n^0(\widetilde{K}_n);$$

If $\widetilde{v}_n = \alpha$, then $\widetilde{K}_{n+1} = g_n(\widetilde{K}_n) + \alpha h_n(\widetilde{K}_n) < \widetilde{K}_n$ and

$$\widetilde{\Psi}_n \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\widetilde{K}_n)] \leqslant \lambda_n U_n'[\lambda_n(\varphi_n(\widetilde{K}_n) - \widetilde{K}_n)] = \Psi_n^0(\widetilde{K}_n).$$

e) Let $\widetilde{\Psi}_n = \Psi_n^0(\widetilde{K}_n)$, i.e.

$$\widetilde{\Psi}_n = \lambda_n U_n'[\lambda_n(\varphi_n(\widetilde{K}_n) - \widetilde{K}_n)].$$

It results

$$\frac{1}{\lambda_n} W_n\left(\frac{\tilde{\Psi}_n}{\lambda_n}\right) = \varphi_n(\tilde{K}_n) - \tilde{K}_n.$$

On the other hand, we have $\tilde{v}_n \in (\alpha,\beta)$ which imples

$$\tilde{K}_{n+1} = \varphi_n(\tilde{K}_n) - \frac{1}{\lambda_n} W_n\left(\frac{\tilde{\Psi}_n}{\lambda_n}\right), \text{ therefore } \tilde{K}_{n+1} = \tilde{K}_n.$$

Proposition 4 is proved.

Proposition 5

If the utility functions satisfy the condition

$$\lambda_{n-1} U'_{n-1}(c) \leqslant \lambda_n U'_n(c).$$

for all $c \geqslant 0$ and if $\tilde{K}_n \leqslant K_n^*$ $\tilde{v}_n \in (\alpha,\beta)$ and $\tilde{v}_{n-1} < \beta$ then $\tilde{c}_n \geqslant \tilde{c}_{n-1}$, where $\tilde{c}_n = \lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)$ is the per capita consumption (when $\lambda_n = \frac{1}{P_n}$)

Proof

According to Proposition 4(a) or (b), $\tilde{\Psi}_n \leqslant \tilde{\Psi}_{n-1}$. Let $\tilde{v}_n \in (\alpha, \beta)$, hence

$$\tilde{\Psi}_n = \lambda_n U'_n[\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)].$$

If $\tilde{v}_{n-1} < \beta$, then $\tilde{\Psi}_{n-1} \leqslant \lambda_{n-1} U'_{n-1}[\lambda_{n-1}(1 - \tilde{v}_{n-1})h_{n-1}(\tilde{K}_{n-1})]$.

Therefore $\lambda_n U'_n(\tilde{c}_n) = \tilde{\Psi}_n \leqslant \tilde{\Psi}_{n-1} \leqslant \lambda_{n-1} U'_{n-1}(\tilde{c}_{n-1}) \leqslant \lambda_n U'_n(\tilde{c}_{n-1})$, whence $\tilde{c}_n \geqslant \tilde{c}_{n-1}$.

Remarks

1. If $\lambda_n = \lambda^{-n}$ and $U_n(c) = \delta^n U(c)$, then the hypothesis of Proposition 5 (reffering to the utility functions) becomes: $\delta \geqslant \lambda$. If $\lambda = 1$, then this hypothesis becomes $\delta \geqslant 1$. If $\lambda_n = \lambda^{-n}$ and $U_n(c) = \delta^n \lambda^n U(c)$, then the condition referring to the utility functions becomes also $\delta \geqslant 1$.

Proposition 6

If the utility functions satisfy the condition

$$\lambda_{n-1} U'_{n-1}(c) \geqslant \lambda_n U'_n(c)$$

and if $\tilde{K}_n \geqslant K_n^*$, $\tilde{v}_{n-1} \neq \alpha$, and $\tilde{v}_n \in (\alpha, \beta)$, then $\tilde{c}_n \leqslant \tilde{c}_{n-1}$.

Proof

According to Proposition 4 (a) or (b), $\tilde{K}_n \geqslant K_n^*$ implies $\tilde{\Psi}_n \geqslant \tilde{\Psi}_{n-1}$.

Let us suppose that $v_n \in (\alpha, \beta)$. Then $\tilde{\Psi}_n = \lambda_n U_n'(\tilde{c}_n)$, and $v_{n-1} > \alpha$ imply $\tilde{\Psi}_{n-1} \geqslant \lambda_{n-1} U_{n-1}'(\tilde{c}_{n-1})$.

Hence

$$\lambda_n U_n'(\tilde{c}_n) = \tilde{\Psi}_n \geqslant \tilde{\Psi}_{n-1} \geqslant \lambda_{n-1} U_{n-1}'(\tilde{c}_{n-1}) \geqslant \lambda_n U_n'(\tilde{c}_{n-1}),$$

whence $\tilde{c}_n \leqslant \tilde{c}_{n-1}$.

Q.E.D.

4. ALGORITHM FOR CALCULATING THE OPTIMAL TRAJECTORY

As it follows by using Pontryagin's Maximum Principle, the optimal trajectory for K_n is given by:

$$\tilde{K}_{n+1} = \begin{cases} \varphi_n(\tilde{K}_n) - \dfrac{1}{\lambda_n} W_n\left(\dfrac{\tilde{\Psi}_n}{\lambda_n}\right), & \text{if } \tilde{v}_n \in (\alpha, \beta) \\ g_n(\tilde{K}_n) + \beta h_n(\tilde{K}_n), & \text{if } \tilde{v}_n = \beta \\ g_n(\tilde{K}_n) + \alpha h_n(\tilde{K}_n), & \text{if } \tilde{v}_n = \alpha \end{cases}$$

K_0 given; $n = 0,1 \ldots, N-1$.

The optimal trajectory for the adjoint variable is given by the following formulas:

$$\tilde{\Psi} = \begin{cases} \dfrac{\tilde{\Psi}_{n-1}}{\varphi_n'(\tilde{K}_n)} & \text{if } \tilde{v}_n \in (\alpha, \beta) \\ \dfrac{\tilde{\Psi}_{n-1} - \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)](1-\beta)h_n'(\tilde{K}_n)}{g_n'(\tilde{K}_n) + \beta h_n'(\tilde{K}_n)} & \text{if } \tilde{v}_n = \beta \\ \dfrac{\tilde{\Psi}_{n-1} - \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\tilde{K}_n)](1-\alpha)h_n'(\tilde{K}_n)}{g_n'(\tilde{K}_n) + \alpha h_n'(\tilde{K}_n)} & \text{if } \tilde{v}_n = \alpha \end{cases}$$

$\tilde{\Psi}_{N-1} \geqslant 0$ and $\tilde{\Psi}_{N-1}[\tilde{K}_N - \omega] = 0$ where $\varphi_n(K) = g_n(K) + h_n(K)$ and $W_n(\cdot)$ is the function inverse to $U_n'(\cdot)$.

We note that if K_0 is given and we know Ψ_0 then, by using the above formulas we are able to obtain the optimal trajectories of capital and of the dual variable Ψ_n.

If $\tilde{v}_n \in (\alpha, \beta)$ then, from the optimality condition

$$\tilde{\Psi}_n = \lambda_n U_n'[\lambda_n(1 - \tilde{v}_n)h_n(\tilde{K}_n)]$$

it follows

$$\tilde{v}_n = 1 - \frac{W_n\left(\frac{\tilde{\Psi}_n}{\lambda_n}\right)}{\lambda_n h_n(\tilde{K}_n)}.$$

We know from the optimality conditions that if $\tilde{v}_n = \beta$ then we have

$$\tilde{\Psi}_n \geqslant \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)]$$

where

$$\tilde{\Psi}_n = \frac{\tilde{\Psi}_{n-1} - \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)](1-\beta)h_n'(\tilde{K}_n)}{g_n'(\tilde{K}_n) + \beta h_n'(\tilde{K}_n)}.$$

Whence it follows that

$$\tilde{\Psi}_{n-1} \geqslant \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)][h_n'(\tilde{K}_n) + g_n'(\tilde{K}_n)]$$

or

$$\frac{\tilde{\Psi}_{n-1}}{\varphi_n'(\tilde{K}_n)} \geqslant \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)].$$

Conversely, if we have

$$\frac{\tilde{\Psi}_{n-1}}{\varphi_n'(\tilde{K}_n)} \geqslant \lambda_n U_n'[\lambda_n(1-\beta)h_n(\tilde{K}_n)]$$

then it results $\tilde{v}_n = \beta$.
We can prove also that $\tilde{v}_n = \alpha$ is equivalent to

$$\frac{\tilde{\Psi}_{n-1}}{\varphi'(\tilde{K}_n)} \leqslant \lambda_n U_n'[\lambda_n(1-\alpha)h_n(\tilde{K}_n)].$$

We shall denote

$$\overline{\Psi}_n = \frac{\tilde{\Psi}_{n-1}}{\varphi_n'(\tilde{K}_n)} \text{ and } \xi_n^v(\tilde{K}_n) = \lambda_n U_n'[\lambda_n(1-v)h_n(\tilde{K}_n)].$$

Then we have the following equivalences:

$$\tilde{v}_n \in (\alpha, \beta) \Leftrightarrow \xi_n^\alpha(\tilde{K}_n) < \overline{\Psi}_n < \xi_n^\beta(\tilde{K}_n)$$

$$\tilde{v}_n = \beta \quad \Leftrightarrow \quad \overline{\Psi}_n \geqslant \xi_n^\beta(\tilde{K}_n)$$

$$\tilde{v}_n = \alpha \quad \Leftrightarrow \quad \overline{\Psi}_n \leqslant \xi_n^\alpha(\tilde{K}_n).$$

Let us choose an arbitrary $\Psi_0 > 0$; we shall calculate

$$\tilde{K}_1(\Psi_0), \ldots, \tilde{K}_N(\Psi_0); \tilde{\Psi}_1(\Psi_0), \ldots, \tilde{\Psi}_{N-1}(\Psi_0)$$

according to the following formulas:

$$\text{(II)}\ \widetilde{K}_{n+1}(\Psi_0) = \begin{cases} \varphi_n[\widetilde{K}_n(\Psi_0)] - \dfrac{1}{\lambda_n} W_n\left[\dfrac{\widetilde{\Psi}_n(\Psi_0)}{\lambda_n}\right], \text{ if } \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] < \widetilde{\Psi}_n(\Psi_0) < \\ \qquad\qquad < \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ g_n[\widetilde{K}_n(\Psi_0)] + \beta h_n[\widetilde{K}_n(\Psi_0)], \quad \text{if } \overline{\Psi}_n(\Psi_0) \geqslant \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ g_n[\widetilde{K}_n(\Psi_0)] + \alpha h_n[\widetilde{K}_n(\Psi_0)], \text{ if } \overline{\Psi}_n(\Psi_0) \leqslant \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] \end{cases}$$

where $K_0(\Psi_0) = K_0$ is given and $n = 0,1\ ...,\ N-1$;

$$\text{(III)}\ \widetilde{\Psi}_n(\Psi_0) = \begin{cases} \overline{\Psi}_n(\Psi_0), \quad \text{if } \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] < \overline{\Psi}_n(\Psi_0) < \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ \Psi_n^{\beta}(\Psi_0), \quad \text{if } \overline{\Psi}_n(\Psi_0) \geqslant \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ \Psi_n^{\alpha}(\Psi_0), \quad \text{if } \overline{\Psi}_n(\Psi_0) \leqslant \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] \end{cases}$$

with

$$\widetilde{\Psi}_0(\Psi_0) = \Psi_0; \quad n = 0,\ 1,\ ...,\ N-1,$$

where

$$\overline{\Psi}_n(\Psi_0) = \frac{\widetilde{\Psi}_{n-1}(\Psi_0)}{\varphi_n'[\widetilde{K}_n(\Psi_0)]}$$

$$\Psi_n^{v}(\Psi_0) = \frac{\widetilde{\Psi}_{n-1}(\Psi_0) - \lambda_n U_n'[\lambda_n(1-v)h_n(\widetilde{K}_n(\Psi_0))]\,(1-v)h_n'[\widetilde{K}_n(\Psi_0)]}{g_n'[\widetilde{K}_n(\Psi_0)] + v h_n'[\widetilde{K}_n(\Psi_0)]}$$

for $v = \alpha$ or $v = \beta$

If $\widetilde{\Psi}_N(\Psi_0) \geqslant 0$ then the sequences

$$K_0,\ \widetilde{K}_1(\Psi_0),\ ...,\ \widetilde{K}_N(\Psi_0) \text{ and } \Psi_0,\ \widetilde{\Psi}_1(\Psi_0),\ ...,\ \widetilde{\Psi}_{N-1}(\Psi_0)$$

stand for the optimal trajectories of capital and the adjoint variable, respectively, for model I, with $\omega = \widetilde{K}_N(\Psi_0)$. In this case, the optimal control is:

$$\text{(IV)}\quad \widetilde{v}_n(\Psi_0) = \begin{cases} 1 - \dfrac{W_n\left(\dfrac{\widetilde{\Psi}_n(\Psi_0)}{\lambda_n}\right)}{\lambda_n h_n[\widetilde{K}_n(\Psi_0)]}, \quad \text{if} \quad \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] < \overline{\Psi}_n(\Psi_0) < \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ \beta \qquad\qquad , \quad \text{if } \overline{\Psi}_n(\Psi_0) \geqslant \xi_n^{\beta}[\widetilde{K}_n(\Psi_0)] \\ \alpha \qquad\qquad , \quad \text{if } \overline{\Psi}_n(\Psi_0) \leqslant \xi_n^{\alpha}[\widetilde{K}_n(\Psi_0)] \end{cases}$$

$n = 0,\ 1,\ ...,\ N-1$

Proposition 7

The functions $\widetilde{K}_n(\Psi_0)$, $\widetilde{\Psi}_n(\Psi_0)$ and $\widetilde{v}_n(\Psi_0)$ are continuous and monotonously increasing.

Proof

Let v_0 be in the interval $[\alpha, \beta]$ and let $\Psi_0(v_0)$ be the solution of equation

$$1 - \frac{W_0\left(\frac{\Psi_0}{\lambda_0}\right)}{\lambda_0 h_0(K_0)} = v_0.$$

Hence

$$\Psi_0(v_0) = \lambda_0 U_0'[\lambda_0(1 - v_0)h_0(K_0)].$$

We have

$$\Psi_0'(v_0) = -\lambda_0^2 U_0''[\lambda_0(1 - v_0)h_0(K_0)]h_0(K_0) > 0$$

hence $\Psi_0(v_0)$ is an increasing function. Therefore, if $v_0 \in [\alpha, \beta]$, then

$$\Psi_0(\alpha) \leqslant \Psi_0(v_0) \leqslant \Psi_0(\beta).$$

If $\Psi_0 \leqslant \Psi_0(\alpha)$ then $\Psi_0 \leqslant \lambda_0 U_0'[\lambda_0(1 - \alpha)h_0(K_0)]$

and hence $\widetilde{v}_0(\Psi_0) = \alpha$; analogously, if $\Psi_0 \geqslant \Psi_0(\beta)$.
then $\widetilde{v}_0(\Psi_0) = \beta$.

Let $\varrho_0(\cdot)$ be the function inverse to $\Psi_0(\cdot)$; we have:

$$\lim_{\Psi_0 \to \Psi_0(\alpha) - 0} \widetilde{v}_0(\Psi_0) = \alpha \text{ and}$$

$$\lim_{\Psi_0 \to \Psi_0(\alpha) + 0} v_0(\Psi_0) = \lim_{\Psi_0 \to \Psi_0(\alpha) + 0} \left(1 - \frac{W_0\left(\frac{\Psi_0}{\lambda_0}\right)}{\lambda_0 h_0(K_0)}\right) = 1 - \frac{W_0\left[\frac{\Psi_0(\alpha)}{\lambda_0}\right]}{\lambda_0 h_0(K_0)} =$$

$$= 1 - \frac{W_0\left[\frac{\lambda_0 U_0'[\lambda_0(1 - \alpha)h_0(K_0)]}{\lambda_0}\right]}{\lambda_0 h_0(K_0)} = 1 - \frac{\lambda_0(1 - \alpha)\, h_0(K_0)}{\lambda_0 h_0(K_0)} = \alpha.$$

Therefore $v_0(\Psi_0)$ is continuous in $\Psi_0(\alpha)$. One can prove analogously that it is continuous also in $\Psi_0(\beta)$. It is obvious that in every other point the function $v_0(\Psi_0)$ is continuous.

When $\Psi_0 \leqslant \Psi_0(\alpha)$ or $\Psi_0 \geqslant \Psi_0(\beta)$ the monotonicity of $\widetilde{v}_0(\Psi_0)$ is obvious. When $\Psi_0(\alpha) < \Psi_0 < \Psi_0(\beta)$ we have

$$\widetilde{v}'(\Psi_0) = -\frac{W_0'\left(\frac{\Psi_0}{\lambda_0}\right)}{h_0(K_0)} > 0.$$

Hence, for $\Psi_0(\alpha) < \Psi_0 < \Psi_0(\beta)$, the function $\tilde{v}_0(\Psi_0)$ is strictly increasing.

The graph of the function $\tilde{v}_0(\Psi_0)$ has the following shape:

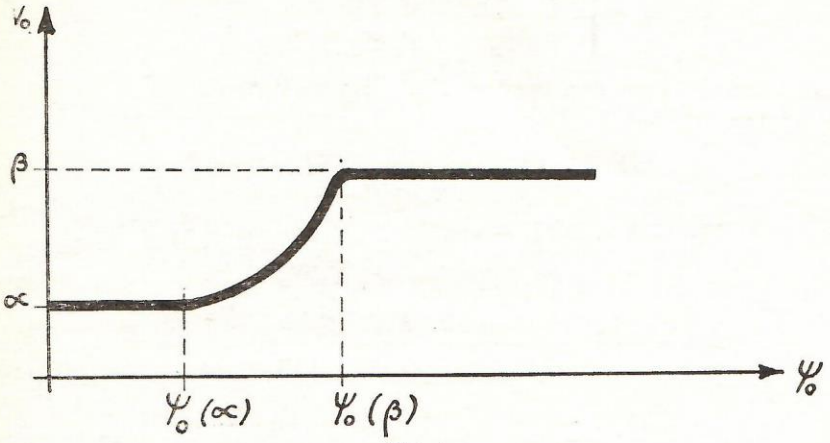


Fig. 1

As concerns function $\tilde{K}_1(\Psi_0)$, we can see from relation $\tilde{K}_1(\Psi_0) = g_0(K_0) + \tilde{v}_0(\Psi_0)h_0(K_0)$ that it is continuous and monotonously increasing.

Let us deduce the behaviour of $\overline{\Psi}_1(\Psi_0)$. Let $\Psi_0^{1,\alpha}$ be the solution of equation

$$\overline{\Psi}_1(\Psi_0) = \xi_1^{\alpha}[\tilde{K}_1(\Psi_0)]$$

or

$$\frac{\Psi_0}{\varphi_1'[\tilde{K}_1(\Psi_0)]} = \lambda_1 U_1'[\lambda_1(1-\alpha)h_1(\tilde{K}_1(\Psi_0))].$$

Taking into account that $\tilde{K}_1(\Psi_0)$ is an increasing function and that $U_1'(\cdot)$ and $\varphi_1'(\cdot)$ are decreasing functions, it follows the unicity of $\Psi_0^{1,\alpha}$. It results also:

$$\Psi_0 \leqslant \Psi_0^{1,\alpha} \Rightarrow \overline{\Psi}_1(\Psi_0) \leqslant \xi_1^{\alpha}[\tilde{K}_1(\Psi_0)]$$

and

$$\Psi_0 \geqslant \Psi_0^{1,\alpha} \Rightarrow \overline{\Psi}_1(\Psi_0) \geqslant \xi_1^{\alpha}[\tilde{K}_1(\Psi_0)].$$

Analogously, defining $\Psi_0^{1,\beta}$ as the unique solution of equation

$$\overline{\Psi}_1(\Psi_0) = \xi_1^{\beta}[K_1(\Psi_0)],$$

it follows:

$$\Psi_0 \leqslant \Psi_0^{1,\beta} \Rightarrow \overline{\Psi}_1(\Psi_0) \leqslant \xi_1^{\beta}[\tilde{K}_1(\Psi_0)]$$

and

$$\Psi_0 \geqslant \Psi_0^{1,\beta} \Rightarrow \overline{\Psi}_1(\Psi_0) \geqslant \xi_1^{\beta}[\tilde{K}_1(\Psi_0)].$$

Taking into account these results, we can redefine $\widetilde{\Psi}_1(\Psi_0)$ in the following way:

$$\widetilde{\Psi}_1(\Psi_0) = \begin{cases} \Psi_1^{\alpha}(\Psi_0), & \text{if } \Psi_0 \leqslant \Psi_0^{1,\alpha} \\ \overline{\Psi}_1(\Psi_0), & \text{if } \Psi_0^{1,\alpha} < \Psi_0 < \Psi_0^{1,\beta} \\ \Psi_1^{\beta}(\Psi_0), & \text{if } \Psi_0 \geqslant \Psi_0^{1,\beta} \end{cases}$$

Let us show the continuity of $\widetilde{\Psi}_1(\Psi_0)$ in $\Psi_0^{1,\alpha}$.
We have

$$\lim_{\Psi_0 \to \Psi_0^{1,\alpha} - 0} \widetilde{\Psi}_1(\Psi_0) = \lim_{\Psi_0 \to \Psi_0^{1,\alpha} - 0} \Psi_1^{\alpha}(\Psi_0) =$$

$$= \frac{\Psi_0^{1,\alpha} - \lambda_1 U_1'[\lambda_1(1-\alpha)h_1(\widetilde{K}_1(\Psi_0^{1,\alpha}))](1-\alpha)h_1'[\widetilde{K}_1(\Psi_0^{1,\alpha})]}{g_1'[\widetilde{K}_1(\Psi_0^{1,\alpha})] + \alpha h_1'[\widetilde{K}_1(\Psi_0^{1,\alpha})]} =$$

$$= \frac{\lambda_1 U_1'[\lambda_1(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]]\,\varphi_1'[\widetilde{K}_1(\Psi_0^{1,\alpha})]}{g_1'[K_1(\Psi_0^{1,\alpha})] + \alpha h_1'(\widetilde{K}_1(\Psi_0^{1,\alpha})]} -$$

$$- \frac{\lambda_1 U_1'[\lambda_1(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]](1-\alpha)h_1'[K_1(\Psi_0^{1,\alpha})]}{g_1'[K_1(\Psi_0^{1,\alpha})] + \alpha h_1'[\widetilde{K}_1(\Psi_0^{1,\alpha})]} =$$

$$= \lambda_1 U_1'[\lambda_1(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]]$$

and

$$\lim_{\Psi_0 \to \Psi_0^{1,\alpha} + 0} \widetilde{\Psi}_1(\Psi_0) = \overline{\Psi}_1(\Psi_0^{1,\alpha}) = \lambda_1 U_1'[\lambda_1(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]]$$

and therefore $\widetilde{\Psi}_1(\Psi_0)$ is continuous in $\Psi_0^{1,\alpha}$. One proves analogously its continuity in $\Psi_0^{1,\beta}$. It is obvious that in every other point $\widetilde{\Psi}_1(\Psi_0)$ is continuous.

In order to prove the monotonicity of $\widetilde{\Psi}_1(\Psi_0)$, we note that $\Psi_1^{\alpha}(\Psi_0)$, $\Psi_1^{\beta}(\Psi_0)$ and $\overline{\Psi}_1(\Psi_0)$ are monotonously increasing. This fact follows from the properties of functions $\widetilde{K}_1(\Psi_0)$, $U_1'(\cdot)$, $h_1'(\cdot)$, $g_1'(\cdot)$ and $\varphi_1'(\cdot)$, the first of them being an increasing function, and the others being decreasing functions.

Let us prove the continuity and monotonicity of $\widetilde{v}_1(\Psi_0)$. We have

$$\widetilde{v}_1(\Psi_0) = \begin{cases} \alpha, & \text{if } \Psi_0 \leqslant \Psi_0^{1,\alpha} \\ \beta, & \text{if } \Psi_0 \geqslant \Psi_0^{1,\beta} \\ 1 - \dfrac{W_1\left(\dfrac{\Psi_1(\Psi_0)}{\lambda_1}\right)}{\lambda_1 h_1[\widetilde{K}_1(\Psi_0)]}, & \text{if } \Psi_0^{1,\alpha} < \Psi_0 < \Psi_0^{1,\beta} \end{cases}$$

$$\lim_{\Psi_0 \to \Psi_0^{1,\alpha} - 0} \widetilde{v}_1(\Psi_0) = \alpha$$

$$\lim_{\Psi_0 \to \Psi_0^{1,\alpha} + 0} \widetilde{v}_1(\Psi_0) = 1 - \frac{W_1\left[\dfrac{\Psi_1(\Psi_0^{1,\alpha})}{\lambda_1}\right]}{\lambda_1 h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]};$$

on the other hand,

$$\Psi_1(\Psi_0^{1,\alpha}) = \lambda_1 U_1'[\lambda_1(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]].$$

Hence

$$\lim_{\Psi_0 \to \Psi_0^{1,\alpha}+0} \widetilde{v}_1(\Psi_0) = 1 - \frac{\lambda_1 \Psi[(1-\alpha)h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]]}{\lambda_1 h_1[\widetilde{K}_1(\Psi_0^{1,\alpha})]} = \alpha$$

and therefore the function $\widetilde{v}_1(\Psi_0)$ is continuous at $\Psi_0^{1,\alpha}$. The proof of continuity at $\Psi_0^{1,\beta}$ is analogous.

Taking into account that $W_1(\cdot)$ is a decreasing function, while $\Psi_1(\cdot)$, $h_1(\cdot)$ and $\widetilde{K}_1(\cdot)$ are monotonously increasing functions, it follows that $\widetilde{v}_1(\Psi_0)$ is a monotonously increasing function.

One proves analogously that if $\widetilde{K}_n(\Psi_0)$, $\widetilde{\Psi}_n(\Psi_0)$ and $\widetilde{v}_n(\Psi_0)$ are continous functions, then $\widetilde{K}_{n+1}(\Psi_0)$, $\widetilde{\Psi}_{n+1}(\Psi_0)$ and $\widetilde{v}_{n+1}(\Psi_0)$ are also continous

Q.E.D.

Corollary 1

Let ${}^1\Psi_0$ be the least Ψ_0 for which $\widetilde{\Psi}_{N-1}({}^1\Psi_0) = 0$ and let $\omega^1 = \widetilde{K}_N({}^1\Psi_0)$ be the corresponding final state.

Let ${}^2\Psi_0$ be the least Ψ_0 for which all $\widetilde{v}_n({}^2\Psi_0) = \beta$ $(n = 0,1 \ldots, N-1)$ and $\omega^2 = \widetilde{K}_N({}^2\Psi_0)$ the corresponding final state.

For each given final state $\omega \in [\omega^1, \omega^2]$ there exists a $\Psi_0 \in [{}^1\Psi_0, {}^2\Psi_0]$ such that $\widetilde{K}_N(\Psi_0) = \omega$ on the optimal trajectory.

In order to justify the above assertion, first we shall note that $\omega^2 = \check{K}_N$ where $\check{K}_N$ is that defined in Proposition 1. As concerns ${}^2\Psi_0$ we note that it is the least value satisfying the following system of inequalities:

$$\Psi_0 \geqslant \lambda_0 U_0'[\lambda_0(1-\beta)h_0(K_0)]$$

$$\Psi_0 \geqslant \varphi_1(\check{K}_1)\lambda_1 U_1'[\lambda_1(1-\beta)h_1(\check{K}_1)]$$

. . . . . .

$$\Psi_0 \geqslant \varphi_1(\check{K}_1)\varphi_2(\check{K}_2) \ldots \varphi_N(\check{K}_N)\lambda_{N-1}U_{N-1}'[\lambda_{N-1}(1-\beta)h_{N-1}(\check{K}_{N-1})]$$

where $\check{K}_1, \check{K}_2, \ldots, \check{K}_N$ follow by the following relation

$$\check{K}_{n+1} = g_n(\check{K}_n) + \beta h_n(\check{K}_n), \text{ with } \check{K}_0 = K_0 \text{ given.}$$

The algorithm

Based on the facts presented till now, we are able to describe an algorithm for calculating the optimal trajectory of model I.

Step 1

We choose any $\Psi_0 > 0$*) and calculate $\widetilde{K}_n(\Psi_0)$, $\widetilde{\Psi}_n(\Psi_0)$ and $\widetilde{v}_n(\Psi_0)$ according to formulas II, III and IV.

When there exists an $n \in \{1, 2, ..., N-1\}$ such that $\widetilde{\Psi}_n(\Psi_0) < 0$, then we choose another Ψ_0, bigger then the initial one and we resume the calculus. If $\widetilde{\Psi}_n(\Psi_0) \geqslant 0$ for $n = 1, 2, ..., N-1$, then we pass to step 2.

Step 2

Let us consider $\widetilde{K}_n(\Psi_0)$ $(n = 0,1, ..., N)$ with $\widetilde{K}_0(\Psi_0) = K_0$ and $\widetilde{\Psi}_n(\Psi_0)(n = 0, 1, ..., N-1)$ with $\widetilde{\Psi}_0(\Psi_0) = \Psi_0$ and $\widetilde{\Psi}_{N-1}(\Psi_0) \geqslant 0$, as calculated at step 1.

We check the final condition

$$\widetilde{\Psi}_{N-1}(\Psi_0)[\widetilde{K}_N(\Psi_0) - \omega] = 0.$$

If it holds, then the optimal solution is:

$$\widetilde{K}_n = \widetilde{K}_n(\Psi_0) \qquad n = 0,1, ..., N$$

$$\widetilde{\Psi}_n = \widetilde{\Psi}_n(\Psi_0) \qquad n = 0, 1, ..., N-1$$

$$\widetilde{v}_n = \widetilde{v}_n(\Psi_0) \qquad n = 0, 1, ..., N-1.$$

If the final condition is not satisfied, i.e. $\widetilde{\Psi}_{N-1}(\Psi_0) > 0$ and $\widetilde{K}_N(\Psi_0) - \omega \neq 0$, then we go back to step 1, taking a bigger Ψ_0, if $\widetilde{K}_N(\Psi_0) < \omega$ and a less value of Ψ_0, if $\widetilde{K}_N(\Psi_0) > \omega$. If there are two values Ψ_0' and Ψ_0'' so that $K_N(\Psi_0') < \omega$ and $K_N(\Psi_0'') > \omega$ then we choose $\Psi_0''' = \dfrac{\Psi_0' + \Psi_0''}{2}$.

Proposition 7 guarantees us that by acting in this way, we improve our result, while the continuity properties proved for $\widetilde{K}_n(\Psi_0)$ and $\widetilde{\Psi}_n(\Psi_0)$ insure that for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ fixed, we can obtain, after finitely many iterations:

$$| \widetilde{K}_N(\Psi_0) - \omega | < \varepsilon_1$$

or

$$| \widetilde{\Psi}_{N-1}(\Psi_0) | < \varepsilon_2.$$

---

*) In order to have an appropriate initial value for Ψ_0 we can choose, in the first iteration

$$\Psi_0 = \lambda_0 U_0[\lambda_0(1 - v_0)h_0(K_0)]$$

with v_0 conveniently chosen.

5. THE SENSITIVITY OF THE OPTIMAL TRAJECTORY

In the previous section we showed a procedure for calculating actually the optimal trajectory of model I.

Based on the above algorithm, the initial state K_0 and the final condition ω being fixed, we can find a $\Psi_0(\omega)$ such that the optimal trajectory of capital, strating from K_0 would reach $\tilde{K}_N(\Psi_0(\omega)) \geqslant \omega$.

From Proposition 7 it follows that $\Psi_0(\omega)$ is a monotonously increasing function of ω.

From the same Proposition it follows that the controls $\tilde{v}_n$ are monotonously increasing function of Ψ_0. Taking into account these facts, we conclude that all of the optimal controls $\tilde{v}_0(\tilde{\Psi}_0(\omega))$, $\tilde{v}_1(\tilde{\Psi}_0(\omega))$, ..., $\tilde{v}_{N-1}(\Psi_0(\omega))$ are monotonously increasing functions of the final condition ω.

This conclusion is very important from the economic point of view; it shows that a bigger final condition imposed on capital implies a greater effort in investment in each period of the planning period under consideration.

The following proposition concerns the dependence of the trajectory of per-capita consumption

$$\bar{c}_n = \lambda_n(1 - \tilde{v}_n)h_n(\bar{K}_n)$$

on the final condition ω.

Proposition 8

Let ω_1 and ω_2 two final conditions imposed on the capital in model I and let $\tilde{c}_n$ be the per-capita consumption in period n:

$$\bar{c}_n(\omega_i) = \lambda_n(1 - \bar{v}_n(\omega_i))\, h_n(\bar{K}_n(\omega_i)); \quad i = 1,2$$

where

$$\bar{v}_n(\omega_i) = \tilde{v}_n(\Psi_0(\omega_i)),\ \bar{c}_n(\omega_i) = \tilde{c}_n(\Psi_0(\omega_i)) \text{ and } \bar{K}_n(\omega_i) = \tilde{K}_n(\Psi_0(\omega_i)).$$

If $\bar{v}_n(\omega_i) \in (\alpha, \beta)$ for $i = 1,2$, then we have:

$$(\omega_2 - \omega_1)[\bar{c}_n(\omega_2) - \bar{c}_n(\omega_1)] \leqslant 0;$$

if $\bar{v}_n(\omega_i) = \alpha$ or $\bar{v}_n(\omega_i) = \beta$ $(i = 1, 2)$ or $\bar{v}_n(\omega_i) = \beta$ $(i = 1, 2)$ then

$$(\omega_2 - \omega_1)[\bar{c}_n(\omega_2) - \bar{c}_n(\omega_1)] \geqslant 0.$$

Proof

If $\bar{v}_n(\omega_i) \in (\alpha, \beta)$ then the optimality condition gives

$$\overline{\Psi}_n(\omega_i) = \lambda_n U_n'[\lambda_n(1 - \bar{v}_n(\omega_i))h_n(\bar{K}_n(\omega_i))]$$

where

$$\overline{\Psi}_n(\omega_i) = \tilde{\Psi}_n(\Psi_0(\omega_i)).$$

Hence

$$\bar{c}_n(\omega_i) = W_n\left(\frac{\overline{\Psi}_n(\omega_i)}{\lambda_n}\right).$$

Taking into account that $W_n(\cdot)$ is a decreasing function, while $\overline{\Psi}_n(\omega)$ is a monotonously increasing function, the first statement of the proposition is proved.

If $\bar{v}_n(\omega_i) = \alpha$ or $\bar{v}_n(\omega_i) = \beta$

we have:

$$\bar{c}_n(\omega_i) = (1 - \alpha)h_n(\overline{K}_n(\omega_i))$$

or

$$\bar{c}_n(\omega_i) = (1 - \beta)h_n(\overline{K}_n(\omega_i)).$$

Taking into account that $\overline{K}_n(\omega)$ is an increasing function, we get the second statement of the proposition.

Remark 1

To summarize the results of this section, we note that if the final level imposed to capital (or, equivalently to the national income or social product $h_N(\overline{K}_N)$) increases, then the value of the optimal control $\tilde{v}_n$ increases also, for each n, while the level of per capita consumption decreases when $\tilde{v}_n$ is in the open interval (α, β) and increases when $\tilde{v}_n = \alpha$ or $\tilde{v}_n = \beta$.

Remark 2

From Proposition 8 it follows that if $\tilde{v}_n(\omega) \in (\alpha, \beta)$ $n = 0, 1, \ldots, N-1$, then the optimal value of the performance

$$J(v) = \sum_{n=0}^{N-1} U_n[\lambda_n(1 - v_n)h_n(K_n)]$$

is a decreasing function of the final condition ω.

The following proposition gives the sensitivity of the optimal value of the performance with respect to the initial value of the capital and its final value, when $\tilde{v}_n \in (\alpha, \beta)$ for $n = 0,1, \ldots, N-1$.

Proposition 9

Let us consider

$$J(v) = \sum_{n=0}^{N-1} U_n[\lambda_n(1 - v_n)h_n(K_n)]$$

and

$$\tilde{v}_n \in (\alpha, \beta) \quad n = 0,1 \ldots, N-1.$$

In the case when only one of the values K_N and K_0 is fixed we have one or the other of the following formulas:

$$\frac{\partial J(\tilde{v})}{\partial K_0} = \varphi_0'(K_0)\tilde{\Psi}_0 > 0 \text{ and } \frac{\partial J(\tilde{v})}{\partial K_N} = -\tilde{\Psi}_{N-1} < 0$$

where $\tilde{\Psi}_n$ is the optimal value of the dual variable and

$$\varphi_0(K_0) = g_0(K_0) + h_0(K_0).$$

Proof

As $c_n = \lambda_n(1 - v_n)\, h_n(K_n)$, we have

$$\frac{\partial J(\tilde{v})}{\partial \beta} = \sum_{n=0}^{N-1} U_n'(\tilde{c}_n)\, \frac{\partial \tilde{c}_n}{\partial \beta}$$

where β is an arbitrary variable. On the other hand,

$$K_{n+1} = \varphi_n(K_n) - \frac{c_n}{\lambda_n},$$

whence

$$c_n = \lambda_n[\varphi_n(K_n) - K_{n+1}]$$

Therefore:

$$\frac{\partial J(\tilde{v})}{\partial \beta} = \sum_{n=0}^{N-1} U_n'(\tilde{c}_n)\lambda_n\left[\varphi_n'(\tilde{K}_n)\frac{\partial \tilde{K}_n}{\partial \beta} - \frac{\partial \tilde{K}_{n+1}}{\partial \beta}\right] =$$

$$= \sum_{n=0}^{N-1} \tilde{\Psi}_n\left[\varphi_n'(\tilde{K}_n)\frac{\partial \tilde{K}_n}{\partial \beta} - \frac{\partial \tilde{K}_{n+1}}{\partial \beta}\right] = \sum_{n=1}^{N-1} \tilde{\Psi}_{n-1}\frac{\partial \tilde{K}_n}{\partial \beta} - \sum_{n=0}^{N-1} \tilde{\Psi}_n \frac{\partial \tilde{K}_{n+1}}{\partial \beta} +$$

$$+ \Psi_0\, \varphi_0'(K_0)\frac{\partial \tilde{K}_0}{\partial \beta} = \tilde{\Psi}_0 \varphi_0'(K_0)\frac{\partial K_0}{\partial \beta} - \Psi_{N-1}\frac{\partial \tilde{K}_N}{\partial \beta}.$$

By replacing β by K_0, $\tilde{K}_N$ respectively, we obtain the formulas set forth in proposition 9.

Remarks

1) When $\tilde{v}_n \in (\alpha, \beta)$ only for $n = n_1, n_1 + 1, \ldots, n_2 - 1$, then the following formulas hold.

$$\frac{\partial J_{n_1, n_2}(\tilde{v})}{\partial K_{n_1}} = \varphi_{n_1}'(K_{n_1}) \cdot \tilde{\Psi}_{n_1} \text{ and } \frac{\partial J_{n_1, n_2}(\tilde{v})}{\partial K_{n_2}} = -\tilde{\Psi}_{n_2 - 1}$$

respectively, where

$$J_{n_1, n_2}(v) = \sum_{n=n_1}^{n_2 - 1} U_n[\lambda_n(1 - v_n)h_n(K_n)].$$

6. THE CASE OF THE AUTONOMOUS MODEL

In this section we consider the following hypotheses concerning the elements of the model:

a) the dynamic equation of the capital is:

$$K_{n+1} = (1 - a)K_n + v_n F(L_n, K_n)$$

$$K_0 \text{ given}$$

$$K_N \geqslant \omega_1,$$

where $F(L, K)$ is a homogenous of the first degree production function and a is the coefficient of physical and moral wear and tear.

b) The evolution of the labour force in the productive sphere is described by the following equation:

$$L_{n+1} = lL_n,\ L_0 = 1 \quad \text{i.e. } L_n = l^n.$$

c) Within the population as a whole the share of the labour force in the productive sphere is constant: $L_n = bP_n$, $b \in (0,1)$; therefore, the dynamic equation of the population is

$$P_n = \frac{1}{b} l^n.$$

d) The performance of the model is

$$max\ J(v) = max \sum_{n=0}^{N-1} P_n \delta^n U\left[\frac{1}{P_n}(1 - v_n)F(L_n, k_n)\right]$$

where $\delta \in (0,1]$; $\frac{1-\delta}{\delta}$ is the discount coefficient of the consumption and $U(\cdot)$ is the utility function of the consumption, an increasing and concave function.

Taking into account the homogeneity of first degree of the production function, the model takes the following form:

$$\bar{V}\begin{cases} k_{n+1} = gk_n + v_n h(k_n); \quad k_0 - \text{given},\ k_N \geqslant \omega \\ max\ J(v) = max \sum_{n=0}^{N-1} \gamma^n U[\lambda(1 - v_n)h(k_n)] \\ \alpha \leqslant v_n \leqslant \beta \end{cases}$$

where:

$$k_n = \frac{K_n}{L_n}$$

is the per capita amount of capital (per worker);

$$h(k_n) = \frac{1}{l} F(1, k_n);\ g = \frac{1-a}{l};\ \lambda = bl;\ \omega = \frac{\omega_1}{l_N};$$

$$\lambda = l\delta.$$

Applying the results obtained in the first part of this paper, we find the following optimality conditions:

a) if $\tilde{v}_n \in (\alpha, \beta)$, then $\tilde{\Psi}_n = \lambda\gamma^n U'[\lambda(1 - \tilde{v}_n)h(\tilde{k}_n])$;

b) if $\tilde{v}_n = \alpha$, then $\tilde{\Psi}_n \leqslant \lambda\gamma^n U'[\lambda(1 - \alpha)h(\tilde{k}_n)]$;

c) if $\tilde{v}_n = \beta$, then $\tilde{\Psi}_n \geqslant \lambda\gamma^n U'[\lambda(1 - \beta)h(\tilde{k}_n)]$

where $\tilde{\Psi}_n$ is the optimal value of the dual variable; it is the solution of the equation:

$$\tilde{\Psi}_{n-1} = \tilde{\Psi}_n[g + \tilde{v}_n h'(\tilde{k}_n)] + \lambda\gamma^n U'[\lambda(1 - \tilde{v}_n)h(\tilde{k}_n)] \cdot (1 - \tilde{v}_n)h'(\tilde{k}_n).$$

Noting $q_n = \lambda^{-1}\Psi_n\gamma^{-n}$ we have:

a') $\tilde{v}_n \in (\alpha, \beta)$ then $\tilde{q}_n = U'[\lambda(1 - \tilde{v}_n)h(\tilde{k}_n)]$

b') $\tilde{v}_n = \alpha$ then $\tilde{q}_n \leqslant U'[\lambda(1 - \alpha)h(\tilde{k}_n)]$

c') $\tilde{v}_n = \beta$ then $\tilde{q}_n \geqslant U'[\lambda(1 - \beta)h(\tilde{k}_n)]$

and the dynamic equation for $\tilde{q}_n$ is:

$$\tilde{q}_{n-1} = \gamma\tilde{q}_n[g + v_n h'(\tilde{k}_n)] + \gamma U'[\lambda(1 - \tilde{v}_n)h(\tilde{k}_n)](1 - \tilde{v}_n)h'(\tilde{k}_n)$$

which becomes, in the particular case when $\tilde{v}_n \in (\alpha, \beta)$:

$$\tilde{q}_{n-1} = \gamma\tilde{q}_n[g + h'(\tilde{k}_n)].$$

Noting $gk + h(k) = \varphi(k)$ the above equation becomes:

$$\tilde{q}_{n-1} = \gamma q_n \varphi'(\tilde{k}_n).$$

For the autonomous model, we have the following properties, which result as particular cases of Proposition 4 or can be proved in a similar way.

Proposition 4'

Let k^* be the solution of equation

$$\gamma\varphi'(k) - 1 = 0$$

and

$$q^\circ(k) = U'[\lambda(\varphi(k) - k)]$$

a) if $k_n \neq k^*$ and $U'(\lambda(1 - \beta)h(\tilde{k}_n)) > \tilde{q}_n > U'[\lambda(1 - \alpha)h(k_n)]$, then

$$(\tilde{k}_n - k^*)(\tilde{q}_n - \tilde{q}_{n-1}) > 0;$$

if $\tilde{k} \leqslant k^*$ and $\tilde{q}_n \leqslant U'(\lambda(1 - \alpha)h(\tilde{k}_n))$ then $\tilde{q}_{n-1} \geqslant \tilde{q}_n$; if $\tilde{k}_n \geqslant k^*$ and

$$\tilde{q}_n \geqslant U'(\lambda(1 - \beta)h(\tilde{k}_n)) \text{ then } \tilde{q}_{n-1} \leqslant \tilde{q}_n.$$

b) if $\tilde{k}_n = k^*$ and $U'(\lambda(1 - \beta)h(\tilde{k}_n)) > \tilde{q}_n > U'[\lambda(1 - \alpha)h[\tilde{k}_n)]$, then

$$\tilde{q}_n = \tilde{q}_{n-1}.$$

c) if $\tilde{k}_n = \tilde{k}^*$ and $\tilde{q}_n \leqslant U'[\lambda(1 - \alpha)h(\tilde{k}_n)]$, then

$$\tilde{q}_n \leqslant \tilde{q}_{n-1};$$

$$\text{if } \tilde{q}_n \geqslant U'(\lambda(1 - \beta)h(\tilde{k}_n)) \text{ then } \tilde{q}_n \geqslant \tilde{q}_{n-1};$$

d) if $\tilde{q}_n \neq q_0(\tilde{k}_n)$, and $\alpha \leqslant \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)} \leqslant \beta$, then

$$[\tilde{q}_n - q_n^0(\tilde{k}_n)][\tilde{k}_{n+1} - \tilde{k}_n)] > 0;$$

e) if $\tilde{q}_n = q^0(k_n)$, and $\alpha < \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)} < \beta$, then $\tilde{k}_{n+1} = \tilde{k}_n$.

Proposition 10

Let k^* be the solution of the equation

$$\gamma\varphi'(k) - 1 = 0$$

a) if $\tilde{k}_n \leqslant k^*$, $\alpha \leqslant \dfrac{(1-g)\tilde{k}_{n-1}}{h(k_{n-1})}$ and $\alpha \leqslant \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)}$, then $\tilde{v}_n = \alpha$ implies

$$\tilde{v}_n = \alpha;$$

b) if $\tilde{k}_n \geqslant k^*$, $\beta \geqslant \dfrac{(1-g)\tilde{k}_{n-1}}{h(\tilde{k}_{n-1})}$ and $\alpha \leqslant \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)} \leqslant \beta$

then $\tilde{v}_{n-1} = \beta$ implies $\tilde{v}_{n-1} = \beta$;

c) if $\tilde{k}_n \geqslant k^*$, $\alpha \leqslant \dfrac{(1-g)\tilde{k}_{n-1}}{h(\tilde{k}_{n-1})} \leqslant \beta$, $\alpha \leqslant \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)} \leqslant \beta$

and $\tilde{q}_{n-1} > q^0(\tilde{k}_{n-1})$, then

$$\tilde{v}_{n-1} > \alpha \text{ implies } \tilde{v}_n > \alpha.$$

d) if $\tilde{k}_n \leqslant k^*$, $\alpha \leqslant \dfrac{(1-g)\tilde{k}_{n-1}}{h(\tilde{k}_{n-1})} \leqslant \beta$, $\alpha \leqslant \dfrac{(1-g)\tilde{k}_n}{h(\tilde{k}_n)} \leqslant \beta$ and

$$\tilde{q}_{n-1} < q^0(\tilde{k}_{n-1}) \text{ then } \tilde{v}_{n-1} < \beta \text{ implies } \tilde{v}_n < \beta.$$

Proposition 11

If $\lim\limits_{k \downarrow 0} \varphi'(k) > \dfrac{1}{\gamma}$; $\lim\limits_{k \to \infty} \varphi(k) < \dfrac{1}{\gamma}$, $\alpha \leqslant \dfrac{(1-g)k^*}{h(k^*)} \leqslant \beta$ (where k^* is the solution of the equation $\gamma\varphi'(k) - 1 = 0$) then the Hamiltonian system:

$$\begin{cases} \tilde{k}_{n+1} = \varphi(\tilde{k}_n) - \dfrac{1}{\lambda} W(\tilde{q}_n) \\ \tilde{q}_{n-1} = \gamma\varphi'(\tilde{k}_n)\tilde{q}_n. \end{cases}$$

has a saddle point.

Proof

In order to insure the existence of the stationary point, i.e. in order to have $\tilde{q}_{n-1} = \tilde{q}_n = q^*$, it is necessary that the equation $\gamma\varphi'(k) = 1$ had a solution k^*; the existence of this solution is ensured by the hypothesis of the proposition.

In order to have

$$\tilde{k}_{n+1} = \tilde{k}_n = k^*,$$

we must have

$$\varphi(k^*) - \frac{1}{\lambda} W(q^*) = k^*,$$

whence

$$q^* = U'[\lambda(\varphi(k^*) - k^*)].$$

Now, we show that (k^*, q^*) is a saddle point for the Hamiltonian system. We note

$$\tilde{y}_{n-1} = \tilde{q}_n\,, \quad y^* = q^*.$$

The Hamiltonian system becomes:

$$\begin{cases} \tilde{k}_{n+1} = \varphi'(\tilde{k}_n) - \dfrac{1}{\lambda} W(\tilde{y}_{n+1}) \\ \tilde{y}_n = \gamma\varphi'(\tilde{k}_n)\tilde{y}_{n+1} \end{cases}$$

By expanding in Taylor series around the stationary point we have:

$$\begin{cases} k^* + \xi_{n+1} = \varphi(k^*) + \varphi'(k^*)\xi_n - \dfrac{1}{\lambda} W(y^*) - \dfrac{1}{\lambda} W'(y^*)\eta_{n+1} + R_1(\cdot) \\ y^* + \eta_n = [\gamma\varphi'(k^*) + \gamma\varphi''(k^*)\xi_n](y^* + \eta_{n+1}) + R_2(\cdot), \end{cases}$$

whence it follows:

$$\begin{cases} \xi_{n+1} = \dfrac{1}{\gamma}\,\xi_n - \dfrac{1}{\lambda} W'(y^*)\eta_{n+1} + R_1(\cdot) \\ \eta_n = \eta_{n+1} + \gamma\varphi''(k^*)\, y^*\, \xi_n + R_3(\cdot) \end{cases}$$

or

$$\xi_{n+1} = \left[\frac{1}{\gamma} + \frac{\gamma}{\lambda}\,\varphi''(k^*)y^*\, W'(y^*)\right]\xi_n - \frac{1}{\lambda} W'(y^*)\eta_n + R_4(\cdot)$$
$$\eta_{n+1} = -\gamma\varphi''(k^*)y^*\xi_n + \eta_n + R_3(\cdot).$$

The characteristic equation is:

$$s^2 - \left[1 + \frac{1}{\gamma} + \frac{\gamma}{\lambda}\,\varphi''(k^*)y^*W'(y^*)\right]s + \frac{1}{\gamma} = 0.$$

The discriminant of the equation is:

$$\Delta = \left[1 + \frac{1}{\gamma} + \frac{\gamma}{\lambda}\varphi''(k^*)y^*W'(y^*)\right]^2 - \frac{4}{q} > \left(1 + \frac{1}{\gamma}\right)^2 - \frac{4}{\gamma} = \left(1 - \frac{1}{\gamma}\right)^2 > 0.$$

Hence, the characteristic equation has real roots. One shows that $|s_1| > 1$ and $|s_2| < 1$, and therefore the stationary point is a saddle point.

From the qualitative theory of difference equations, we know the following property:

Proposition 12

Through the saddle point (k^*, q^*) there pass two singular trajectories; one of them converges to the stationary point and corresponds to that characteristic root of the liniarized Hamiltonian system, which is less than 1; the other trajectory diverges and corresponds to the other characteristic root (wihich is greater than 1).

Based on the previous propositions, we represent graphically in the plane (k, q) the behaviour of the optimal trajectories for the case $\alpha = 0$ and $\beta = 1$. In order to do this, we shall draw the curves corresponding to equations

$$k_{n+1} = k_n$$

and

$$q_{n-1} = q_n$$

which define the singular point (k^*, q^*).

Equation $k_{n+1} = k_n = k$ is equivalent to $q = U'[\lambda(\varphi(k) - k)]$. The function $q = U'[\lambda(\varphi(k) - k)]$ has a minimum in $\hat{k}$, the solution of equation $\varphi'(k) - 1 = 0$. If we make the additional assumption that $\lim_{c \downarrow 0} U'(c) = \infty$, then we have

$$\lim_{k \downarrow 0} U'[\lambda(\varphi(k) - k)] = \lim_{k \uparrow \bar{k}} U'[\lambda(\varphi(k) - k)] = \infty$$

where $\bar{k}$ is the non-zero solution of equation

$$\varphi(k) - k = 0.$$

We shall refer to the graph of function

$$q = U'[\lambda(\varphi(k) - k)]$$

as the curve of simple reproduction (**SR**) (Fig. 2).

According to Proposition 4'(d), above the curve **SR**

$$q_n > q^0(k_n) = U'[\lambda(\varphi(k_n) - k_n)]$$

we have $\tilde{k}_{n+1} > \tilde{k}_n$, hence the domain above the curve **SR** corresponds to expanded reproduction; the domain below **SR** corresponds to under-reproduction.

Equation $q_{n-1} = q_n = q$ is equivalent to $k = k^*$, if $q \geqslant U'(\lambda h(k))$ and

$$\text{to} \quad q = \frac{\gamma U'[\lambda h(k)]h'(k)}{1 - \gamma g}, \text{if } q < U'(\lambda h(k)).$$

In Fig. 2 we noted by **ZA** the curve which stands for the graph of function $q = U'[\lambda h(k)]$ which limitates from above the domain where $\tilde{v}_n = 0$ (Zero Accumulation); by **QC** (q constant) we noted the graph of function

$$\begin{cases} k = k^*, \text{ for } q \geqslant U'[\lambda h(k)] \\ q = \dfrac{\gamma U'[\lambda h(k)]h'(k)}{1 - \gamma g}, \text{ for } q < U'[\lambda h(k)]. \end{cases}$$

According to Proposition 4' (a) if $k_n < k^*$ and $q_n \geqslant U'(\lambda h(\tilde{k}_n))$, then $q_n < q_{n-1}$ and if $\tilde{k}_n > k^*$ then $\tilde{q}_n > \tilde{q}_{n-1}$.

In Fig. 2 we have drawned the curves SR and QC, which determine in the plane (k, q) four domains. It is also shown the behaviour of the optimal trajectories in each domain, as it results from Proposition 4'.

In the case when the interval of admissible controls is $[\alpha, \beta] \subset [0,1]$, then the domain of attainable optimal trajectories (k_n, q_n) is limited by the curves $q = U'[\lambda(1 - \alpha)h(k)]$ and $q = U'[\lambda(1 - \beta)h(k)]$.

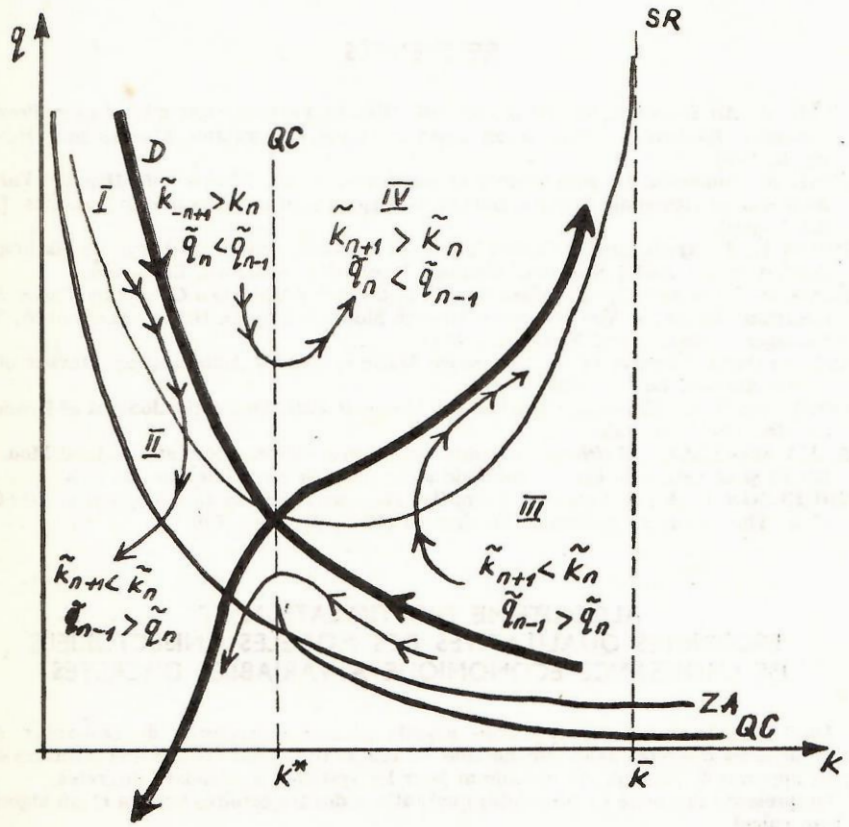


Fig. 2

We note that, when $\alpha > (1-g)\frac{k^*}{h(k^*)}$ or $\beta < (1-g)\frac{k^*}{h(k^*)}$ (where k^* is the solution of equation $\gamma\varphi'(k) - 1 = 0$), the stationary point (k^*, q^*) does not belong to the admissible domain.

We note also that the greater α or the smaller β respectively, the narrower the admissible domain.

In some cases, instead of imposing constraints on the range of admissible controls, we can impose a constraint concerning the minimal acceptable level of per-capita consumption for each period. Such a model should contain the constraint

$$c_n = \lambda(1 - v_n)h(k_n) \geqslant m$$

m being the minimal level to be reached. This constraint being of isoperimetric type, the necessary optimality conditions for the new model can be easily inferred.

As concerns the domain of admissible trajectories, it will be limited from above by the line $q = \overline{m}$, where $\overline{m} = U'(m)$, i.e. the the admissible trajectories will lie in the domain $q \leqslant U'(m)$.

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ALGORITHME D'OPTIMISATION ET PROPRIETES QUALITATIVES DES MODELES UNISECTORIELS DE CROISSANCE ECONOMIQUE A VARIABLES DISCRETES

Dans l'article on fait l'analyse d'un modèle général unisectoriel de croissance économique à variables discrètes ayant un horizon de temps fini. Pour déduire les conditions d'optimum, on applique le principe du maximum pour les systèmes à variables discrètes.

On présente une série de propriétés qualitatives des trajectoires optima et un algorithme pour leur calcul.

La partie finale contient l'analyse du cas d'un modèle autonome.

КАЧЕСТВЕННЫЕ СВОЙСТВА И ОПТИМИЗАЦИОНЫЙ АЛГОРИТМ ДЛЯ ОДНОСЕКТОРНЫХ МОДЕЛЕЙ ЭКОНОМИЧЕСКОГО РОСТА С ДИСКРЕТНЫМИ ПЕРЕМЕННЫМИ

В настоящей статье рассматривается общая модель экономического роста с дискретными переменными и конечным горизонтом.

Для вывода условий оптимальности применяется приниципп максимума для систем с дискретными переменными.

Излагаются ряд качественных свойств оптимальных траекторий и алгоритм для вычисления этих траекторий.

В последней части статьи изучается случай автономной модели.

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