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On Optimum Potentials

Necessary and sufficient conditions for a finite solution of the optimum potential problem are given for the following cases: 1) variation intervals of potentials are unlimited: 2) certain variation intervals are limited.

Let us consider the system of inequalities

$$x_j - x_i \geqslant k_{ij}, \quad (i,j) \in K \subset N \times N, \tag{1}$$

where $N = \{1, 2, ..., n\}$ and K are given sets and k_{ij} given real numbers. We know the necessary and sufficient condition for the compatibility of system (*1*) ([1], [2]).

Supposing (*1*) is compatible, let us note by $X = \{x_i, i \in N\}$ one of these solutions (problem of potential). B. Roy points out an algorithm ([1], [2]) for the solution $X = \hat{X}$ of system (*1*) which minimizes a given function:

$$f(X) = \sum_{i \in N} h_i x_i \quad h_i \text{ real (problem of optimum potential)} \tag{2}$$

B. Roy also gives a necessary condition for the existence of a finite minimum of $f(X)$ (*f min*):

$$\sum_{i \in N} h_i = 0 \tag{3}$$

This condition is not sufficient. Here is the simplest counter-example. Let us consider the unknowns x_1 and x_2 meeting the condition

$$x_2 - x_1 \geqslant 1$$

We find their values minimizing the function

$$f = x_1 - x_2$$

Now the numbers

$$x_1 = -k \ (k \text{ arbitrary positive})$$
$$x_2 = 1$$

verify the condition given and we obtain

$$f = -(1 + k), \text{ as small as required.}$$

Roy's algorithm acknowledges such a case. However, a sufficient (and necessary) condition for the existence of f_{min} would be useful, for it would allow us to avoid the rather slow and toilsome computations of the algorithm. We shall indicate this condition.

We shall previously set forth certain preliminary considerations.

Definition. Let us consider graph $G\ [N, K]$ and one of its subgraphs $G'\ [N', K']$. Let $\overline{K} = K - K'$. The subset N' is called an anterior (posterior) part of N in graph G, if there are no inward (outward) incident arcs to N' proceeding from K. *

Let A be an anterior part of N. Then $P = N - A$ is a part posterior of N and corresponding to A. It is obvious that N is an anterior as well as a posterior part of N. Therefore, any graph should have at least one anterior and one posterior non-void parts.

We shall attach graph $G\ [N,K]$ to system (*1*), as follows:

the vertices of G are marked by the numbers $i \in N$;

the arcs of G are marked by ordinated couples of numbers $(i,j) \in K$;

we assign to arc (i,j) the length k_{ij}.

We know that if system (*1*) is compatible, it is equivalent to system

$$x_j - x_i \geqslant l_{ij} \cdot (i,j) \in L \tag{4}$$

with

$$K \subset L \subset N \times N$$

Here the couple (i,j) belongs to set L if and only if there is a path in G connecting vertex i with vertex j. The number l_{ij} is equivalent to the length of the longest path from i to j.

LEMMA 1. Let A be an anterior part of N and X a solution to (1). By setting

$$x'_i = \begin{cases} x_i - k, i \in A \\ x_i, i \in N \setminus A \end{cases} \tag{5}$$

with k arbitrary non-negative number, vector $X' = \{x_i \mid i \in N\}$ *also becomes a solution to system* (*1*)

Analogically, we establish the formula corresponding to (*5*) for each posterior part P of N:

$$x'_i = \begin{cases} x_i + k, i \in P \\ x_i,\ i \in N \setminus P \end{cases} \tag{5 bis}$$

with k arbitrary non-negative number.

Proof. We have to prove that for any $(i,j) \in K$

$$x_j - x_i \geqslant k_{ij} \Rightarrow x'_j - x'_i \geqslant k_{ij}$$

If $i \in A$, $j \in A$ or $i \notin A$, $j \notin A$, then

$$x'_j - x'_i = x_j - x_i \geqslant k_{ij}$$

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* The set of arcs K' constitutes in this case a co-circuit

If $i \in A$, $j \notin A$, we have

$$x'_j - x'_i = x_j - x_i + k \geqslant 0$$

The case $i \notin A$, $j \notin A$ is not possible, for then A would not be an anterior part.

Theorem 1. *In order to have f_{min} it is necessary and sufficient that for any anterior part A of N, we should have*:

$$\sum_{i \in A} h_i \leqslant 0 \tag{6}$$

and that for any posterior part P of N we should have:

$$\sum_{i \in P} h_i \geqslant 0 \tag{7}$$

Proof. The condition is necessary. For indeed, if X is a solution to system (*1*), then by calculating another solution X' with the aid of formula (*5*), we obtain:

$$f(X') = f(X) - k \sum_{i \in A} h_i$$

If condition (*6*) is broken — that is to say if $\Sigma_{i \in A} h_i > 0$ — then function $f(X)$ has no inferior limit. Proof for formula (*7*) is analogous. We shall mention that, if we first consider set N as an anterior part, then as a posterior part, condition (*3*) results from formulae (*6*) and (*7*). Likewise, conditions (*3*) and (*6*) imply condition (*7*). etc. When applying the theorem, it would be more suitable to use (*3*) and (*6*), or else (*3*) and (*7*) instead of (*6*) and (*7*). In the subsequent proceedings, we shall consider conditions (*3*) and (*7*) as verified.

The condition is sufficient. We shall write function (*2*) as follows:

$$f(X) = -\sum_{\alpha \in a} c_\alpha x_\alpha + \sum_{\beta \in P} c_\beta x_\beta, \tag{8}$$

where $P \cup Q = N$, $P \cap Q = \emptyset$ and $c_i \geqslant 0$, $i \in N$, with

$$\sum_{\alpha \in a} c_\alpha = \sum_{\beta \in P} c_\beta \tag{9}$$

Variables x_i, $i \in N$ verify the inequalities (*4*). We attach to them graph $G[N,L]$ which is the transitive closure of the graph associated with system (*1*).

We note by

$S(i) = \{j \mid (i, j) \in L\}, i \in N$ — the set of vertices following vertex i in graph $G[N,L]$

$V(i) = \{j \mid (j, i) \in L\}, i \in N$ — the set of vertices preceding vertex i in graph $G[N,L]$

We shall call the arcs (i, j), $i \in Q$, $j \in P$ open arcs, while all the others will be closed arcs.

Having $q \subset Q$, we note by

$$p(q) = \{\beta(\alpha, \beta) \in L,\ \beta \in P,\ \alpha \in Q\} \subset P$$

the representation of set q given by the multivo que applications generated by the open arcs. We shall call set $n = p \cup q \subset N$ a simple part of N in graph G. We shall prove:

LEMMA 2. *If the conditions of theorem 1 are verified by the coefficients c of function f(X), then for any simple part of N.n = q∪p(q) we shall have:*

$$-\sum_{\alpha \in q} c_\alpha + \sum_{\beta \in p(q)} c_\beta \geqslant 0 \qquad (10)$$

Proof. Suppose there is a simple part n in graph G for which relation (*10*) is not verified. Let us have $qccQ$, as in the opposite case we should have $q = Q, p(q) = P$ and condition (*10*) would then be verified (see (9)).

1. If there is no outward incident arc to n, then n is not a posterior part and condition (7) is not satisfied. Hence the lemma is true for this case.

2. In the opposite case, these arcs would be inward incident arcs to $Q - q$. For the existence of an incident arc to $P - p$ supposes the existence of a path going from q to $P - p$, which contradicts the definition of representation p.

We shall therefore suppose that there are outward incident arcs to n and inward incident arcs of a subset $q_0 c Q$, $q_0 \cap q = \emptyset$. We shall have for $n' = n \cup q_0$ ($n' = p \cup q'$, $q' = q \cup q_0$):

$$-\sum_{\alpha \in q'} c_\alpha + \sum_{\beta \in p} c_\beta \leqslant -\sum_{\alpha \in q} c_\alpha + \sum_{\beta \in p} c_\beta < 0$$

2.1. Subset n' is a posterior part. Then condition (7) is not verified and the lemma is true.

2.2. Subset n' has outward incident arcs. These are not inward incident arcs to Q (that is to $Q - q'$) for then a path connecting q with $Q - q$ would exist, which would contradict definition q_0. Hence these arcs should be inward incident arcs to P (that is to $P - p$). But then there would be a path going from q to $P - p$, in contradiction with the definition of set $p(q)$. Therefore this case is not possible. The lemma is proved.

We draw up a transport nework R in graph $G[N,L]$ as follows:

entry vertex a

entry arcs (a,α), $\alpha \in Q$ provided with capacities $C\alpha$

exit vertex b

exit arcs (β, b), $\beta \in P$ provided with capacities $C\beta$

other vertices $i \in N$

other arcs $(i, j) \in L$ endowed with infinite capacities for open arcs and with null capacities for closed arcs.

We shall draw up a maximum flow $\Phi = \{\varphi ij \mid (i, j) \in L'\}$ $(L' = L \cup \cup \{(a, \alpha) \mid \alpha \in Q\} \cup \{(\beta, b) \mid \beta \in P\}$ in network R saturating the entry arcs (and of course the exit arcs as well).

For this flow we get:

$$c_\alpha = \sum_{i \in S(\alpha)} \varphi_{\alpha i} - \sum_{i \in V(\alpha)} \varphi_{i\alpha},\ \alpha \in Q$$

$$c_\beta = \sum_{i \in V(\beta)} \varphi_{i\beta} - \sum_{i \in S(\beta)} \varphi_{\beta i},\ \beta \in P$$

Considering that we have $\varphi_{ij} = 0$ for the closed arcs, we obtain

$$\begin{cases} c_\alpha = \sum\limits_{\beta \in P \cap S(\alpha)} \varphi_{\alpha\beta}, \ \alpha \in Q \\ c_\beta = \sum\limits_{\alpha \in Q \cap V(\beta)} \varphi_{\alpha\beta}, \ \beta \in P \end{cases} \tag{11}$$

There is Gale's criterium ([1], [2]) for the existence of such a flow. We are going to use it later on, but we shall previously suppose that flow Φ exists.

Taking into account formulas (*11*), we can write down

$$c_\alpha x_\alpha = \sum_{\beta \in P \cap S(\alpha)} \varphi_{\alpha\beta} x_\alpha , \ \alpha \in Q \tag{12}$$

$$c_\beta x_\beta = \sum_{\alpha \in a \cap V(\beta)} \varphi_{\alpha\beta} x_\beta , \ \beta \in P. \tag{13}$$

By virtue of inequalities (*4*) we have

$$x_\beta - x_\alpha \geqslant l_{\alpha\beta}, \ (\alpha, \beta) \in L \cap (Q \times P) = L_0$$

It therefore results from formula (*13*):

$$c_\beta x_\beta \geqslant \sum_{\alpha \in Q \cap V(\beta)} \varphi_{\alpha\beta} x_\alpha + C', \tag{14}$$

C' being a constant which does not depend on the values of x_i. By using formulae (*12*) and (*14*), we obtain from (*8*)

$$f(X) \geqslant g(X) = \sum_{\beta \in P} \sum_{\alpha \in Q \cap V(\beta)} \varphi_{\alpha\beta} x_\alpha - \sum_{\alpha \in Q} \sum_{\beta \in P \cap S(\alpha)} \varphi_{\alpha\beta} x_\alpha + C = P(X) - Q(X) + C,$$

where C is another constant.

But the two functions $P(X)$ and $Q(X)$ are identical since they are equal with the expression:

$$\sum_{(\alpha,\beta) \in L_0} \varphi_{\alpha\beta} x_\alpha$$

resulting from it by reversing the order of the addition.

We have consequently obtained

$$f(X) \geqslant C$$

which proves the existence of a finite f_{min}, if we prove that flow Φ exists.

Gale's condition, necessary and sufficient for the existence of a flow saturating the entry arcs requires that

$$e(q) \leqslant s(q), \text{ for any } q \subset Q.$$

Here $e(q)$ is the maximum flow coming out from the entry vertex which can enter q;

$s(q)$ is the maximum flow coming out from q, which can enter the exit vertex.

In this case we have $e(q) = \sum_{\alpha \in q} c_\alpha$ and $s(q) = \sum_{\beta \in p(q)} c_\beta$, that is:

$$-\sum_{\alpha \in q} c_\alpha + \sum_{\beta \in p(q)} c_\beta \geqslant 0.$$

This relation being verified on the ground of lemma 2, it comes out that flow $\varnothing$ exists. We have thus proved that conditions (*6*) and (*7*) are sufficient. Theorem 1 is proved.

The problem we have studied so far has an extension: if the interval of variation of some of the variables x_i is limited, then we have to add to system (*1*) a supplementary condition: the value of one of the unknowns is established.

Let us consider the system

$$\begin{cases} x_j - x_i \geqslant k_{ij}, \ (i, j) \in K \subset N \times N \\ x_1 = 0 \end{cases} \tag{14}$$

We shall presume that it is compatible and note by $\overline{X} = \{x_i, \ i \in N\}$ one of its solutions. Being given the function:

$$f(\overline{X}) = \sum_{i \in N} h_i x_i \tag{15}$$

we seek the solution $\overline{X} = \hat{X}$ of system (*14*) which minimizes function $f(\overline{X})$.

We first notice that here lemme 1 is valid under the following form:

LEMMA 3. Let $\overline{X}$ be a solution to system (14) and A an anterior part of N in graph $G[N,K]$ which does not contain vertex 1. If

$$x_i' = \begin{cases} x_i - k, i \in A \\ x_i, \ i \in N \setminus A \end{cases}$$

with an arbitrary non-negative number k, vector $\overline{X}' = \{x_i' \mid i \in N\}$ is also a solution to (14).

We shall proceed analogically for a posterior part of N.

We notice that set $\bar{x} = \{\overline{X}\}$ of the solutions to system (*14*) is contained in set $\bar{x} = \{\overline{X}\}$ of the solutions to system (*1*). Therefore, if function $f(X)$ is limited on its inferior level on x, it is also limited on $\bar{x}$. Hence, conditions (*6*), (*7*) are also sufficient for the existence of a minimum limited on $\bar{x}$ of the function $f(\overline{X})$.

However, less restrictive conditions can be formulated.

Theorem 2. *In order that function $f(X)$ have an inferior limit on the set of solutions to system (14), it is necessary and sufficient to have, for any anterior part A of N which does not contain vertex 1:*

$$\sum_{i \in A} h_i \leqslant 0 \tag{16}$$

and for any posterior part P of N which does not contain vertex 1:

$$\sum_{i \in P} h_i \geqslant 0 \qquad (17)$$

A direct consequence of this theorem is the case in which there are no anterior or posterior arcs which do not contain vertex 1. In this case, the minimum of $(f(\overline{X})$ exists, with no restrictions for coefficients h_i.

Proof.

Conditions (16), (17) are necessary.
This is proved by a reasoning analogous to that proving theorem 1, but this time we take lemma 3 as a basis.

The conditions are sufficient. We shall prove that, if conditions (*16*) and (*17*) are fulfilled, conditions (*6*) and (*7*) are verified as well and then function $f(\overline{X})$ admits, by virtue of theorem 1, a finite minimum.

We shall previously make three remarks (which are obvious, in fact):

1. We can assign to coefficient h_1 an arbitrary value. Condition (*3*) is thus verified, in any case.

2. If set $M_1 \subset N' = N - \{1\}$ is an anterior (posterior) part of N, then set $M_2 = N \setminus M_1$ $(1 \in M_2)$ is a posterior (anterior) part of N and vice-versa.

3. If we note by $C = \{M_1\}$ the class of all anterior (posterior) parts of N which do not contain vertex 1, then $C' = \{M_2\}$ is the class of all posterior (anterior) parts of N which do not contain vertex 1.

We suppose, on the ground of remark 1 that

$$\sum_{i \in N} h_i = 0$$

If M_1 is an anterior (posterior) part of N which does not contain vertex 1 and for which condition (*16*) (*17*) is fulfilled, then $M_2 = N - M_1$ is a posterior (anterior) part of N containing vertex 1 (remark 2) and for which condition (*7*) (*6*) is fulfilled, the sum of the coefficients h_1 being null. This condition is then verified, by virtue of remark 3, for any posterior (anterior) part containing vertex 1.

The first enunciation of the proof is true then, if there are anterior or posterior parts which do not contain vertex 1. If the only anterior or posterior part which does not contain vertex 1 is the void set, then the only non-void posterior and anterior part is set N itself, for which conditions (*6*) and (*7*) are fulfilled. Theorem 2 is proved.

As an application of theorem 2, we shall mention the following particular case which occurs sometimes: vertex 1 is an anti-basis (that is to say that there is a path from vertex 1 to any vertex $i \in N'$) and we have $(x, i \in N)$ as a solution to system (*14*)

$$f(X) = \sum_{i \in N} c_i x_i, \quad c_i \geqslant 0$$

In this case, since there are no anterior parts which do not contain vertex 1, and conditions (*17*) being fulfilled, the minimum of function $f(X)$ exists and we do not require Roy's algorithm to achieve this minimum: we obtain it through the universal minimum solution ([1], [2]) of the problem of potential we have considered.

REFERENCES

1. BERGE, C. and GHOUÏLA-HOURI, A., *Programmes. Jeux et réseaux de transport*, Dunod, Paris, 1962.
2. ROY, B., *Cheminement et connexité dans les graphes. Application aux problèmes d'ordonnancement*. Thèse, Paris. 1962.

SUR LES POTENTIELS OPTIMAUX

B. Roy a étudié le problème suivant:
Considérons le système d'inégalités

$$x_j - x_i \geqslant k_{ij},\ (i, j) \in K \subset N \times N,$$

où $N = (1, 2, \ldots, n)$ et K sont des ensembles données et k_{ij} des nombres réels donnés. Supposons que le système soit compatible et notons une de ses solutions par $X = \{x_i, i \in N\}$.

On donne la fonction linéaire $f(X) = \sum_{i \in N} h_i x_i$, ($h_i$ des nombres réels) et on cherche la solution $X = \hat{X}$ du système (1) pour laquelle $f(X)$ devient minimum. C'est toujours B. Roy qui a indiqué un algorithme pour trouver la solution du problème qui conduit soit à la solution recherchée, soit à la constatation qu'il n'existe pas une solution finie (c'est-à-dire que $f(X)$ n'est pas bornée inférieurement sur l'ensemble des solutions X du système (1).

Ayant en vue cette possibilité du deuxième cas, un théorème concernant l'existence d'un minimum fini de la fonction $f(X)$ peut être utile.

La condition nécessaire pour l'existence de $f_{min} \sum_{i \in N} h_i = 0$ est bien connue.

Dans l'étude présentée, on donne la condition nécessaire et suffisante pour l'existence de f_{min}, tout en démontrant en même temps que la dernière relation dérive de la condition donnée par nous-mêmes.

ОБ ОПТИМАЛЬНЫХ ПОТЕНЦИАЛАХ

Б. Рой исследовал следующую задачу:
Предположим, что дана система неравенств

(1) $$x_j - x_i \geqslant k_{ij}, (i,j) \in k \subset N \times N,$$

где $N = \{1,2, \ldots n,\}$ и K являются данными совместностями и k_{ij} — данными реальными числами. Предположим, что система является совместимой и запишем ее решение $X = \{x_i / i \in N\}$. Дана линейная функция $f(X) \sum_{i \in N} h_i x_i$, ($h$ реальные числа) и нужно найти такое решение $X = \hat{X}$ системы (1) для которого $f(X)$ становится минимальной. Б. Рой также указал алгоритм для нахождения решения этой задачи, который приводит к искомому решению, или в выводу о том, что нет законченного решения, то есть, $f(X)$ не имеет нижнего предела в ансамбле решении системы (1).

Имея в виду возможности этого второго случая, может стать полезной теорема существования для (законченного) минимума функции $f(x)$.

Известно условие, необходимое для существования $f_{min} \sum_{i \in N} h_i = 0$

В настоящей статье дается необходимое и достаточное условие для существования f_{min} и одновременно указывается, что последнее отношение вытекает из условия, поставленного нами.