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Some Considerations on the Technique of Optimum Filter in Projecting and Analysing the Economic-Cybernetic Systems

1. THE ECONOMIC UNITS CONSIDERED AS CYBERNETIC SYSTEMS

In economy, the systemic way of approaching problems leads to the consideration of any economic unit, starting with the work places, workshops, sections, enterprises, branches and even the economy as a whole as being sets of systems reunited according to some rules so as to attain a well established goal. The cybernetics of the economic systems is based on the extension in economy of the analogy made by N. Wiener between communication and control at beings and machines, using selfadjustment as a specific characteristic of any cybernetic system.

At the limit, any selfadjustment process is achieved by reuniting an effectuating system, which is in direct connection with environment, with a control system which controls the outputs and transmits orders to the input (feedback).

Noting with $[X]$ the set of inputs to a certain system $[S]$ and with $[Y]$ the set of outputs, the most general mathematical statement of a cybernetic system becomes:

$$S = \{X, Y \mid \mathcal{A}\} \tag{1}$$

where $\mathcal{A}$[*)] represents a transforming structure characterized by operatorial sets describing the activities specific of the system.

A first approximation liable to applications in the quantification of the economic system can be made, supposing a similitude between the behaviour of the time transforming structure and that of optimum filters; in essence, if

*) L. A. Zadeh, E. Polak — General Theory of Systems, Editura tehnică, Bucharest.

an economic unit has an answer $g(t)$ to a unit step function (for example, a leu invested into fixed assets), the output signal $f_0(t)$ is linked to the input one $f_i(t)$ by a convolution integral of the form:

$$f_0(t) = \int_0^t f_i(\tau)\, g(t - \tau)\, d\tau \tag{2}$$

where τ is an integration variable.

Under these conditions, we consider that $X = f_i(t)$ and $Y = f_0(t)$ respectively the inputs and outputs are time functions, which also happens in the case of the economic units. Always in practice, owing to internal or external perturbations, the economic unit obtains an output different from the desired one. The difference between the desired and the real output is an error which we shall conventionally note with (e). It is inferred from the theory of the optimum filter and from the criterion of the least squares proposed by N. Wiener and Kalmogorov that the square mean error e^{-2} can be calculated with ratio:

$$\bar{e}^2 = lim \frac{1}{2T} \int_{-T}^{T} [f_0(t) - f_0'(t)]^2\, dt \tag{3}$$

where $f_0'(t)$ is the desired output.

In practice, the economic units have several inputs and outputs, with a structure of multivariable complex transformation. In these cases, it is advisable to apply to vectorial matricial descriptions of the form:

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{4}$$

where

$\mathbf{x}(t)$ — the condition vector depending on time $(n \times 1)$
$\mathbf{u}(t)$ — the input vector (depending on time) $(m \times 1)$
$\mathbf{y}(t)$ — the output vector (1×1)
$\mathbf{A}(t)$ — the condition matrix $(n \times n)$
$\mathbf{B}(t)$ — the input-condition matrix $(1 \times m)$
$\mathbf{C}(t)$ — the condition-output matrix $(1 \times n)$

and $\dot{\mathbf{x}}(t) = \dfrac{d\mathbf{x}}{dt}$ that is a vectorial function defined on $\mathbf{x}$, $\mathbf{u}$ between a space R^n.

At the same time, from the cybernetic point of view, inputs $U(t)$ also include the controls which are given for the desired functioning of the system.

An adequate application of the rules in the theory of systems may lead to the construction of the control matrix **(B)** and of the matrix **(C)** which links the outputs to the conditions.

If the construction of the above matrices is achieved, the first problem which must be solved is to find a solution to the differential equation.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

In the hypothesis of linearity which, generally, is valid in economy, it can be proved that also this equation has as a solution a convolution integral which includes control, namely:

$$\mathbf{x}_{(t)} = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)} \cdot \mathbf{B}\mathbf{u}(\tau)\, d\tau \tag{5}$$

Knowing the solution of equation (5), the outputs of the system can be established with the aid of ratio (4) and on this basis, estimate whether it functions or not satisfactorily. *)

However, it is very important to use an adequate methodology for defining the basic elements of any economic unit, through sizes and measure units specific of the theory of systems, a problem which we shall try to present briefly in the following.

2. THE ECONOMIC SIGNIFICANCE OF THE REPLY SIGNAL TO THE UNIT STEP FUNCTION

An essential element used in the application of cybernetic concepts to economy is the reply to a unit step function $g(t)$.

To understand the economic significance of this element, it should be pointed out that in a certain economic unit, the unit step function or the unit impulse (as it is defined in automatics) is, according to case, a unit size of the investments put into operation which does not sensibly depreciate in a longer time interval, a unit of manpower, a quantity which is also considered unitary measuring the input flow of materials, a unit of installed power-or any input size which can be approximated by a time constant value.

The answer to the unit impulse expresses the evolution in time of the transformation taking place in the system.

Consequently, the integration of function $g(t)$ determines the effects of the transformation measured in physical and value production, in expenses, consumption reductions, etc.

If we simulate the behaviour of an enterprise with that of a system which in a period T answers a unit signal (for example, production to 1 000 lei investment) with transforming operators which are developing according to a function $g(t)$, three families of curves can be imagined in principle, as in figure 1.

These curves can be established analytically, drawn graphically or calculated by repeated approximations on the basis of the knowing of some points $g(t_1)$, $g(t_2)$, ... $g(t)$. It is noticed that the first family $g_1(t)$ would be specific of a type of industrial systems able to attain the planned parameters (the moment of oblateness) in a short time interval. The second family is specific of $g_2(t)$ the systems requiring more industrial experience and therefore, the planned parameters are attained more slowly and the third family

*) We recommend: S. Călin, C. Belea "Complex Automatic Systems", Editura tehnică, 1968 and Roger W. Brockett "Finite Dimensional Linear Systems", John Wiley and Sons Inc.— New York, London, Sydney-1970.

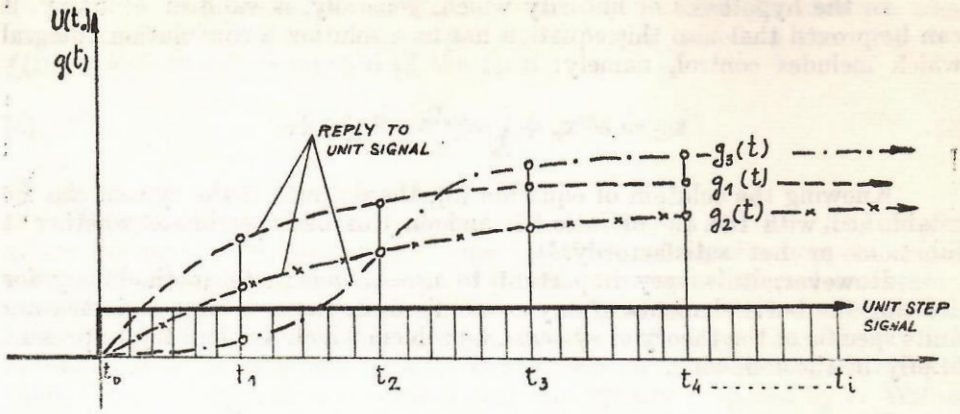

Fig. 1.

Reply curves g(t) of an industrial system initiated to moment t_0 for a unit step function

belongs to the category of systems including new technological methods, successive tests being necessary also after the putting into operation until the moment when the inflexion point appears. Of course, the examples in Fig. 1 are arbitrary, the system designer being able to imagine a multitude of such curves. Therefore, by knowing a previous experience or having curves $g_i(t)$ established analytically, the transformation occuring in the system at a certain unit impulse can be established.

3. THE ECONOMIC SIGNIFICANCE OF THE INPUT FUNCTION $F(T)$ AND OF THE OUTPUT FUNCTION $F_o(T)$

Generally, to describe the functioning of an economic unit as a cybernetic system, the outputs and its transforming structure are considered in anticipation. A characteristic of the economic systems is that, as a rule, inputs can be approximated for certain time intervals with the aid of scale functions. For example, the number of workers working in an enterprise in a shift can be considered as a step function and the variation of this number within a month is approximated with scale functions (the variation of the scale). Similarly, the problem can be put for investments, batches or for parts to be produced or other sizes.

Considering the step function $f_i(t)$ as an economic constant value equivalent to an input at moment i and which momentarily remains in the system, the scale function is the sum of steps, thus

$$f(t) = \sum_{i=1}^{i=n} f_i(t) \tag{6}$$

where n represents the number of time intervals in which the change in the step takes place. Practically, it can be considered that on an average $f(t) \equiv \equiv x(t)$, respectively it is equivalent to the variation of the input signal on each time interval (t_i). Hence, the instantaneous answer to a step function, thus

to a certain input (if $Y_0(t_i) = f(t) \cdot g(_it)$ are not delay factors) is where $g(t_i)$ plays the part of operator for changing the economic system at moment (i) and $Y_0(t_i)$ is the output from the system at the same moment. Thus, it results. that the value $Y_0(t)$ is a variation curve of the outputs from the systemc As a rule, the analyst is interested in the size of the output from the economi, unit not at every moment, but, after a certain time interval (t). For example he is interested in how many spare parts are produced by a number of workers within a month, what production is obtained at the total value of investments in one year, etc.

Consequently, a sum or integration operator must be introduced which determines the output in an absolute value:

$$Y(t) = \int_0^t Y_0(t)\, dt$$

which after the effectuation of calculations leads to a convolution integral (rel. 2), where $Y(t) = F_0(t)$ expresses the equivalence of the absolute output to the surface below the output variation curve.

In economy, the analytical forms for these types of curves are known especially for the big aggregate systems, leading to economic growth models of an exponential or logistic type. The analytical forms are more difficult to be used in the activity of the economic units. Therefore, the problem is tackled in parts, by introducing linear approximations. For facility, we shall apply to the elements in Figure 2.

Considering as known the outputs in the form of a physical value production, or any other sizes for a time interval $t = t_i - t_{i-1}$ conventionally noted $Y(t_{i-1}, t_i)$, the problem arises of finding the answer $g(x)$ to the unit impulse.

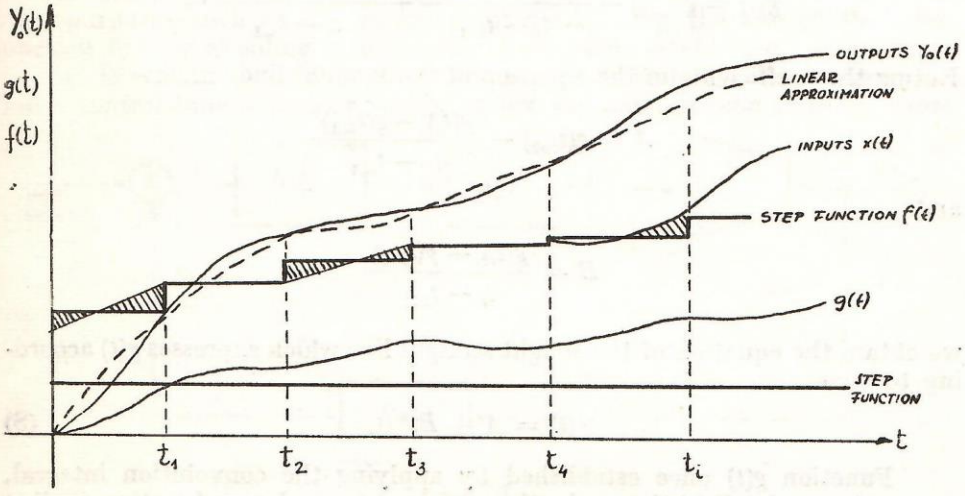

Fig. 2

Approximations by linear and step functions of the parameters of an economic system.

First, the ordinates of the curve expressing the output variation are calculated step by step, by using ratio

$$Y_0(t_i) = \frac{2Y(t_{i-1}, t_i)}{t_i - t_{i-1}} - Y_0(t_{i-1}) \tag{7}$$

a ratio which results from the surface of the trapeziums formed by $Y_0(t)$ and which can be used in the successive calculation of ordinates $Y_0(t_1)$, $Y_0(t_2)$ ... ... $Y_0(t_i)$ considering the starting point as the origin of the coordinate axes.

Knowing the size of the step function (investments, number of workers, materials, etc.) introduced into the manufacturing process which determines the total output for the given time interval $t = t_i - t_{n-1}t_i$ one can establish for a certain moment (t_i) the size of the answer to the unit signal. For example,

if $Y_0(t_i) = \frac{\text{production}}{\text{time } i}$ and $f(t) =$ number of workers (constant) it results that

$$g(t_i) = \frac{(t_i)}{f(t)} \frac{\text{production}}{\text{time worker}}$$

which represents the output signal at moment (t_i) owed to a worker (labour productivity).

It results from the hypothesis of linearity done for $Y_0(t_{i-1}, t_i)$ that in the same interval $t_{i-1} \leqslant t \leqslant t_i$ and $g(t)$ will be linear.

Consequently, we shall write that function $g(t)$ is a straight line passing through points $g(t_{i-1})$ and $g(t_i)$.

The equation of the straight line passing through these two points is:

$$g(t) = g(t_{i-1}) - \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}} + \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}} \cdot t$$

Noting the coefficients in the equation of the straight line

$$A = g(t_{i-1}) - \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}$$

and

$$B = \frac{g(t_i) - g(t_{i\ 1})}{t_i - t_{i-1}}$$

we obtain the equation of the sought straight line which expresses $g(t)$ according to time.

$$g(t) = A + Bt \tag{8}$$

Function $g(t)$ once established by applying the convolution integral, we may know size $F_0(t)$ the production obtained for each step function applied on the input of the system. This means that from the economic point of view,

at another enterprise which preserves the same probable evolution of the unit signal, the probable output can be established if its scale function is known, by applying the convolution integral

$$F_0(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} f(t) \cdot (A + B(t_i - \tau)\, d\tau$$

just as $f(t)$, A, B, do not depend on (τ) by effectuating the necessary calculations, we obtain

$$F_0(t_{i-1}, t_i) = [f(t) \cdot [t_i - t_{i-1}] \cdot [A + B(t_i + 1/2(t_i + t_{i-1})] \qquad (9)$$

a ratio having a general character which can be used either for forecasting the future development of an economic system or for the control of the optimal functioning of an existing unit.

4. ESTABLISHING THE CONTROL FOR THE OPTIMAL FUNCTIONING OF THE ECONOMIC UNITS

The essential element offered by cybernetics in applying the theory of the optimum in the activity of the economic units is the possibility of introducing a control factor which should pursue the elimination of perturbations when a reference element exists. Considering the answer signal to the unit impulse as a reference element, the establishment of control is possible if we have a scheme similar to that in Figure 3 and an adequate ratio for calculating the error to the unit signal.

We can use either the Wiener and Kolmogaroff criterion (2) or a ratio which establishes the difference between the signals of reply to the unit impulse — those which are desired and those which are effectively achieved.

In this case $e_i = g(t_i) - g'(t_i)$ where $g'(t_i)$ is calculated with the aid of ratio (9) respectively implies the knowing before hand of function $F_0(t_i)$ (the desired production). The cybernetic scheme in Fig. 4 can be used for other purposes, such as, for example, for establishing the size of the step function $f(t)$ for avoiding a deviation from the established programme.

It is certain that errors e_1, e_2 ... e_i once established, it is necessary to find a control function which should ensure the minimization of these errors

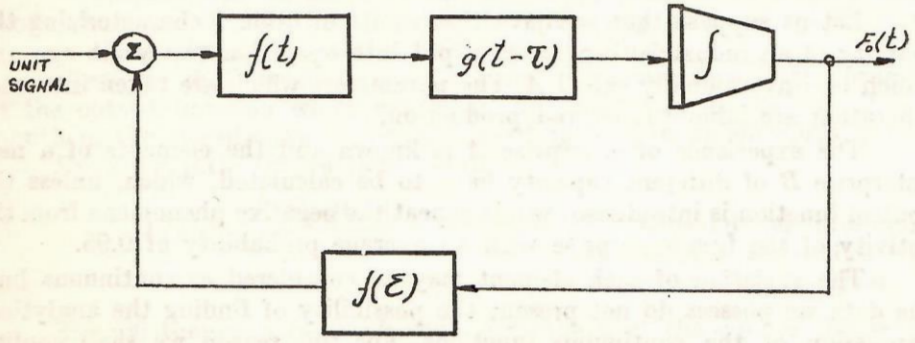

Fig. 3

Cybernetic scheme of adjusting an industrial system by error detection of reply signal for the unit step function.

by changing the inputs. Consequently, the performance criterion of the system (the control function) will be:

$$F_d = \sum_{i=1}^{i=n} e_i^{-2} = minim \tag{10}$$

Practically, by applying the above methodology, the decision function (F_d) will be quantified by changing the input on each interval (i). If it is considered that each control must give a nul error at the end of the next interval for the noticed reason, on each interval, the following system of linear equations results:

$$\begin{aligned}
&Fd_1(e_1 + e_2 + \dots e_n) = \int_{t=0}^{t=n} [Y_0(t) - Y_0'(t)]\,dt \\
&(Fd_1 + Fd_2)\,(e_2 + e_3 \dots e_n) = \int_{t=1}^{t=n} [Y_0(t) - Y_0'(t)]\,dt \\
&\quad\vdots \qquad\qquad\qquad\quad \vdots \qquad\quad \ddots \\
&(Fd_1 + Fd_2 + \dots Fd_i)\,(e_i + e_{i+1} + \dots e_n) = \int_{t=i}^{t=n} [Y_0(t) - Y_0'(t)]\,dt
\end{aligned} \tag{11}$$

The solving of the system is of course laborious but it can be achieved rather easily if one applies to an electronic computer into which only the input data are introduced.

In order to understand the method, we shall further present an example with developed calculations and a few comments for putting at the reader's disposal the facilities offered by this method.

5. EXAMPLE OF APPLYING THE CYBERNETIC THEORY OF OPTIMUM FILTER IN PROJECTING AND ANALYSING THE ACTIVITY OF AN INDUSTRIAL UNIT OPERATING WITH TWO PARAMETERS

5.1. Basic Data

Let us suppose that we have the results in table 1 characterizing the activity of an industrial organization put into operation two years ago and which is conventionally called *A*. The parameters which are taken into consideration are labour force and production.

The experience of enterprise *A* is known and the elements of a new enterprise *B* of different capacity have to be calculated, which, unless the control function is introduced, would repeat the negative phenomena from the activity of the first enterprise with an average probability of 0.95.

The variation of each element may be considered as continuous but, the data we possess do not present the possibility of finding the analytical expression of the continuous functions. For this reason we shall confine ourselves to tackling the problem in parts considering that an approximation with scale functions for periods of three months is possible.

Table 1

Basic Data Regarding A Existent Enterprise and B the Enterprise to Be Designed.

I. Enterprise A — initiated 2 years ago

Time intervals where the results are known (quarters)	t_1 3	t_2 3	t_3 3	t_4 3	t_5 3	t_6 3	t_7 3	t_8 3
Employees planned on an average (persons) $f(t_i)$	1 000			1 500				
Planned production (thousand/ lei/ quarter) $Y = F_0(t_i)$	500	1 000	1 200	1 350	1 500	1 600	1 700	1 800
Persons employed at the beginning of each quarter	800	300	100	200	250	200	100	100
Persons left at the end of each quarter	0	50	60	100	50	50	0	30
Production really achieved (thousand lei/quarter)	250	800	1 200	1 600	1650	1 700	1 800	1 900

II. Enterprise B — is to be initiated

Desired production	1 800	1 900	2 000	2 200	2 400	2 800	2 900	3 000

5.2. Establishing the Basic (Planned) Parameters of the Cybernetic System (A)

As it has already been shown, the basic parameters of a cybernetic system are:

— inputs (persons employed and persons departed) $f(t)$

— outputs: the variation function $Y_0(t)$ (the curve of the planned production) and $Y'_0(t)$ (the curve of the effectively achieved production).

— the reply signal to the unit impulse $g(t)$.

First, we shall proceed to establishing the dimensions of the ordinate of the output function which expresses the production at the end of each month in the conditions established according to plan at enterprise A.

By using ratio (8) we shall obtain

$$Y_0^{A_1}(t_1) = \frac{2Y_0^{A_1}[0, t_1]}{t_1 - t_0} - Y_0^{A_1}(t_0) = 333 \text{ thousand lei production per month at the end of the first quarter of the year}$$

333.666 thousand lei production per month at the end of the second quarter of the year is obtained in the same way and $Y_0^{A_1}(t_8) = \frac{2 \cdot Y_0^{A_1}[t_7, t_8]}{t_8 - t_7} -$

$- Y_0^{A_1}(t_7) = 566$* at the end of the two years of activity. The data obtained were written in the first line of table 2.

Table 2

enterprise A

Parameters \ Time intervals	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
1 $Y_0^{A_1}(t_i)$	333	333.6	467	433	567	499	634	566
2 $f(t)$	1 000	1 000	1 000	1 000	1 500	1 500	1 500	1 500
3 $g^{A_1}(t_i)$	0.33	0.33	0.467	$\frac{0.433}{0.2886}$	0.378	0.332	0.4226	0.4716

A first remark is offered by the fact that variation $Y_0^{A_1}(t)$ (production on time) under the planned conditions presents moments of decline during the two years, which may obviously represent a first condition for using the method — respectively, the industrial system designer may estimate *the correctness of elaborating the plan targets*. Generally, it is desirable to imprint such a rate on production so as to ensure a continuously increasing evolution of curve $Y_0(t)$. It results from the initial table (table 1) that although the planned production shows an increase from a quarter of year to another, the stress of the enterprise is not uniform because curve $Y_0(t)$ can be also considered *as a lever of this stress*. Preserving however the data as they are in table 1 we shall continue the calculations trying to avoid those happening at enterprise A, when designing enterprise B.

For outputs, in the planned case, the scale function $f(t)^{A_1}$ represents just the number of planned workers.

Further, the third parameter will be calculated, namely the size of the reply signal to the unit impulse by using to this purpose ratio (8).

It can be seen that the scale function $f(t)^{A_1}$ is constant for the period of one year (1000 workers for the first year and 1500 workers in the second year), according to those written in the second line of table 2.

Consequently at the moment t_4 we shall have two values for the unit signal, namely:

$$g^{A_1}(t_4)_{[t_3,\, t_4]^*} = \frac{f_0^{A_1}(t_4)}{f[t_3,\, t_4]} = \frac{433}{1\,000} = 0.433$$

$$g^{A_1}(t_4)_{[t_4,\, t_5]^*} = \frac{f_0^{A_1}(t_4)}{f[t_4,\, t_5]} = \frac{433}{1\,000} = 0 \cdot 2886$$

*) Notation $Y_0^{A_1}(t_i)$ was used to express that ordinate $Y_0(t)$ refers to enterprise A in the first variant (that is the planned or desired production).

*) Notation $g^{A_1}(t_4)$ $[t_3, t_4]$ is read: the reply to the unit signal at moment t_4 for enterprise A in the first variant, with the scale function on interval t_3, t_4 and $g^{A_1}(t_4)$ $[t_2, t_4]$ is the same answer to the unit signal but in the case of another scale function $[t_4, t_5]$ (respectively 1 000 and 1 500 in our case).

After calculating the reply to the unit signal, the system designer can make the second application namely to examine the reasons why it presents fluctuations. They may have a subjective character and in this case the conclusion is confirmed that planning and the decisions connected with it are incorrect, or, if they have an objective character (changes in products, technologies, the worsening of the price conditions for supplying materials, etc.) they will be recorded as such.

5.3. Establishing the Real Parameters of the Cybernetic System A

The first parameter is represented by inputs $f^{A_2}(t)$ calculated as a difference between the workers employed and those who went away on each time interval (tabe 3, line 1). Then, proceeding similary as in the previous case, for $Y_0(t)^{A_2}$, we obtain the data in table 3 line 2.

Table 3

The real parameters of enterprise *A*

1	$f^{A_2}(t_i)$		800	1 050	1 090	1 190	1 390	1 540	1 640	1 710
2	$Y_0^{A_2}(t_i)$		166.6	366.67	433.33	633.333	466.6	666.6	533.3	733.32
3	$g^{A_2}(t_i)$	$\frac{t_{i-1}, t_i}{t_i + t_{i+1}}$	0.2082 0.1586	0.3492 0.4583	0.3975 0.4126	0.532 0.5810	00.3357 0.3921	0.4328 0.4795	0.3352 0.3462	0.4280 0.4471

It is noticed that the variation of this curve is better than in the previous case, but, no conclusion can be drawn for the time being unless the unit signal $g^{A_2}(t_i)$ is calculated. In contrast with the previous case when the planner operated with annually constant impulse functions, the concrete conditions in which the enterprise functioned are different — respectively, fluctuations in the labour force have occured in each quarter of the year, a fact which changes the scale functions with which we are operating.

By using ratio 9 for $g^{A_2}(t_i)$ we shall obtain the values written in the third line in the same table. It should be mentioned that since the impulse functions are varying from one interval to another, the answer to the unit signal will be variable too. Thus, in the interval $[0, t]$, we shall have an answer

$$g(t)^{A_2}_{[t_0, t_1]} = \frac{f_0(t_1)}{f(t_1)} = \frac{166.66}{800} = 0.282$$

for the first stage $[t_0, t]$) and suddenly the second answer for the stage in the next interval:

$$g(t)^{A_2}_{[t_1, t_2]} = \frac{f_0(t_1)}{1\,050} = \frac{166 \cdot 66}{1\,050} = 0 \cdot 1586$$

Similarly, for any moment t_i we shall have distinct values for $g(t)^{A_2}_{[t_{i-1}, t_i]}$ and $g(t_i)^{A_2}_{[t_i, t_{i+1}]}$

These values may occasion brief remarks, to the effect that if we consider an input signal (impulse function) as correctly established, in this case, the changes occuring in the answer to a unit signal represent the fluctuations of labour productivity. In the first year, although the deviations are positive, that is the enterprise has worked with a higher labour productivity but it has not fulfilled the plan target, in the second year, we notice an oscillation in labour productivity which if considered as normal by the decision maker (the change in some technologies, the assimilation of more complex products, etc. may objectively cause such oscillations at the beginning), in this case he can take it as a probable behaviour of similar systems in the future.

5.4. Establishing the Parameters of Enterprise A in Case it Would Have Functioned with Optimal Control

First, it is necessary to introduce a working hypothesis about the evolution in time of the training of manpower. We shall suppose that each employee attains the normal level of productivity after three months. To make things easier, we shall graphically represent the various sizes calculated so far according to figure and we shall make a few remarks.

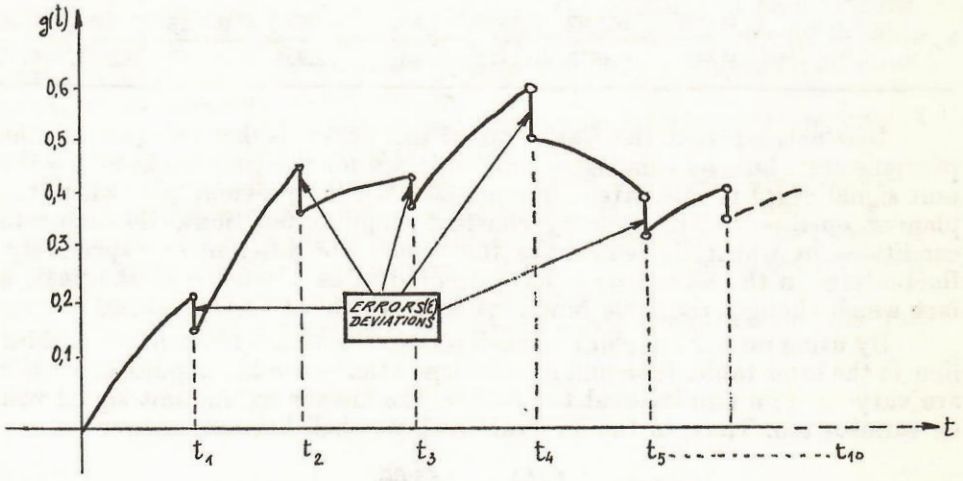

Fig. 4

Variation curve of reply signal to unit step function, for an industrial system without optimal control, functioning in accordance to data in table 3.

By applying to the cybernetic scheme in Figure 2 we can consider that at the end of each time interval an error e_i appears, equal to:

$$e_i = g(t)_{[t_{i-1}, t_i]} - g(t)_{[t_i, t_{i+1}]}$$

Obviously, we limit the total perturbations in the functioning of the system to an exclusive reason, namely labour force. The error we refer to derives from the fact that at the end of each quarter of the year, labour force is employed whose normal productivity is obtained with a delay equal to t_i. Had the system been optimally controlled, it would have tended to obtain at each moment a minimum error which would have observed a condition imposed by the decision-maker. This would have implied that the start in the next cycle be made by correcting the error at the unit signal (for example the training of manpower beforehand). To facilitate the calculations we shall make $e_i = 0$ which implies the accepting of the idea that the input into a new cycle is equal to the real output to the previous cycle. Thus, for t_1 (the moment of start) $E_0 = 0$; for the cycle $[t_1, t_2]$

$$E_1 = 0.2080 - 0.1586 = 0.0494$$

Admitting that we have anticipated an optimal control which made $E_1 = 0$, the answer to this input would have been equivalent to

$$g^A(t_1)^* = 0 \cdot 0494 + 0 \cdot 1586 = 0 \cdot 2080$$

for the beginning of the interval and maintaining the correction until the end of t_2 a size $g(t_2) = 0 \cdot 4583 + 0 \cdot 0494 = 0 \cdot 5087$ would have resulted, similarly

$$e_2 = 0 \cdot 4583 = 0 \cdot 3492 = 0 \cdot 1091$$

that is the error derives from the real productivity given by the workers who worked in the $[t_1, t_2]$ interval, in comparison with the workers employed at the end of this interval. The optimum answer to a controlled system would have eliminated this error

$$g^A(t_3) = 0 \cdot 5217$$

By continuing the calculation, the data in table 4, line 1 will be obtained; graphically representing the optimum answer to the unit signal, its considerable improvement can be noticed, as results from figure 5.

Table 4

Th Cybernetic Interpretation of the Functioning of Enterprise A by Using Optimal Control

Enterprise *A*

1 $g^A(t)$	0.2082	0.5087	0.5217	0.5962	0.4411	0.5359	0.4581
2 A^A	0	−0.0921	0.4829	0.2985	1.2166	−0.0329	1.0021
3 B^A	0.0694	0.1001	0.0043	0.0248	−0.0517	0.0316	−0.00259
4 $F_0^A(t)$	249.840	1 128	1 684.7	1 955	2162.5	2 256.87	2 445.24
5 ΔF_0	0	328	484	395	512.00	556.0	645

*) Notation $g^A(t_i)$ represents the optimum answer to the unit impulse signal.

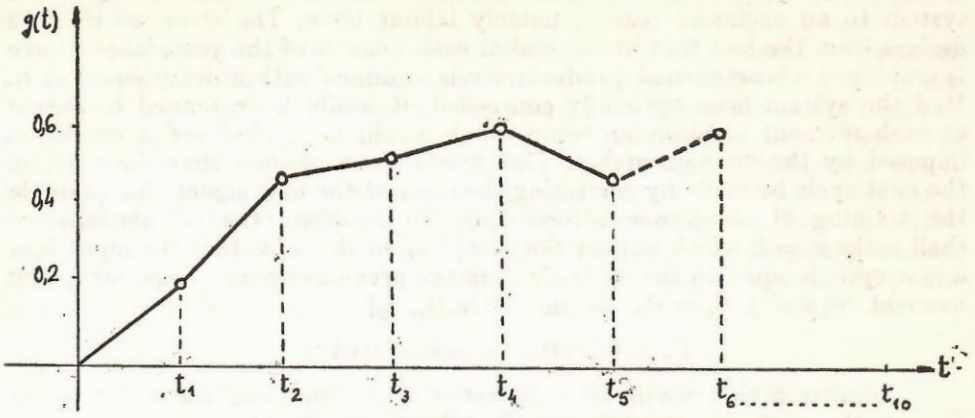

Fig. 5

Optimum reply to a unit signal – as a result of introducing cybernetic orders according to the scheme. Data are included in table 4.

Further on, to establish what would have been the production obtained in optimum conditions, it is necessary to solve the convolution integral. The coefficients of equation $g(t)$ optimizing the functioning of the system are calculated: A^A, respectively B^A. By applying ratio (10), the data resulting are those in table 4, lines 2 and 3. *)

Further, considering that the impulse function is a constant value, but taking as a unit signal that which can be achieved if labour force had no fluctuations, we can establish the size of production in the conditions in which the enterprise functions as a cybernetic system with optimal control. **

We shall write the convolution integral with the result obtained through ratio (10). For the first interval $[0, t_1]$, we shall have:

$$F_0^A[t_0,\ t_1] = \int_0^{t_1} f(\tau) \cdot g(t-\tau)\, d\tau = [f^A(t)_{[t_0,\, t_1]} \cdot (t_1 - t_0)] \cdot$$

$$\cdot [A^A_{[t_0,\, t_1]} + B^A_{[t_0,\, t_1]} \cdot (t_1 - {}^1/_2(t_1 - t_0)]$$

where

$$A^A_{[t_0,\, t_1]} = 0, \quad B^A_{[t_0,\, t_1]} = 0 \cdot 0694; \quad f^A(t)_{[t_0,\, t_1]} = 1800;$$

$$t_0 = 0; \quad t_1 = 3.$$

By carrying out the replacements, we obtain:

$$F^A_{0\,[t_0,\, t_1]} = 249 \cdot 840 \text{ thousand lei}$$

*) It is noticed that the calculation stops at interval 7 because we must know the desired output at the end of interval 8 in order to observe the anticipating character of the decision of eliminating the error.

**) The same method could have been applied in case the impulse function would have represented investments. In this case, the solutions of the problem would have been the quantified establishment of the negative or positive effects generated by the putting into operation with delay and respectively ahead of schedule.

which gives approximately the real result because the control function was nil at the start of fabrication. Acting similarly

$$F_{0[t_1,t_2]}^{A} = 1128 \text{ thousand lei}$$

It results thereof that through the optimal control of the system, a production increase $\Delta_2 = 1128 - 800 = 328 \cdot 96$ thousand lei is obtained for the second cycle.

Continuing the calculations, the data are obtained in table 4, lines 4 and 5. It results that by using a control which eliminates the difference of productivity engendered by the fluctuation of labour force, a production increase of 2920 thousand lei might be achieved in comparison with the results for two years obtained by enterprise A.

5.5. Designing the Optimal Control System for Enterprise B

After the analysis of enterprise A, the system designer can proceed to the examination of enterprise B. The problem is stated as follows: knowing the values $F_0^B(t)_{[t_0,t_1]} \dots F_0^B(t)_{[t_7,t_8]}$ and the unit signals with optimal control at enterprise A $g^A(t)_{[t_0,t_1]} \dots g^A(t)_{[t_7,t_8]}$ (table 5, lines 1 and 2) the impulse functions will be established $f^B(t)_{[t_0,t_1]} \dots f^B(t)_{[t_7,t_8]}$ at enterprise B simulated in relation to A, by using ratio (9).

The results were written in table 5, line 3.

Table 5

Results of Determining the Functioning Parameters of the Designed Enterprise B

Enterprise B

Time Intervals	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
Desired production $Y_t = F_0^B(t)$	1 800	1 900	2 000	2 200	2 400	2 800	2 900	3 000
Optimum reply to a unit signal $g^A(t)$	0.208	0.5087	0.5217	0.5962	0.4411	0.5359	0.4581	—
$f^B(t)$	5 769*	1 406	1 312	1 542	1 683	1 492	1 934	**
Planned labour force	1 435	1 710	1 970	1 970	***			
Planned production	413	1 620	2 115	3 105	***			

* the number has no significance for the optimal order because $E_0 = 0$;
** the step function can not be determined without knowing the desired reply for t_9;
*** It is left for the reader's exercise.

As it is noticed, the data obtained contain an optimum fluctuation except for the first and last intervals in which the situations are not conclusive from this point of view. Of course, for the technical systems, such fluctuations are achievable and the system designer may consider the problem as solved. For economic structures, things are different in general, because, as we speak of investments which, as a rule, cover several cycles and employees whose fluctuation on a cycle would raise social problems, the results obtained must be corrected. Particularly, we want to draw attention to the fact that if the impulse fluctuations refer to other sizes which are entirely consumed in the fabrication process (materials, energy, financial means, etc), the optimum input signal can be that in table 5. The correction of the results can be made in two ways, namely: by leaving aside the unimportant results and by imposing a constant impulse function without changing the optimum answer to the unit signal, a fact which implies the correction of production and the second way, by changing the answer to the unit signal within convenient limits so as to ensure a desired output all the time.

By applying the first method, the calculations will be the following:

A balanced average impulse function will be calculated for the first year:

$$f_m^B(t)_{[0,\, t_5]} = 1435$$

Then, integrating on parts with the optimum signal

$$F^{B_1}_{0[t_0,\, t_1]} = 413 \cdot 2 \text{ thousand lei}$$

which means that remaining uncovered for the first period is a production

$$F^{B_1}_{0_1[t_0,\, t_1]} = 1\,387 \text{ thousand lei}$$

similarly

$$F^{B_1}_{0[t_1,\, t_2]} = 1\,542 \text{ thousand lei}$$

wherefrom

$$\Delta F^{B_1}_{0[t_1,\, t_2]} = 357 \text{ thousand lei}$$

$$F^{B_1}_{0[t_2,\, t_3]} = 2\,217 \text{ thousand lei}$$

$$\Delta F^{B_1}_{0[t_2,\, t_3]} = 217 \text{ thousand lei}$$

$$F^{B_1}_{0[t_3,\, t_4]} = 2\,406 \cdot 6$$

$$\Delta F^{B_1}_{0[t_3,\, t_4]} = 206 \cdot 06 \text{ thousand lei}$$

The general result for the first year is a decrease of the desired production equal to

$$-\Delta F^B(t_0,\, t_5) = -1\,321 \text{ thousand lei}$$

which, of course, cannot be acceptible for the decision-maker.

By putting the condition that the production which was not obtained for each period be recuperated by the end of the year and by using ratio 13, the following system of equations will be formed:

$$Fd_1(e_1 + e_2 + e_3) = 1\,387 - 423$$

$$(Fd_1 + Fd_2)\ (e_1 + e_2) = 357$$

where Fd_1 represents the number of workers employed in addition at moment t_1 and Fd_2, the same element at moment t_2 and e_1, e_2, e_3 represent the result of integration of the optimum answer to the unit signal,

By effectuating the calculations, we obtain:

$$Fd_1(0.2880 + 1.0752 + 1.5466) = 964$$

$$Fd_1 = \frac{964}{3.5876} = 269$$

workers supplementarily employed at the end of interval t_1

$$Fd_2 = \frac{357}{1.3632} = 261 \text{ workers.}$$

The corrected production will be:

$$F^{B_2}_{0[t_0,\ t_1]} = 413 \cdot 2 \text{ thousand lei}$$

$$F^{B_2}_{0[t_1,\ t_2]} = F^{B_1}_{0[t_1,\ t_2]} + 269 \cdot e_1 = 1619 \cdot 47 \text{ thousand lei}$$

$$F^{B_2}_{0[t_2,\ t_3]} = F^{B_1}_{0[t_2,\ t_3]} + 269e_2 + 261e_1 = 2712 \cdot 24 \text{ thousand lei}$$

$$F^{B_2}_{0[t_3,\ t_4]} = F^{B_1}_{0[t_3,\ t_4]} + 269e_2 + 261e_2 = 3\,102 \cdot 77 \text{ thousand lei}$$

It results thereof that $F^{B_2}_{0[t_0,\ t_4]} = 7\,847$ in comparison to the initially desired one, that is an error of 5%. It is noticed that by counting off the calculus values, the result of the method gives a very small error. Consequently, the newly designed system (enterprise B) will have as plan parameters for the first year the data in table 5, lines 4 and 5. Similarly, calculations are made for the second year (which is left to the reader in the form of a drill).

CONCLUSIONS

The proposed method, which was previously described in the example of calculus for establishing the decisions linked with the internal evolution of complex industrial structures, has the advantage to be simple and easy to apply. On a more careful examination, it can be noticed that its first disadvantage is that it operates only with two parameters (investments and production, labour force production, materials-production, investments — labour force, etc.)

Nevertheless, it allows the much more rigorous establishment, than other techniques, of the elements for designing and planning the economic activity of a unit, if there are adequate reference elements. In order to avoid the afore-mentioned disadvantage, it is recommendable to use the description of the economic unit by matricial-vectorial equations. The economic significance of the parameters of the economic units considered as cybernetic systems is not changing. At the same time, the computing technique is generally the same with the only difference that as it is very laborious, its application is almost inconceivable without the contribution of the electronic computer. Anyway, the expounded cybernetic methods present a multitude of advantages so that, in our opinion, it is worth making the efforts of calculation implied by their use. The reader's opinion is requested.