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A Simple Algorithm for Determining the Number of Paths in a Finite Graph

The proposed algorithm is the association of each node of each value resulting from the amount of the values associated to the preceding nodes starting from the hypothesis that it is possible to associate the value 1 to each initial node.

The various applications of the graph theory to the co-ordination and planning of economic processes are, on the main, fairly well-known, since they cover an ever wider area. In order to apply the method of the critical path (with all its variants: CPM, SPM, KOPPER, RAMS, LESS, etc.) and determine the minimum or maximum flow, certain constructing procedures are required as well as some calculation techniques which enable us to find the best economic solutions and eliminate at the same time any source of errors.

Overlooking one of the paths contained in a graph constitutes a serious source of errors.

Leaving out a path, when analyzing a complex economic process in the network, may lead to erroneous or incomplete conclusions as to the re-distribution of resources and the optimization of the execution term.

With a view to emphasize the importance of this problem and make it as clear as possible, I shall furher point out that:

1. In order to co-ordinate and rationalize certain economic processes, we frequently resort to the critical path method. Its use will be the more effective as the investment design, or the respective technological process, is more complex — that is to say, the more activities or transport ways it contains, the more necessary it will be to construct a graph with multiple arcs and nodes.

2. Network analysis of a complex process by means of the critical path method consists of two main working procedures, which might induce the following errors, in case one or more paths were omitted:

2.1. If calculations are operated directly on the graph without the aid of the associated matrix — which quite often occurs in practice — there is always a chance of passing over a path. The comparative analysis of the sum of the values assigned to the arcs would thus be rendered incomplete and lead to conclusions liable to be refuted afterwards (if it were ascertained, for instance, that the omitted path was precisely the one of maximum duration).

2.2. Even by using the matrix associated to the graph — a method which rules out any possible error as regards determination of the critical path strictly speaking — we are sometimes compelled to deal with so-called 2nd and 3rd order critical paths; in this case, the slightest reduction of the time of the critical activities might engender either a 1st order critical path or, so to speak, a "supercritical" path, which would be all the more serious *.

3. There are also cases when, in order to determine the maximum or minimum flow of a graph, it is necessary to know the exact number of paths which can be performed from the initial node to any other node which can, or cannot coincide with the final node of the whole graph.

In all such cases, establishing the highest number of paths in a graph is particularly useful, for it provides an efficient control means.

The algorithm I suggest for this purpose refers, from the mathematical point of view, to a particular case of the graphs. However, in economic practice, this case is generally valid.

In mathematical terms, this particular case of a finite graph may be thus formulated:

a) this graph is to have neither twists nor circuits; actually, this condition is bound to be satisfied by any graph representing the model of an economic process, since in practice one cannot revert to an already due date;

b) the second condition to be fulfilled by this graph is to exclude the existence of an arc (x_j, x_i), on account of the existence of an arc (x_i, x_j), that is:

$$(x_i, x_j) \in U \Rightarrow (x_j, x_i) \notin U$$

where x_i and x_j represent two nodes whatever, U — the set of arcs of the respective graph. This condition only adjusts the graph to the requirements of economic practice since, on the one hand, the path (carrying out) of an activity in a certain direction logically bars the carrying out of the same activity in the opposite direction; and on the other hand, it meets the condition a), reinforcing it;

c) the finite graph of that type has only one node x_i for which $U^-_{x_i} = \varnothing$, and one single node x_j for which $U^+_{x_j} = \varnothing$; in other words, the graph we

* The critical path strictly speaking, i.e. the one of maximum value, is also called sometimes "1st order critical path" so as to distinguish it from 2nd and 3rd order critical paths whose values are very close to the maximum value; some authors describe the latter as "subcritical paths", considering that they have at their disposal a minimum spare time. To all these terms — rather frequently used by experts, I have preferred the one of "supercritical path", in order to define that or those paths which, although disposing in the original graph of certain spare time, have been brought — by reduction of the duration of critical activities — to lose their respective spare times and, moreover, run short of time, enter a crisis or, as they say, "have a negative reserve" (Author's note)

are dealing with must have only one network entry node and one single network exit node (see Fig. 1 and 2).

In fact, all these conditions are just setting the boundaries of the class of graphs specific to the problems tackled by network analysis with the aid of the critical path method, with all its variants. For indeed, any investment

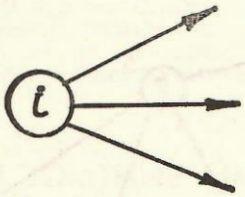

Fig. 1

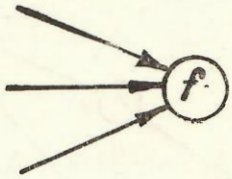

Fig. 2

project, industrial construction or production process, any scientific research or economic and administrative action can be modelled with the help of this class of graphs.

As regards problems of domestic transport or conveyance on external ways (highroads, railways etc.) these cannot always be solved by means of graphs meeting all the above mentioned conditions.

However, for a large range of cases — and chiefly for internal transport processes — this class of graphs can tally with the practical requirements, provided certain conventional procedures be used for constructing transport networks. I shall not insist though, on this matter, since it does not enter the object of this paper.

Having set down the conditions which define what is meant by that "particular case of finite graph" I am dealing with, I shall further present an operative proceeding for establishing the number of paths existing in such a graph. In other words, it is required to find the number of variants which would enable us to proceed from the initial node to the final one or to any other node. The following example will help to explain this process:

It can be noticed that we have here a graph composed of 10 nodes and 17 arcs and containing — naturally — several paths. In order to determine their number we shall use the following algorithm;

1. We associate with the initial node the value 1 and assign that same value 1 to all nodes directly bound to the initial node only. Referring to Fig. 3 we find that we have associated to the initial node (*0*) the value 1 (see the numbers beside each ringlet representing a node); likewise, we have associated this value to the node (*1*) and (*2*) because they are directly connected to the node (*0*) only. We must see that the nodes directly connected with the initial node do not have at the same time an inner incident arc resulting from another node (as in the case of node (*3*) of our example).

2. For nodes which are no longer directly bound to the initial node only, we calculate the associated values under the form of the sum of the values associated to the nodes directly preceding them. That means that, if for instance node (*3*) receives inner incident arcs from node (*1*) and (*2*) — previously marked, each one separately, by the value 1 — then we shall associate the sum of these values, that is 2, to node (*3*).

3. We repeat these operations, proceeding from one node to another, and are careful not to calculate the value associated to a certain node before previously calculating the values associated to all the preceding nodes.

4. The value obtained for the final node of the whole graph represents the highest number of paths that can be covered on a certain route of the

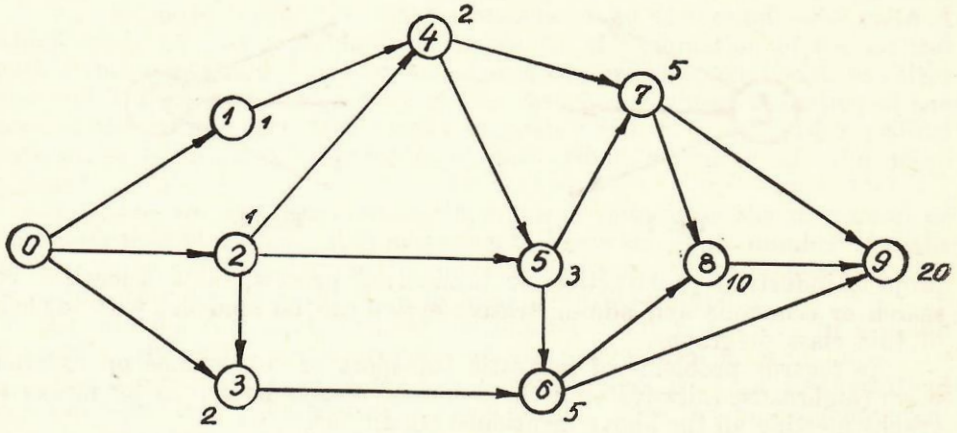

Fig. 3

respective graph (in our example in Fig. 3, the number of paths performed is 20).

This proceeding can be interrupted at any node. We thus obtain the maximum number of paths up to this particular node (for instance, 5 paths may be covered up to node (7)).

We can also use this algorithm for determining the highest number of paths between any two nodes, which are not necessarily initial or final nodes of the whole economic process which is being modelled. I shall point out though, that in this case, the subgraph comprised between any two nodes must be looked upon as an independent graph. Consequently, the algorithm will be applied this time beginning with the initial node of the respective subgraph : hence, we shall associate to the initial node of subgraph the value 1, and carry out the algorithm identically.

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UN ALGORITHME SIMPLE POUR LA DETERMINATION DU NOMBRE DE CHEMINS DANS UN GRAPHE FINI

L'étude propose un procédé simple et rapide pour déterminer le nombre de chemins dans un graphe fini non symétrique, satisfaisant les conditions suivantes :

a) $(x_i, x_j) \in U \Rightarrow (x_j, x_i) \notin U$

où x_i et x_j sont deux noeuds quelconques et U la multitude des arcs du graphe respectif ;

b) le graphe dispose d'un seul noeud x_i pour lequel $U_{x_i}^{-} = \emptyset$ et d'un seul noeud x_j pour lequel $U_{x_j}^{+} = \emptyset$; ici, U^{-} est la multitude des arcs incidents à l'intérieur et U^{+} la multitude des arcs incidents à l'exterieur ;

c) le graphe définit du point de vue mathématique un cas particulier des graphes, qui du point de vue des problèmes économiques représente cependant, un cas généralement valable.

L'algorithme proposé revient à associer chaque noeud à chacune des valeurs qui s'ensuivent de la somme des valeurs associées aux noeuds précédents, en partant de l'hypothèse qu'au noeud principal on associe la valeur 1.

La valeur associée qui s'ensuit pour le noeud final indique le nombre de chemins du graphe.

ПРОСТОЙ АЛГОРИТМ ДЛЯ ОПРЕДЕЛЕНИЯ ЧИСЛА ПУТЕЙ В КОНЕЧНОМ ГРАФЕ

В статье предлагается простой и быстрый способ определения числа путей в несимметричном графе, удовлетворяющем следующие услвия:

а) $$(x_i, x_j) \in U \Rightarrow (x_j, x_i) \notin U$$

где x_i и x_j — какие-либо две вершину, а U — множество дуг данного графа;

б) граф имеет одну вершину x_i, для которой $U_{\bar{x}i} = \varnothing$ и одну вершину x_j для которой $U_{xj}^{+} = \varnothing$; здесь U^{-} представляет собой множеством заходящих дуг, а U^{+} —множество исходящих дуг;

в) граф не содержит ни петель, ни цепей.

Этот тип графа математически отграничивает частный случай графов, который, однако, применительно к экономическим вопросам представляет собой действительный общий случай.

Предлагаемый алгоритм является обобщением каждой вершины той или иной величины, вытекающей из суммы величин, соответствующих предыдущим вершинам, исходя из предположения, что первоначальной вершине соответствует величина 1.

Получаемая величина, соответствующая конечной вершине, дает число путей данного графа.