M. VATCU
Department of Economic Cybernetics
Academy of Economic Studies

# A Method of Solving a Linear Programming Problem with Several Objective Functions*)

*The models of linear programming with several objective functions are ever more met with in practice, a fact which has determined the appearance of several methods for solving them.*

*In the present study, the author proposes to use a rapid method for solving such models and in the end, this method is exemplified on a practical problem.*

## GENERAL DESCRIPTION OF THE PROBLEM

In the complexity of economic phenomena and processes, the taking of best decisions implies the analysis of several objectives or variants very often substantiated scientifically with the aid of the linear programming theory.

To describe the general framework which is at the basis of decision making in case of problem of this kind, we shall consider the following:

Consider a linear programming problem with the following constraints:

$$\sum_{j=1}^{n} a_{ij}x_j \lesseqgtr b_i; \quad \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix} \tag{1}$$

$$x_j \geqslant 0; \tag{2}$$

and with $r$ efficiency functions.

$$F_h = \sum_{j=1}^{n} c_{hj}x_j; \quad h = 1, 2, ..., r \tag{3}$$

for which the optimum is sought.

*) In working out the proposed method, I got a substantial help from Mr. V IONESCU and I avail myself of this opportunity to thank him for it.

Each of the $r$ efficiency functions has a distinct economic significance, which distinguishes it from the others and together with constraints (1) and (2) forms a linear programming problem the solving of which allows the obtaining of an optimum from the point of view of the considered function.

The multicriterion linear programming consists in finding a vector

$$X^* = (x_1^*, x_2^*, ..., x_n^*) \tag{4}$$

which should verify system (1) and (2) and give for each efficiency function values having minimum deviations from their optimum. In other words, it is necessary to solve the system composed of constraints (1) and (2) according to a function $F^*$ synthetizing the set of the objectives of functions $F_h$.

## PREMISES OF THE PROBLEM

As it was pointed out, the problem arises to find a vector $X^*$ which should verify system (1) and (2) and ensure the optimum according to the set of the objectives of functions $F_h$.

Reffering to linear programming with several objective functions [3] and [11], the literature of speciality points out that as the $r$ efficiency functions represent distinct sizes, the problem arising, in the first place, is to find a form through which the $r$ functions may be expressed through a common size. This might be achieved by setting up a synthesis function $F^*$, by totalling the functions having a concrete significance, after they had previously been brought to the same common size, after which we pass on to solving the system made up of (1) and (2), according to this function and thus to obtain vector $X^*$.

## USING THE METHOD OF ARRAYS

To exemplify the use of a certain common size for expressing the efficiency functions, we consider to have a linear programming problem made up of constraints (1) and (2) and the linear vectors

$$X = (x_1, x_2, ..., x_5) \tag{5}$$

$$C_1 = (10, 20, 30, 40, 50) \tag{6}$$

$$C_2 = (10, 10, 20, 20, 50) \tag{7}$$

where $X$ represents the vector of the unknown variables

$C_1$ — the coefficients of an efficiency function $F_1$ which must be minimized,

$C_2$ — the coefficients of a function $F_2$ which must also be minimized (optimized).

If we consider and analyse the components of vector $C_1$, we notice that the least value corresponds to variable $x_1$ and represents the smallest increase of function $F_1$ (an objective which interests us), while variable $x_5$ has the hi-

ghest value and respectively the biggest increase of function $F_1$ (an objective which interests us in the least).

Analysing the components of vector $C_2$, we notice that variable $x_1$ and $x_2$ have the lowest and identical values (thus $x_1$ nonpreferential to $x_2$ and vice versa), whereas variable $x_5$ has the maximum value and gives the biggest increase of function $F_2$.

What was noticed above was done in the hypothesis that the type of constraints is entirely excluded and the interpretation took place in a way leading to the attaining of the minimum of functions $F_1$ and $F_2$.

On this basis, it can be said that in solving function $F_1$, variable $x_1$ is most advantageous and is followed by variable $x_2$, is continued with $x_3$, $x_4$ and respectively $x_5$.

Similarly, but based on function $F_2$, variables $x_1$ and $x_2$ are most advantageous and they are followed by variables $x_3$ and $x_4$ ($x_3$ nonpreferential to $x_4$ and inversely) and continued with $x_5$.

In order to argue the use of a comon size for expressing each efficiency function by observing as such the conditions given and underlined, the following is proposed:

a) The separate array of coefficients $C_j$ within each efficiency function and the finding of the highest value $M$, so that we should have:

$$M = max(C_j) \tag{8}$$

If the vector containts negative elements too, we add to the maximum positive value that is denoted now by $M'$ that negative element $m$ which in the modulus has the highest value. In this case:

$$M = M' + m. \tag{9}$$

This was done to establish the whole interval in which the coefficients of the respective function assume values.

b) The approach of a unitary array $r_j$ to each variable $j$ on the basis of the array of coefficients $C_j$ in relation to $M$, where:

$$r_j \in R \text{ and} \tag{10}$$

$$R = (r_1, r_2, ..., r_j, ..., r_n). \tag{11}$$

Table 1 presents the coefficients $C_j$ of functions $F_1$ and $F_2$ as well as the unitary arays of $r_j$ of each variable. It is noticed that the arrays obtained within each function preserve as such the preference of each variable to the other ones.

c) The obtaining of the percentage arrays $r_j^*$ where:

$$r_j^* \in R^* \tag{12}$$

and

$$R^* = (r_1^*, r_2^*, ..., r_j^*, ..., r_n^*) \tag{13}$$

by relating each coefficient $C_j$ to the maximum value $M$, thus:

$$r_j^* = \frac{C_j}{M}. \tag{14}$$

This is necessary because the unitary arrays of the variables give but a preferential order, while the percentage arrays also allow an order of quantitative separation of a variable in relation to the other ones.

By obtaining these percentage arrays, the coefficients of the efficiency functions are only turned into some conventional sizes.

Figures 1 and 2 graphically represent the positions of arrays $r_j^*$ corresponding to each variable in relation to the right parallel to $Ox$ brought through point $y = 1$, which marks the position of the highest array.

d) The replacement of coefficients $C_j$ in each function by the percentage arrays $r_j^*$, so that:

$$\mathcal{F}_h(r_j^*) = F_h(C_j). \tag{15}$$

It is noticed that the solving of any system made up of (1) and (2) according to an efficiency function $\mathcal{F}$ whose elements are the percentage arrays $r_j^*$ allows the obtaining of the same solution as in solving the system with the efficiency function initial with coefficients $C_j$. Table 1 also includes these percentage arrays. It should be pointed out that the unitary arrays $r_j$ have but an analysis significance whereas the percentage arrays $r_j^*$ also have a calculus significance. Thus, by changing the coefficients $C_j$ of function $F_2$, the same unitary arrays are obtained in table 2, by changing only the percentage arrays and implicitely their sum vector.

Table 1

| Variables / Symbols | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| Coefficients of function $F_1$ | $c_{1j}$ | 10 | 20 | 30 | 40 | 50** |
| Coefficients of function $F_2$ | $C_{2j}$ | 10 | 10 | 20 | 20 | 50** |
| Unitary arrays for $F_1$ | $r_{1j}$ | 1 | 2 | 3 | 4 | 5 |
| Unitary arrays for $F_2$ | $r_{2j}$ | 1 | 1 | 2 | 2 | 3 |
| Percentage arrays for $F_1$ | $r_{1j}^*$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| Percentage arrays for $F_2$ | $r_{2j}$ | 0.2 | 0.2 | 0.4 | 0.4 | 1.0 |
| Sum of percentage arrays $\mathcal{F}^*$ | $\sum_{i=1}^{2} r_{ij}^*$ | 0.4 | 0.6 | 1.0 | 1.2 | 2.0 |

*Table* 2

| Variables / Symbols | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| Coefficients of function $F_1$ | $C_{1j}$ | 10 | 20 | 30 | 40 | 50** |
| Coefficients of function $F_2$ | $C_{2j}$ | 100 | 100 | 200 | 200 | 300** |
| Unitary arrays for $F_1$ | $r_{1j}$ | 1 | 2 | 3 | 4 | 5 |
| Unitary arrays for $F_2$ | $r_{2j}$ | 1 | 1 | 2 | 2 | 3 |
| Percentage arrays for $F_1$ | $r^*_{1j}$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| Percentage arrays for $F_2$ | $r^*_{2j}$ | 0.333 | 0.333 | 0.666 | 0.666 | 1.0 |
| Sum of percentage arrays $\mathcal{F}^*$ | $\sum_{i=1}^{2} r^*_{ij}$ | 0.533 | 0.733 | 1.266 | 1.466 | 2.0 |

** repesents $M = max\ (C_j)$

*Table* 3

| Efficiency function | Minimum value | Maximum value | $\Delta_r = (max - min)$ | $\alpha_h$ coefficient | Importance coefficients |
|---|---|---|---|---|---|
| $F_1$ | 1 079 879 789 | 1 253 232 488 | 173 352 699 | 0,0000000057 | 0,5 |
| $F_2$ | 2 980 | 6 610 | 3 630 | 0,00027 | 0,2 |
| $F_3$ | 29 488 327 | 37 731 822 | 8 243 495 | 0,00000012 | 0,3 |

Returning to the problem of linear programming with several objective functions, the proposed method consists in replacing coefficients $C_j$ of each function with a concrete significance by adequate percentage arrays $r^*_j$ and their totalling with a view to obtaining a vector $\mathcal{F}^*$ which represents the synthesis function according to which the system is solved.

Each $\mathcal{F}^*$ obtained is given in tables 1 and 2. According to figures 1 and 2, this means the totalling of the dotted segments described by the arrays corresponding to each variable and through the minimization of the newly formed vector, the minimum deviation is ensured from the abscissa and which corresponds to the theoretical minimum of the solution. That is:

$$min\mathcal{F}^* = [\Sigma(h^*_{1j} - 0)x_j + \Sigma(r^*_{2j} - 0)x_j] = \sum_{h=1}^{2}\sum_{j=1}^{n} r^*_{hj}x_j. \qquad (16)$$

This way of calculus is achieved in case that the user did not specify and preference order for a certain efficiency function.

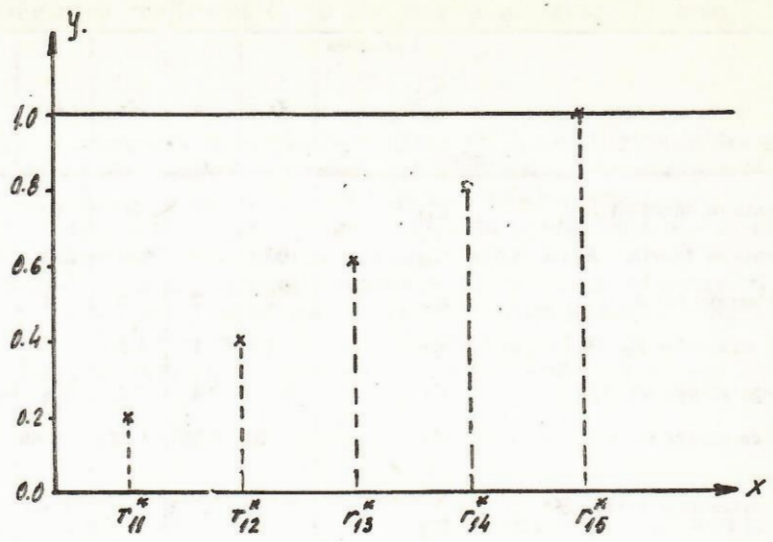


*Fig. 1*

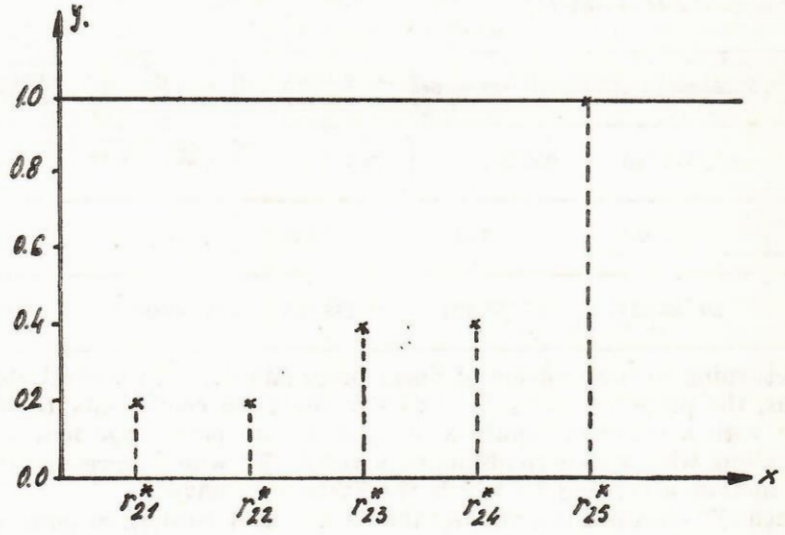


*Fig. 2*

In case the user wants a preference order of the efficiency functions, he must provide importance coefficients $\lambda_h$ for each function. Thus $F_1$, $F_2$,..., $F_h$, ..., $F_l$, for the efficiency functions the importance coefficients will, be $\lambda_1$, $\lambda_2$, ..., $\lambda_h$, ..., $\lambda_l$ where:

$$\sum_{h=1}^{l} \lambda_h = 1. \tag{17}$$

Table 4

| | | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ | $X_{16}$ | $X_{17}$ | $X_{18}$ | $X_{19}$ | $X_{20}$ | $X_{21}$ | $X_{22}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_1$ | $C_{1j}$ | 855 | 845 | 810 | 820 | 820 | 810 | 850 | 841 | 934 | 885 | 865 | 935** | 905 | 850 | 862 | 885 | 900 | 880 | 910 | 870 | 825 | 885 |
| $F_2$ | $C_{2j}$ | 12 | 15 | 20 | 20 | 20 | 30 | 22 | 26 | 31 | 70** | 50 | 38 | 70** | 30 | 40 | 10 | 10 | 30 | 30 | 30 | 40 | 30 |
| $F_3$ | $C_{3j}$ | 22,5 | 22 | 21 | 24,5 | 21 | 23 | 25 | 28,1 | 36,2** | 22 | 24 | 24,5 | 25 | 23 | 23,6 | 28 | 29 | 23,5 | 24,5 | 23,5 | 24,5 | 30,2 |
| $F_1$ | $r_{1j}$ | 7 | 5 | 1 | 2 | 2 | 1 | 6 | 4 | 16 | 12 | 9 | 17 | 14 | 6 | 8 | 12 | 13 | 11 | 15 | 10 | 3 | 12 |
| $F_2$ | $r_{2j}$ | 2 | 3 | 4 | 4 | 4 | 7 | 5 | 6 | 8 | 12 | 11 | 9 | 12 | 7 | 10 | 1 | 1 | 7 | 7 | 7 | 10 | 7 |
| $F_3$ | $r_{3j}$ | 3 | 2 | 1 | 8 | 1 | 4 | 9 | 11 | 14 | 2 | 7 | 8 | 9 | 4 | 6 | 10 | 12 | 5 | 8 | 5 | 8 | 13 |
| $F_1$ | $r^*_{1j}$ | 0,91443 | 0,90374 | 0,86631 | 0,87700 | 0,87700 | 0,86631 | 0,90909 | 0,89946 | 0,99893 | 0,94652 | 0,92513 | 1,00000 | 0,96791 | 0,90909 | 0,92192 | 0,94652 | 0,96256 | 0,94117 | 0,97326 | 0,93048 | 0,88235 | 0,94652 |
| $F_2$ | $r^*_{2j}$ | 0,17142 | 0,21428 | 0,28571 | 0,28571 | 0,28571 | 0,42875 | 0,31428 | 0,37142 | 0,44285 | 1,00000 | 0,71428 | 0,54285 | 1,00000 | 0,42875 | 0,57142 | 0,14285 | 0,14285 | 0,42875 | 0,42875 | 0,42875 | 0,57142 | 0,42875 |
| $F_3$ | $r^*_{3j}$ | 0,62154 | 0,60773 | 0,58011 | 0,67679 | 0,58011 | 0,63635 | 0,69060 | 0,77624 | 1,00000 | 0,60773 | 0,66298 | 0,67679 | 0,69060 | 0,63535 | 0,65193 | 0,77348 | 0,80110 | 0,64917 | 0,67679 | 0,64917 | 0,67679 | 0,83425 |
| $\mathcal{F}^*_1$ | $\sum_{h=1}^{n} r^*_{hj}$ | 1,70739 | 1,72575 | 1,73213 | 1,83950 | 1,74282 | 1,93141 | 1,91397 | 2,04712 | 2,44178 | 2,55425 | 2,30239 | 2,21964 | 2,65851 | 1,97319 | 2,14527 | 1,86285 | 1,90651 | 2,01909 | 2,07880 | 2,00840 | 2,13046 | 2,20952 |
| $\mathcal{F}^*_2$ | $\sum_{h=1}^{n} \lambda_h \cdot r^*_{hj}$ | 0,57796 | 0,57704 | 0,66433 | 0,69868 | 0,66967 | 0,70947 | 0,72458 | 0,75689 | 0,88693 | 0,85558 | 0,80431 | 0,81161 | 0,89113 | 0,79086 | 0,77082 | 0,73367 | 0,75018 | 0,75105 | 0,77538 | 0,74570 | 0,75850 | 0,80325 |
| $\mathcal{F}^*_2$ | $r_j$ | 4 | 3 | 1 | 5 | 2 | 6 | 7 | 13 | 21 | 20 | 17 | 19 | 22 | 8 | 15 | 9 | 11 | 12 | 16 | 10 | 14 | 18 |

Table 5

| Efficiency function | Minimum value | $F_h$ minimum after $F_1^*$ function | Relative deviation of $F_h$ as against minimum | $F_h$ minimum after $F_2^*$ function | Relative deviation of $F_h$ as against minimum | $F_h$ minimum after $\mathcal{F}$ function | Relative deviation of $F_h$ as against minimum | $F_h$ minimum after $\mathcal{F}_2^*$ function | Relative deviation of $F_h$ as against minimum |
|---|---|---|---|---|---|---|---|---|---|
| $F_1$ | 1079879789 | 1082950282 | 1.00284 | 1080706080 | 1.00076 | 1100345250 | 1.01895 | 1082950281 | 1.00284 |
| $F_2$ | 2980 | 3361 | 1.12785 | 3484 | 1.16912 | 3033 | 1.01778 | 3361 | 1.12785 |
| $F_3$ | 29488327 | 29673507 | 1.00627 | 29585330 | 1.00328 | 30074590 | 1.01988 | 29673507 | 1.00627 |

Within the synthesis function $\mathcal{F}^*$, these coefficients have a significance of increasing the importance of some functions of concrete efficiency to the detriment of the other ones, by moderating each percentage array $r_j^*$ with the coefficient $\lambda_h$ corresponding to the respective function, thus:

$$\mathcal{F}_h = \lambda_h \cdot \mathcal{F}_h = \sum_{j=1}^{n} \lambda_h r_{hj}^* x_j \tag{18}$$

and

$$\mathcal{F}^* = \sum_{h=1}^{l} \mathcal{F}_h = \sum_{h=1}^{l} \sum_{j=1}^{n} \lambda_h r_{hj}^* x_j. \tag{19}$$

On the basis of these considerations, the solving of a linear programming problem can be done in the following steps:

Step 1. The transformation and replacement of coefficients $C_j$ corresponding to each efficiency function by the respective percentage arrays $r_j^*$.

Step 2. The moderating of functions $\mathcal{F}_h$ with the importance coefficients $\lambda_h$. This step is eliminated in case the user wishes an equal preference for all the efficiency functions with a concrete significance.

Step 3. The totalling of functions $\mathcal{F}_h$ and the obtaining of the synthesis function $\mathcal{F}^*$.

Step 4. The solving of the linear programming system made up of (1) and (2) with a synthesis function $\mathcal{F}^*$.

## A PRACTICAL EXAMPLE

In 1974, as part of the technical-scientific collaboration with the Bucharest INCERC Institute, the latter asked the Laboratories of the Department of Economic Cybernetics within the Academy of Economic Studies to solve on the computer a model of linear programming with three objective functions, through the method of global utility maximization. Concurantly with the solving of the model through the requested method, the method described above was also used.

The problem consisted of a system of constraints of type (1) and (2) with the efficiency functions $F_1$ minimum cost price, $F_2$ minimum consumption of steel and $F_3$ minimum consumption of wood.

The unkonown values of the system $x_1,\ldots,x_{22}$ represent the floor space for various sorts of apartment houses (improved comfort apartments, comfort A, with one or several rooms, on types of construction, etc.).

The mathematical model was solved with the aid of the library programme the LINEAR PROGRAMMING SYSTEM, a program which was run on the electronic computer IBM-360 installed within the Laboratories of the Department of Economic Cybernetics.

According to the requirements of the requested method and on the basis of the calculations in table 3, the minimum and maximum values are given of each efficiency function, the differences between the minimum and maximum values noted with $\Delta_h$ and the coefficients $\alpha_h$.

The importance coefficients $\lambda_h$ of each function, supplied by the beneficiary are presented in the last column of the table.

According to the method proposed, coefficients $C_j$ of each efficiency function are given in table 4, the unitary arrays, the percentage arrays, the synthesis function $\mathcal{F}_1^*$ which represents the sum of the percentage arrays not moderated with coefficients $\lambda_h$, the synthesis function $\mathcal{F}_2^*$ which represents the sum of the percentage arrays moderated with coefficients $\lambda_h$ as well as the unitary arrays of $\mathcal{F}_2^*$ which illustrates the order of each variable in the solution, but, taking into consideration the set of the three efficiency functions.

Table 5 presents the solutions obtained as a result of the solving of the system according to $F_1^*$ and $F_2^*$ through the method of global utilitay maximization and the solutions obtained after the system was solved according to $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$, through the method proposed.

The synthesis functions $F_1^*$ and $F_2^*$ are the functions not moderated with the importance coefficients $\alpha_h$. The solving of the system was done with the aid of this function with a view to facilitating the analysis and comparison of the results obtained in the two ways.

On the basis of the data comprised in this table, we can notice the following:

— the relative deviation of each function from the minimum possible according to the synthesis function $\mathcal{F}_1^*$, by way of comparison with the solutions obtained according to function $F_1^*$, is greater at function $F_1$ and $F_3$, but smaller at $F_2$;

— the relative deviations according to function $\mathcal{F}_1^*$ from the minimum possible are more homogenous and under 2%;

— the solutions obtained after $\mathcal{F}_2^*$ are identical with those obtained after function $F_1^*$.

Taking into consideration the tests made with the proposed method, and other exemples the following is estimated:

— it is a relatively simple method;

— it requires to solve the mathematical model only once;

— it is easily adaptable for solving on the computer;

— it does not require the introduction of additional constraints for cases of unlimited optimum;

— it allows the summing of the efficiency functions with a concrete significance into a synthesis function through relative sizes;

— for solving problems of the type presented, it resorts to less steps in contradistinction to other methods.

## REFERENCES


1. BERNARD ROY: *Décisions avec critères multiples* Problèmes et méthodes. METRA. no. I 1972.
2. BOLDUR G.: *Procese informaționale și de decizie.* (Information and Decision-Making Processes). Editura științifică, Bucharest, 1969.
3. BOLDUR G., STANCU-MINASIAN I. M.: *Programarea liniară cu mai multe funcții obiective* (Linear Programming with Several Objective Functions), Studii și cercetări matematice, no. 8, 1972.

4. BOLDUR G., CIOBANU G., RATIU-SUCIU C., STANCU-MINASIAN I. M.: *Cercetarea operațională cu aplicații în economie* (Operations Research with Applications in Economy), first part, notes internal usage, The Academy of Economic Studies, 1973.
5. FRATILA G., NASTASE A., CRISTUINEA M.: *Optimizarea planurilor de producție în agricultură* (Optimizing the Production Plans in Agriculture), Revista de statistică, no. 8, 1973.
6. KAUFMANN A.: *Metode și modele ale cercetării operaționale.* (Operations Research Methods and Models), Vol. I and II, Editura științifică, Bucharest, 1967.
7. MARINESCU I., MOINEAGU C., NICULESCU R., URSEANU V.: *Elemente de statistică matematică și aplicațiile ei* (Fundamentals of Mathematical Statistics and its Applications), Editura științifică, Bucharest, 1966.
8. OSKAR LANGE: *Decizii optime* (Optimum Decisions), Editura științifică, Bucharest, 1970
9. PASTERNAK HANOC, PASSY, URY: *Finding Global Optimum of Bicriterion Mathematical Programs.* Cahiers du centre d'etudes de recherches operationnelle, no. 1, 1974.
10. RESA I. D., LADAR L.: *Considerații privind cumularea criteriilor de optimizat în programarea liniară* (Considerations on the Cumulation of Optimizing Criteria in Linear Programming), Revista de statistică, no. 8, 1973.
11. TEODORESCU, M., BOLDUR G., STOICA M. STANCU-MINASIAN I. M., BANCILA I: *Metode ale cercetării operationale în gestiunea întreprinderilor* (Operations Research Methods in Enterprise Administration), Editura tehnică, Bucharest, 1972.

## METHODE DE RESOLUTION D'UN PROBLEME DE PROGRAMMATION LINEAIRE A PLUSIEURS FONCTIONS OBJECTIVES

Les modèles de programmation linéaire à plusieurs fonctions objectives sont toujours plus fréquents dans la pratique, ce qui détermine l'apparition de plusieurs méthodes pour leur résolution.

L'auteur propose, dans le présent ouvrage, l'utilisation d'une méthode rapide pour résoudre cette sorte de modèles, en l'illustrant d'exemples tirés d'un problème pratique.

## МЕТОД РЕШЕНИЯ ЗАДАЧИ ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ С НЕСКОЛЬКИМИ ЦЕЛЕВЫМИ ФУНКЦИЯМИ

Модели линейного программирования с несколькими целевыми функциями все чаще встречаются на практике, что обусловило появление различных методов их решения.

В настоящей статье автор предлагает применение быстрого метода решения подобных моделей, а в заключительной части статьи дается пример применения этого метода к практической задаче.

28.01.1975