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# An Economic Growth Model with Objective Function Depending Explicitly on the Level of Capital*)

*The paper treats the problem of the optimal rate of investment based on an economic growth model with linear production function and with the utility function depending not only on the level of consumption, but also on the level of capital. The necessary and sufficient conditions for optimality are deduced and, on the basis of these conditions, an algorithm for computing the optimal trajectory is worked out. The last section contains a discussion of the results of some concrete examples.*

## 1. INTRODUCTION

We present herein a one-sector economic growth model with discrete variables. The difference between this model and those usually considered in the economic-mathematical literature consists in the fact that in the performance of our model the level of capital for every year of the prognosis period appears explicitly, whereas, in the majority of the known models, the performance depends only on the level of consumption.

The economic motivation of the introduction of capital in the utility function is the following.

In the classical one -sector models with discrete variables, the performance is generally, of the form

$$V_1 = \sum_{n=0}^{N-1} \mathcal{U}(C_n),$$

where $\mathcal{U}(.)$ stands for the utility function of consumption and $C_n$ for consumption at moment $n$.

*) This paper was discussed in the Seminar on Mathematical Programming at the Center of Mathematical Statistics of Bucharest, where Professors B. Bereanu and V. Urseanu offered valuable suggestions. For this we take the opportunity of expressing our thanks.

If we did not impose a final constraint of the form $K_N \geqslant \omega$ on the level of capital, then the maximum value of the performance would be obtained in the case when the rate of investment, from a moment $n_0 < N$ onwards, would take its minimum admissible value. Obviously, such a solution is unacceptable, because it would hinder the evolution of the economy beyond the prognosis period considered.

However, the choice of the value "$\omega$" is to a great extent arbitrary and, in any case, exogenous to the model.

On the contrary, if we choose a performance of the form

$$V_2 = \sum_{n=0}^{N-1} \mathcal{U}(C_n, K_n)$$

that is, if we include the level of capital in the structure of the utility function, we obtain an optimal trajectory which guarantees a harmonious development of capital during the whole prognosis period.

On the other hand, the higher the economic level (expressed by the level of capital), the greater the possibility of satisfying the needs of the society, therefore the performance of the $V_2$ type offers a better characterization of society's goals.

In view of the above considerations, we make the following assumptions with respect to the utility function $\mathcal{U}(x, y)$:

$$\frac{\partial \mathcal{U}}{\partial x} \geqslant 0; \qquad \frac{\partial \mathcal{U}}{\partial y} \geqslant 0.$$

We suppose also that $\mathcal{U}(x, y)$ submits to the law of diminishing returns with respect to both variables:

$$\frac{\partial^2 \mathcal{U}}{\partial x^2} \leqslant 0; \qquad \frac{\partial^2 \mathcal{U}}{\partial y^2} \leqslant 0.$$

For the MUBELIN* model, which will be described and analysed herein after, we have chosen a special form of the utility function, which has, in addition to the above properties, that of homogeneity of the first degree,; this function is

$$\mathcal{U}(C_n, K_n) = C_n^{1-\alpha} K_n^{\alpha} \qquad 0 \leqslant \alpha < 1.$$

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* The name "MUBELIN" derives from the Romanian: Model Unisectorial cu funcţie de utilitate de tip Bernoulli şi funcţie de producţie Liniară = one-sector model with Bernoulli utility function and linear production function.

## 2.1. Description of the control system

A control system with discrete variables is characterized by the following elements:

a) The dynamic state equation, which is of the form

$$x_{n+1} = f_n(x_n, u_n); \qquad n = 0, 1, ..., N-1$$

where $x_n$ is the vector of state variables at moment $n$, $x_n \in R^q$, and $u_n$ is the vector of control parameters at moment $n$, $u_n \in U \subset R^p$.

b) Constraints (isoperimetric conditions), of the form

$$I_j = \sum_{n=0}^{N-1} \mathcal{L}_{jn}(x_n, u_n) \leqslant 0; \qquad j = 1, ..., s'$$

$$I_j = \sum_{n=0}^{N-1} \mathcal{L}_{jn}(x_n, u_n) = 0; \qquad j = s'+1, ..., s.$$

c) Initial and final conditions, of the forms: $x_0 = \omega_0$, $x_N \leqslant \omega_N$ respectively.

d) The objective — function (performance):

$$I_0 = \sum_{n=0}^{N-1} \mathcal{L}_{0n}(x_n, u_n).$$

## 2.2. Statement of the optimum problem and necessary conditions for optimality

*Definition 1*

The sequence of controls $\{u_n\}_{n=0}^{N-1}$ is called *admissible*, if the corresponding solution of the state equation statisfies the initial and final conditions and the isoperimetric constraints.

*Definition 2*

The sequence of controls $\{u_n\}_{n=0}^{N-1}$ is called *optimal*, if the corresponding solution of the state equation minimizes the performance in the class of the admissible controls.

**Theorem 2.1.** (The Maximum Principle for discrete systems)

Let $\{\tilde{u}_n\}_{n=0}^{N-1}$ be a sequence of optimal controls and $\{\tilde{x}_n\}_{n=0}^{N-1}$ the corresponding solution of the state equation. There exist the multipliers $\lambda_0 \geqslant 0$, $\lambda_j \geqslant 0$ for $j = 1, ..., s+q$, not vanishing simultaneously,

$$\lambda_j I_j(\tilde{u}_n) = 0, \; j = 1, ..., s', \; \lambda_{s+i}(\tilde{x}_N^i - \omega_N^i) = 0^*, \quad i = 1, ..., q,$$

* We noted by $z^i$ the $i$-th element of the vector $z$.

such that, if $\tilde{\psi}_n$ is the solution of the adjoint system:

$$\psi_{n-1} = \psi_n A_n - \sum_{k=0}^{s} \lambda_k b_n^k; \quad \tilde{\psi}_{N-1}^i = -\lambda_{s+i}, \quad i = 1, ..., q$$

where

$$A_n = \frac{\partial f_n}{\partial x}(\tilde{x}_n, \tilde{u}_n)$$

$$b_n^k = \frac{\partial \mathcal{L}_{kn}}{\partial x}(\tilde{x}_n, \tilde{u}_n)$$

and if we define with this $\tilde{\psi}_n$ the Hamiltonian

$$H_n(x, u, \tilde{\psi}_n) = \tilde{\psi}_n f_n(x, u) - \sum_{k=0}^{s} \lambda_k \mathcal{L}_{kn}(x, u)$$

then we have

$$\frac{\partial H_n}{\partial u}(\tilde{x}_n, u_n, \tilde{\psi}_n)(u - \tilde{u}_n) \leqslant 0$$

for any admissible $u$; $0 \leqslant n \leqslant N - 1$.

## 3. DESCRIPTION OF THE MUBELIN MODEL NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

### 3.1. The model

We make the following assumptions about the elements of the model:

a) The production function is linear with respect to the variable $K$ and independent of $L$: $F(K, L) = \beta K$, where $K$ is the level of capital and $L$ the level of the labour force.

b) The utility function has a form which is analogous to that of the Cobb-Douglas production function:

$$\mathcal{U}(C, K) = C^{1-\alpha} K^{\alpha}; \qquad 0 \leqslant \alpha < 1$$

where $C$ denotes consumption. The above utility function is, in fact, a weighted geometric mean of consumption and capital.

c) The control parameter is the rate of investment, $u$, whose range is the closed interval $[\gamma, \delta]$; $0 \leqslant \gamma < \delta \leqslant 1$.

With these specifications, we obtain the following one-sector model:

$$\begin{cases} K_{n+1} = (1 - \mu + u_n\beta)\, K_n \qquad n = 0, 1, ..., N-1; \quad K_0 > 0. \\ u_n \in [\gamma, \delta] \\ max \sum_{n=0}^{N-1} C_n^{1-\alpha} K_n^{\alpha} \end{cases}$$

where $\mu$ is the coefficient of physical and moral wear and tear of capital.

*Remark*

Taking into account the relation $C_n = (1 - u_n)\, \beta K_n$, the performance can be written as:

$$\sum_{n=0}^{N-1} K_n[(1 - u_n)\, \beta]^{1-\alpha}.$$

### 3.2. Necessary conditions for optimality

The MUBELIN model is a special case of a control system, in which neither isoperimetric constraints, nor final condition appear. The Hamiltonian function of the system is:

$$H_n(K, u, \tilde{\psi}_n) = \tilde{\psi}_n(1 - \mu + \beta u)\, K + \lambda_0 \mathcal{U}[(1 - u)\, \beta K, K]$$

where $\tilde{\psi}_n$ is the solution of the dual equation

$$\psi_{n-1} = \psi_n(1 - \mu + \beta\tilde{u}_n) + \lambda_0\left[\frac{\partial \mathcal{U}}{\partial x}(\tilde{C}_n, \tilde{K}_n)\,(1 - \tilde{u}_n)\,\beta + \frac{\partial \mathcal{U}}{\partial y}(\tilde{C}_n, \tilde{K}_n)\right]^*$$

for which $\tilde{\psi}_{N-1} = \lambda_1$ is the final condition.

From theorem 2.1 we have the following information about multipliers $\lambda_0$ and $\lambda_1$: $\lambda_0 \geqslant 0$, $\lambda_1 \geqslant 0$; $\lambda_0 + \lambda_1 > 0$.

The absence of the final condition imposed on the state variable $K$ implies $\lambda_1 = 0$ and, therefore, $\lambda_0 > 0$. Without loss of generality, we may suppose $\lambda_0 = 1$. Then, the Hamiltonian becomes

$$H_n = \tilde{\psi}_n(1 - \mu + \beta u)\, K + \mathcal{U}[(1 - u)\, \beta K, K] =$$

$$= K\{\tilde{\psi}_n(1 - \mu + \beta u) + [\beta(1 - u)]^{1-\alpha}\},$$

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* We noted by $\frac{\partial \mathcal{U}}{\partial x}$, $\frac{\partial \mathcal{U}}{\partial y}$, the partial derivatives of $\mathcal{U}$ (.,.) with respect to the first and the second variable respectively.

and the adjoint equation

$$\text{a)}\quad \tilde{\psi}_{n-1} = \tilde{\psi}_n(1-\mu) + \beta\tilde{u}_n\left[\tilde{\psi}_n - \frac{\partial \mathcal{U}}{\partial x}(\tilde{C}_n, \tilde{K}_n)\right] +$$

$$+ \beta\frac{\partial \mathcal{U}}{\partial x}(\tilde{C}_n, \tilde{K}_n) + \frac{\partial \mathcal{U}}{\partial y}(\tilde{C}_n, \tilde{K}_n).$$

The necessary optimum conditions become:

$$\text{b)}\quad \frac{\partial H_n}{\partial u}(\tilde{K}_n, \tilde{u}_n, \tilde{\psi}_n)(u - \tilde{u}_n) = \beta\tilde{K}_n\left[\tilde{\psi}_n - \frac{\partial \mathcal{U}}{\partial x}(\tilde{C}_n, \tilde{K}_n)\right](u - \tilde{u}_n) \leqslant 0$$

$$\text{c)}\quad \tilde{\psi}_{N-1} = 0.$$

### 3.3. Sufficiency of the necessary conditions for the MUBELIN model

*Theorem 3.1*

The necessary optimum conditions for the MUBELIN model are also sufficient optimum conditions. The optimal solution is unique.

*Proof*

Let $\{\hat{K}_n, \hat{C}_n, \hat{u}_n\}_{n=0}^{N-1}$ be an admissible trajectory and $\hat{\psi}_n$ the corresponding adjoint variable, which satisfy conditions a), b), c) above, and let $\{K'_n, C'_n, u'_n\}_{n=0}^{N-1}$ be an arbitrary admissible trajectory. We shall estimate the difference $\Delta = \sum_{n=0}^{N-1} \mathcal{U}(\hat{C}_n, \hat{K}_n) - \sum_{n=0}^{N-1} \mathcal{U}(C'_n, K'_n)$.

$$\Delta = \sum_{n=0}^{N-1}(\mathcal{U}(\hat{C}_n, \hat{K}_n) - \mathcal{U}(C'_n, K'_n)) =$$

$$= \sum_{n=0}^{N-1}\left\{\mathcal{U}(\hat{C}_n, \hat{K}_n) - \mathcal{U}(C'_n, K'_n) + \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n)(\hat{K}_n - K'_n) - \right.$$

$$- \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n)(\hat{K}_n - K'_n) + \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)[(\beta\hat{K}_n - \hat{C}_n - \hat{u}_n\beta\hat{K}_n) -$$

$$- (\beta K'_n - C'_n - u'_n\beta K'_n)] - \hat{\psi}_n[(\hat{K}_{n+1} - \hat{K}_n + \mu\hat{K}_n - \beta\hat{u}_n\hat{K}_n) -$$

$$\left. - (K'_{n+1} - K'_n + \mu K'_n - \beta u'_n K'_n)]\right\} = \sum_{n=0}^{N-1}\left\{\left[\mathcal{U}(\hat{C}_n, \hat{K}_n) - \mathcal{U}(C'_n, K'_n) - \right.\right.$$

$$\left. - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)(\hat{C}_n - C'_n) - \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n)(\hat{K}_n - K'_n)\right] +$$

$$+\beta\left[\hat{\psi}_n - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)\right](\hat{u}_n\hat{K}_n - u'_nK'_n) +$$

$$+\left[\frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n) + \beta\frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n) - \mu\hat{\psi}_n\right](\hat{K}_n - K'_n)\Big\} -$$

$$-\sum_{n=0}^{N-1}\hat{\psi}_n[(\hat{K}_{n+1} - \hat{K}_n) - (K'_{n+1} - K'_n)].$$

Written in a more convenient way, the last sum in the above expression becomes

$$\sum_{n=1}^{N-1}\hat{\psi}_n(K_{n+1} - K_n) = \sum_{n=1}^{N-1}\hat{\psi}_nK_{n+1} - \sum_{n=1}^{N-1}\hat{\psi}_{n-1}K_n + \sum_{n=1}^{N-1}K_n[\hat{\psi}_{n-1} - \hat{\psi}_n] =$$

$$= \sum_{n=1}^{N-1}\hat{\psi}_nK_{n+1} - \sum_{n=0}^{N-2}\hat{\psi}_nK_{n+1} + \sum_{n=1}^{N-1}K_n[\hat{\psi}_{n-1} - \hat{\psi}_n] =$$

$$= -\hat{\psi}_0K_1 + \hat{\psi}_{N-1}K_N + \sum_{n=1}^{N-1}K_n[\hat{\psi}_{n-1} - \hat{\psi}_n].$$

Taking into account that $\hat{\psi}_{N-1} = 0$, we have

$$\sum_{n=0}^{N-1}\hat{\psi}_n[K_{n+1} - K_n] = \hat{\psi}_0(K_1 - K_0) - \hat{\psi}_0K_1 + \sum_{n=1}^{N-1}K_n[\hat{\psi}_{n-1} - \hat{\psi}_n];$$

$$\sum_{n=0}^{N-1}\hat{\psi}_n[\hat{K}_{n+1} - \hat{K}_n] = -\hat{\psi}_0\hat{K}_0 + \sum_{n=1}^{N-1}\hat{K}_n[\hat{\psi}_{n-1} - \hat{\psi}_n];$$

$$\sum_{n=0}^{N-1}\hat{\psi}_n[K'_{n+1} - K'_n] = -\hat{\psi}_0K'_0 + \sum_{n=1}^{N-1}K'_n[\hat{\psi}_{n-1} - \hat{\psi}_n].$$

Since both trajectories have the same initial condition, we have

$$\sum_{n=0}^{N-1}\hat{\psi}_n[(\hat{K}_{n+1} - \hat{K}_n) - (K'_{n+1} - K'_n)] = \sum_{n=1}^{N-1}(\hat{K}_n - K'_n)(\hat{\psi}_{n-1} - \hat{\psi}_n)$$

Taking into account the above formula, the difference

$$\Delta = \sum_{n=0}^{N-1}\mathcal{U}(\hat{C}_n, \hat{K}_n) - \sum_{n=0}^{N-1}\mathcal{U}(C'_n, K'_n)$$

takes the form

$$\Delta = \sum_{n=0}^{N-1} \Big[ \mathcal{U}(\hat{C}_n, \hat{K}_n) - \mathcal{U}(C'_n, K'_n) -$$

$$- \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)(\hat{C}_n - C'_n) - \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n)(\hat{K}_n - K'_n) \Big] +$$

$$+ \sum_{n=1}^{N-1} \Big\{ - \hat{\psi}_{n-1} + \hat{\psi}_n - \mu \hat{\psi}_n + \hat{u}_n \beta \Big[ \hat{\psi}_n - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n) \Big] +$$

$$+ \beta \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n) + \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n) \Big\} (\hat{K}_n - \hat{K}'_n) +$$

$$+ \sum_{n=0}^{N-1} \beta K'_n \Big[ \hat{\psi}_n - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n) \Big] (\hat{u}_n - u'_n).$$

The second sum in the above expression vanishes, according to condition a), and the third sum is non-negative, according to condition b). It remains to be shown that the first sum is positive. We begin with the following remark: if $\delta < 1$ then, for any admissible trajectory, $(C_n, K_n) \in \mathcal{L}$ for every $n$, where

$$\mathcal{L} = \{(C, K) \in R^2 \mid C > 0,\ K > 0\}$$

Indeed, the minimum possible value of $K_n$ would be obtained for $u_0 = u_1 = \ldots = u_{n-1} = \gamma$; then ,we would have

$$K_n = (1 - \mu + \beta\gamma)^n K_0 > 0$$

for any $\gamma \geqslant 0$.
As concerns $C_n$, we have, for every $n$,

$$C_n = (1 - u_n)\beta C_n \geqslant (1 - \delta)\beta K_n > 0.$$

However, if $\delta = 1$, it is possible to have, on an admissible trajectory, $C_n = 0$ for some $n$ (corresponding to $u_n = 1$).
According to the above remark, we shall distinguish two possible cases in the estimation of the differences

$$D_n = \mathcal{U}(\hat{C}_n, \hat{K}_n) - \mathcal{U}(C'_n, K'_n) - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)(\hat{C}_n - C'_n) -$$

$$- \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_n, \hat{K}_n)(\hat{K}_n - K'_n).$$

namely
i) $\delta = 1$ and $C'_n = 0$;
ii) $\delta < 1$ or $\delta = 1$ and $C'_n > 0$.

In the reasoning which follows, we shall neglect the indices $n$, since the reasoning is valid for every $n$ $(0 \leqslant n \leqslant N - 1)$.

**Case i)**

If $C' = 0$, then $\mathcal{U}(C', K') = 0$. From the homogeneity of the first degree of $\mathcal{U}(C, K)$ we obtain, according to Euler's formula, the following equality:

$$\mathcal{U}(\hat{C}, \hat{K}) = \frac{\partial \mathcal{U}}{\partial x}(\hat{C}, \hat{K})\,\hat{C} + \frac{\partial \mathcal{U}}{\partial y}(\hat{C}, \hat{K})\,\hat{K}.$$

Therefore,

$$D = \mathcal{U}(\hat{C}, \hat{K}) - \mathcal{U}(C', K') - \frac{\partial \mathcal{U}}{\partial x}(\hat{C}, \hat{K})\,(\hat{C} - C') -$$

$$- \frac{\partial \mathcal{U}}{\partial y}(\hat{C}, \hat{K})\,(\hat{K} - K') = \frac{\partial \mathcal{U}}{\partial y}(\hat{C}, \hat{K})\,K' > 0.$$

We note that, from condition b), which trajectory $\{\hat{K}_n, \hat{C}_n, \hat{u}_n\}_{n=0}^{N-1}$ is supposed to satisfy, it follows $\hat{C}_n > 0$, hence $\frac{\partial \mathcal{U}}{\partial x}(\hat{C}_n, \hat{K}_n)$ finite. Indeed, $\hat{C}_n = 0$, that is $\hat{u}_n = 1$, would imply:

$$\hat{\psi}_n \geqslant \frac{\partial \mathcal{U}}{\partial x}(0, \hat{K}_n) = +\infty$$

**Case ii)**

According to the above remark, $(\hat{C}_n, \hat{K}_n) \in \mathcal{L}$ for every $n$.
We have, also, $(C'_n, K'_n) \in \mathcal{L}$, by assumption.
Let us note by $\hat{z}, z'$ the vectors $(\hat{C}, \hat{K})$ and $(C', K')$, respectively.
Since $\mathcal{L}$ is an open convex set in $R$, containing $z'$ and $z$, the set

$$\mathcal{J} = \{\lambda \mid z' + \lambda(\hat{z} - z') \in \mathcal{L}\}$$

is an open interval of the line, containing 0 and 1.
We define on $\mathcal{J}$ the function:

$$g(\lambda) = \mathcal{U}[C(\lambda), K(\lambda)]$$

where

$$C(\lambda) = C' + \lambda(\hat{C} - C')$$

$$K(\lambda) = K' + \lambda(\hat{K} - K').$$

We have

$$\frac{dg}{d\lambda}(\lambda) = \frac{\partial \mathcal{U}}{\partial x}[C(\lambda), K(\lambda)]\,(\hat{C} - C') + \frac{\partial \mathcal{U}}{\partial y}[C(\lambda), K(\lambda)]\,(\hat{K} - K')$$

and

$$\frac{d^2g}{d\lambda^2}(\lambda) = \frac{\partial^2\mathcal{U}}{\partial x^2}[C(\lambda),\ K(\lambda)]\,(\hat{C} - C')^2 +$$

$$+ 2\,\frac{\partial^2\mathcal{U}}{\partial x\,\partial y}[C(\lambda),\ K(\lambda)]\,(\hat{C} - C')\,(\hat{K} - K') + \frac{\partial^2\mathcal{U}}{\partial y^2}[C(\lambda),\ K(\lambda)]\,(\hat{K} - K')^2 =$$

$$= -\,\alpha(1-\alpha)\,[K(\lambda)]^{\alpha-2}\,[C(\lambda)]^{-(\alpha+1)}\,[K(\lambda)\,(\hat{C} - C') - C(\lambda)\,(\hat{K} - K')] \leqslant 0$$

It follows that $\frac{dg}{d\lambda}(\cdot)$ is decreasing on $\mathcal{J}$.

We have

$$\mathcal{U}(\hat{C},\ \hat{K}) - \mathcal{U}(C',\ K') = g(1) - g(0) = \int_0^1 \frac{dg}{d\lambda}\,d\lambda \geqslant \frac{dg}{d\lambda}(1) =$$

$$= \frac{\partial\mathcal{U}}{\partial x}(\hat{C},\ \hat{K})\,(\hat{C} - C') + \frac{\partial\mathcal{U}}{\partial y}(\hat{C},\ \hat{K})\,(\hat{K} - K').$$

Hence, we have shown that for every $n$ the expression $D_n$ is non-negative. This concludes the proof of the "sufficiency" part of the theorem.

In order to prove the uniqueness of the optimal trajectory we shall show that, if the trajectories we compare are distinct, then, at least for one index $n$ $(0 \leqslant n \leqslant N-1)$, the expression $D_n$ is strictly positive. As above, we treat separately cases i) and ii).

i) If $\delta = 1$ and $u_n' = 1$, then

$$D_n = \frac{\partial\mathcal{U}}{\partial y}(\hat{C}_n,\ \hat{K}_n)\,K_n' > 0.$$

ii) Let us suppose $u_n' < 1$, $n = 0, 1, \ldots, N-1$.

Since trajectories $\{\hat{K}_n,\ \hat{C}_n,\ \hat{u}_n\}_{n=0}^{N-1}$ and $\{K_n',\ C_n',\ u_n'\}_{n=0}^{N-1}$ are distinct, there exists at least one index $m$, such that $\hat{u}_m \neq u_m'$.
We have

$$\frac{\hat{C}_m}{\hat{K}_m} = (1 - \hat{u}_m)\,\beta; \qquad \frac{C_m'}{K_m'} = (1 - u_m')\,\beta$$

Therefore

$$\frac{\hat{C}_m}{\hat{K}_m} \neq \frac{C_m'}{K_m'}$$

From the above inequality it follows

$$K_m(\lambda)\,(\hat{C}_m - C_m') \neq C_m(\lambda)\,(\hat{K}_m - K_m')$$

Therefore, function $g_m(\lambda) = \mathcal{U}[C_m(\lambda), K_m(\lambda)]$ has a strictly negative second derivative on $\mathcal{I}$, hence function $\frac{dg_m}{d\lambda}(\cdot)$ is strictly decreasing, which implies:

$$\mathcal{U}(\hat{C}_m, \hat{K}_m) - \mathcal{U}(C'_m, K'_m) = g_m(1) - g_m(0) > \frac{dg_m}{d\lambda}(1) =$$

$$= \frac{\partial \mathcal{U}}{\partial x}(\hat{C}_m, \hat{K}_m)(\hat{C}_m - C'_m) + \frac{\partial \mathcal{U}}{\partial y}(\hat{C}_m, \hat{K}_m)(\hat{K}_m - K'_m).$$

## 4. QUALITATIVE PROPERTIES OF THE MUBELIN MODEL

Considering the optimum condition b) explicitely, we distinguish the following cases:

b.1)

$$\tilde{u}_n = \gamma \Leftrightarrow \tilde{\psi}_n \leqslant (1 - \alpha)[\beta(1 - \gamma)]^{-\alpha}$$

b.2)

$$\tilde{u}_n = \delta \Leftrightarrow \tilde{\psi}_n \geqslant (1 - \alpha)[\beta(1 - \delta)]^{-\alpha}$$

b.3)

$$\tilde{u}_n \in (\gamma, \delta) \Leftrightarrow \tilde{\psi}_n = (1 - \alpha)[\beta(1 - \tilde{u}_n)]^{-\alpha}$$

The corresponding forms for the adjoint equation are:

a.1)

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n[1 - \mu + \beta\gamma] + [\beta(1 - \gamma)]^{1-\alpha}$$

a.2)

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n[1 - \mu + \beta\delta] + [\beta(1 - \delta]^{1-\alpha}$$

a.3)

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n\left[1 - \mu + \frac{\beta}{1 - \alpha} - \frac{\alpha\beta\tilde{u}_n}{1 - \alpha}\right]$$

Since, from the final condition c) we have

$\tilde{\psi}_{N-1} = 0 < (1 - \alpha)[\beta(1 - \gamma)]^{-\alpha}$, it follows that $\tilde{u}_{N-1} = \gamma$.

Supposing that $\tilde{u}_{N-2} = \tilde{u}_{N-3} = \ldots = \tilde{u}_{N-m} = \gamma$, and solving the difference equation a.1), we obtain

i) $\tilde{\psi}_{N-p-1} = \frac{(1 - \mu + \beta\gamma)^p - 1}{\beta\gamma - \mu}[\beta(1 - \gamma)]^{1-\alpha}$

$p = 0, 1, \ldots, m$, if $\beta\gamma \neq \mu$, or

ii) $\tilde{\psi}_{N-p-1} = p[\beta(1 - \gamma)]^{1-\alpha}$

$p = 0, 1, \ldots, m$, if $\beta\gamma = \mu$.

We note that $\tilde{\psi}_{N-p-1}$ increases as $p$ increases, hence it can happen that, for a given $p$, the inequality

$$\tilde{\psi}_{N-p-1} \leqslant (1-\alpha)\,[\beta(1-\gamma)]^{-\alpha}$$

is no longer valid.
Let $m$ be the last index for which the above inequality is valid, i.e.

$$\tilde{\psi}_{N-m-1} \leqslant [\beta(1-\gamma)]^{-\alpha}\,(1-\alpha)$$

$$\tilde{\psi}_{N-m-2} > [\beta(1-\gamma)]^{-\alpha}\,(1-\alpha).$$

We have

i) $$m = \left[\frac{\ln[\beta(1-\alpha\gamma) - \mu(1-\alpha)] - \ln\beta - \ln(1-\gamma)}{\ln(1-\mu+\beta\gamma)}\right]$$

if $\beta\gamma \neq \mu$, where $[x]$ stands for the integer part of number $x$; and

ii) $$m = \left[\frac{1-\alpha}{\beta(1-\gamma)}\right]$$

if $\beta\gamma = \mu$.
Hence, we have $\tilde{u}_{N-m-2} \neq \gamma$.
There exist two possibilities:

1) $(1-\alpha)\,[\beta(1-\gamma)]^{-\alpha} < \tilde{\psi}_{N-m-2} < (1-\alpha)\,[\beta(1-\delta)]^{-\alpha}$

and then $\tilde{u}_{N-m-2} \in (\gamma, \delta)$

2) $\tilde{\psi}_{N-m-2} \geqslant [\beta(1-\delta)]^{-\alpha}\,(1-\alpha)$

and, in this case, $\tilde{u}_{N-m-2} = \delta$.

We shall now prove some propositions which characterize the behaviour of the optimal solution.

**Proposition 4.1**

If $\beta > \mu$, then $\tilde{u}_n \in (\gamma, \delta)$ implies $\tilde{\psi}_{n-1} > \tilde{\psi}_n$

*Proof*

If $\tilde{u}_n \in (\gamma, \delta)$, then we have for $\tilde{\psi}_{n-1}$ equation a.3):

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n\left(1 - \mu + \frac{\beta}{1-\alpha} - \frac{\alpha\beta\tilde{u}_n}{1-\alpha}\right).$$

In order to prove that $\tilde{\psi}_{n-1} > \tilde{\psi}_n$, it is sufficient to show that

$$\frac{\beta}{1-\alpha}(1-\alpha\tilde{u}_n) - \mu > 0.$$

We have

$$\frac{\beta(1-\alpha\tilde{u}_n)}{1-\alpha}-\mu>\frac{\beta(1-\alpha\delta)}{1-\alpha}-\mu>\beta-\mu>0.$$

*REMARK*

The hypothesis $\beta > \mu$ is completely justified from the economic point of view. Indeed, $\beta K_n$ stands for the value of the production obtained with capital $K_n$, whereas $\mu K_n$ stands for capital placed out of operation. The inequality $\beta < \mu$ would mean that the value of production is insufficient even to counterbalance the value of capital placed out of operation.

**Proposition 4.2**

If $\beta > \mu$, then $\tilde{u}_n = \delta$ implies $\tilde{u}_{n-1} = \delta$.

If, moreover, $\beta\delta > \mu$, then $\tilde{u}_n = \delta$ implies $\tilde{\psi}_{n-1} > \tilde{\psi}_n$.

*Proof*

If $\tilde{u}_n = \delta$, then $\tilde{\psi}_n \geqslant (1-\alpha)[\beta(1-\delta)]^{-\alpha}$ and we have for $\tilde{\psi}_{n-1}$ the equation a.2):

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n(1-\mu+\beta\delta)+[\beta(1-\delta)]^{1-\alpha}.$$

From the above relations it follows that $\tilde{\psi}_{n-1} > (1-\alpha)[\beta(1-\delta)]^{-\alpha}$ which implies $\tilde{u}_{n-1} = \delta$.

Indeed,

$$\tilde{\psi}_{n-1} \geqslant (1-\alpha)[\beta(1-\delta)]^{-\alpha}\left[1-\mu+\beta\delta+\frac{\beta(1-\delta)}{1-\alpha}\right] > (1-\alpha)[\beta(1-\delta)]^{-\alpha}$$

since the inequality

$$1-\mu+\beta\delta+\frac{\beta(1-\delta)}{1-\alpha}>1$$

reduces to the obvious inequality

$$\beta\cdot\frac{1-\alpha\delta}{1-\alpha}>\mu.$$

If, moreover, $\beta\delta > \mu$, then the inequality $\tilde{\psi}_{n-1} > \tilde{\psi}_n$ follows immediately from formula a.2).

*REMARK*

The hypothesis $\beta\delta > \mu$ expresses, mathematically, the possibility of expanded reproduction. Indeed, the value of $\beta\delta K_n$ expresses the maximum realizable gross investment at moment $n$, whereas $\mu K_n$ stands for the value of the capital placed out of operation.

The conclusions of the preceding propositions can be synthetized in the following theorem.

**Theorem 4.1**

In the general case, in the behaviour of the optimal control for the MUBELIN model three phases can be distinguished, namely:

Phase 1) $\tilde{u}_n = \delta$, $\quad n = 0, 1, ..., n_1^*$

Phase 2) $\tilde{u}_n \in (\gamma, \delta)$, $\quad n = n_1^* + 1, ..., n_2^*$

Phase 3) $\tilde{u}_n = \gamma$, $\quad n = n_2^* + 1, ..., N - 1.$

According to the values of the parameters of the model, one of the first two phases or both of them can be absent.

**Corollary**

The optimal control $\tilde{u}_n$ is a non-increasing function of $n$.

*Proof*

If phases 1) and 2) are absent, then $\tilde{u}_n$ is constant, equal to $\gamma$. If only phase 2) is absent, then the optimal control is piece-wisely constant, having only one switch, from $\tilde{u}_{n^*} = \delta$ to $\tilde{u}_{n^*+1} = \gamma$.

In the general case, the situation is as follows. At the moment $n_1^*$ of the first switch, the optimal control decreases from $\tilde{u}_{n_1^*} = \delta$ to $\tilde{u}_{n_1^*+1} \in (\gamma, \delta)$. At the moment $n_2^*$ of the second switch, the optimal control decreases from $\tilde{u}_{n_2^*} \in (\gamma, \delta)$ to $\tilde{u}_{n_2^*+1} = \gamma$.

It remains to be shown that in phase 2) $\tilde{u}_n$ is decreasing. Let $n$ and $n + 1$ be two consecutive moments in phase 2). Then, from the optimum condition b. 3), we have:

$$\tilde{\psi}_n = (1 - \alpha)\,[\beta(1 - \tilde{u}_n)]^{-\alpha}$$

$$\tilde{\psi}_{n+1} = (1 - \alpha)\,[\beta(1 - \tilde{u}_{n+1})]^{-\alpha}$$

hence

$$\tilde{u}_n = 1 - \frac{1}{\beta}\left[\frac{1 - \alpha}{\tilde{\psi}_n}\right]^{1/\alpha}$$

$$\tilde{u}_{n+1} = 1 - \frac{1}{\beta}\left[\frac{1 - \alpha}{\tilde{\psi}_{n+1}}\right]^{1/\alpha}.$$

According to proposition 4.1, we have $\tilde{\psi}_{n+1} < \tilde{\psi}_n$. Therefore $\tilde{u}_{n+1} < \tilde{u}_n$.

*REMARK*

As we have mentioned in 3.3 (in the proof of theorem 3.1), in the special case $\delta = 1$, the control $u_n = \delta = 1$ cannot be optimal for any $n$.

## 5. ALGORITHM FOR SOLVING THE MUBELIN MODEL

Step 1)

1. a) We compute the difference $\beta\gamma - \mu$.

1. b) We compute

$$m^* = \left[\frac{\ln\{\beta(1-\alpha\gamma) - \mu(1-\alpha)\} - \ln\beta - \ln(1-\gamma)}{\ln(1-\mu+\beta\gamma)}\right]$$

if $\beta\gamma \neq \mu$, or

$$m^* = \left[\frac{1-\alpha}{\beta(1-\gamma)}\right]$$

if $\beta\gamma = \mu$; $[x]$ stands for the integer part of number $x$.

1. c) We compute $n_2^* = N - m^* - 2$.

We have $\tilde{u}_{N-1} = \tilde{u}_{N-2} = \ldots = \tilde{u}_{N-m^*-1} = \gamma$.

If $m < N - 1$, we pass to 1.d), otherwise the computation process is finished.

1. d) We compute

$$\tilde{\psi}_{N-m^*-2} = \frac{(1-\mu+\beta\gamma)^{m^*+1} - 1}{\beta\gamma - \mu} \cdot \{\beta(1-\gamma)\}^{1-\alpha}$$

if $\beta\gamma \neq \mu$, or

$$\tilde{\psi}_{N-m^*-2} = (m^* + 1)\{\beta(1-\gamma)\}^{1-\alpha}$$

if $\beta\gamma = \mu$.

Step 2)

For $k = 2, 3, \ldots$

2. a) We compare $\tilde{\psi}_{N-m^*-k}$ with $(1-\alpha)\{\beta(1-\delta)\}^{-\alpha}$.

If

$$\tilde{\psi}_{N-m^*-k} \geqslant (1-\alpha)\{\beta(1-\delta)\}^{-\alpha},$$

then we pass to step 3).

If

$$\tilde{\psi}_{N-m^*-k} < (1-\alpha)\{\beta(1-\delta)\}^{-\alpha},$$

then we pass to 2.b).

2.b) We compute

$$\tilde{u}_{N-m^*-k} = 1 - \frac{1}{\beta}\left\{\frac{1-\alpha}{\tilde{\psi}_{N-m^*-k}}\right\}^{1/\alpha}.$$

2.c) We compute

$$\tilde{\psi}_{N-m^*-k-1} = \tilde{\psi}_{N-m^*-k}\left\{1 - \mu + \frac{\beta}{1-\alpha} - \frac{\alpha\beta}{1-\alpha}\,\tilde{u}_{N-m^*-k}\right\}.$$

2.d) We revert to step 2) with $k+1$ instead of $k$.

Step 3)

Let $k^* \geqslant 2$ be the first index $k$ (if it exists) such that

$$\tilde{\psi}_{N-m^*-k} \geqslant (1-\alpha)\{\beta(1-\delta)\}^{-\alpha}.$$

We compute $n_1^* = N - m^* - k^*$.
We have

$$\tilde{u}_{n_1^*} = \tilde{u}_{n_1^*-1} = \ldots = \tilde{u}_0 = \delta.$$

Step 4)

Having obtained the values $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1}$.

4. a) We compute $\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_N$

with the formula

$$\tilde{K}_{n+1} = (1 - \mu + \beta\tilde{u}_n)\,\tilde{K}_n.$$

4. b) We compute $\tilde{C}_0, \tilde{C}_1, \ldots, \tilde{C}_{N-1}$
with the formula

$$\tilde{C}_n = (1 - \tilde{u}_n)\,\beta\tilde{K}_n.$$

4. c) We compute the optimal value of the performance, with the formula

$$\tilde{V} = \sum_{n=0}^{N-1} \tilde{K}_n\{\beta(1-\tilde{u}_n)\}^{1-\alpha}.$$

*REMARK*

For $\delta = 1$, steps 2. a) and 3) have no meaning, hence we pass from step 1) directly to step 2.b) and then to step 4).

On the basis of the above algorithm, a Fortran programme was worked out. The conclusions deduced from its executions will be presented in the next section.

## 6. CONCLUSIONS DEDUCED FROM PROGRAMME EXECUTIONS WITH CONVENTIONAL DATA

Taking into account the economic nature of the practical problem the MUBELIN model is meant to solve, we have to assign to the majority of its parameters values determined on the basis of economic considerations and objective statistical data. Thus, coefficient $\beta$ of the production function, as well as coefficient $\mu$ of physical and moral wear and tear of capital have to be estimated on the basis of existing statistical data. The choice of the values of parameters $\gamma$ and $\delta$, which limit from below and from above, respectively, the range of admissible controls, is essentially a decision of economic policy, which has also to be eventually founded on objective economic data. The only "subjective" parameter of the model is $\alpha$, the elasticity coefficient which appears in the expression of the utility function.

Since we are only in the initial stage of experimenting with the model, we have used, for the first programme executions, conventional data. Consulting various materials connected with our subject, we tried to assign to the parameters of our model values as close to reality as possible. Thus, we assigned the value 0.05 to parameter $\mu$ and values close to 0.3 to parameter $\beta$; however, we have also taken into account some higher values for $\beta$, in order to simulate a greater variety of trajectories of evolution. Obviously, for high values of $\beta$, these trajectories are unrealistic from the economic point of view.

As concerns the choice of the values for parameters $\gamma$ and $\delta$, in a first stage we carried out simulations with their extreme values: $\gamma = 0$ and $\delta = 1$. Obviously, such an approach is not justified from the economic point of view, since we must ensure, at every moment, a minimal level of consumption, in order to make possible the reproduction of the labour force. However, the choice of the interval [0,1] for the values of the control parameter becomes acceptable, if we give a slightly different meaning to the control parameter, namely if we assume that $u$ is the share earmarked for investment not from the whole national income, but only from that part of the national income which remains after the value of the compulsory minimal consumption of the population has been subtracted.

In a later approach, we assigned to parameters $\gamma$ and $\delta$ the values 0.1 and 0.4 respectively. As the results of computations show, in this case we have (except for variant II.5) optimal control of the "bang-bang" type, i.e. the optimal control takes only its extreme admissible values. In order to obtain less trivial solutions, we intend to broaden the range of $u$ in the subsequent simulations.

The choice (somewhat arbitrary) of the prognosis horizon $N$, as well as of the initial value of capital, $K_0$, is justified (mathematically) by the two following properties of the MUBELIN model, properties which result from the homogeneity of the first degree of the utility function.

1) Let us suppose we have two variants of the MUBELIN model, which differ from each other only in the length of the prognosis horizon. Let $N_1$, $N_2$ be the prognosis horizons for variant 1 and variant 2, respectively; $N_1 < N_2$.

Then the sequence of optimal controls for variant 2 in the period $[N_2 - N_1, N_2]$ coincides with the sequence of optimal controls for variant 1.

2) The sequence of optimal controls for the MUBELIN model does not depend on the initial value of capital, $K_0$.

Based on 1) above, we considered programme execution for several values of $N$ irrelevant. We chose only one value, $N = 30$, considering that a relatively vast horizon offers a better possibility of observing the evolution of the optimal trajectory. Property 2) above allowed us to assign arbitrary values to $K_0$, without in so doing altering the conclusions of principle which result from the application of the model.

In the two tables below we synthetised the most significant results obtained from the first two executions of the programme. We removed from among the variants of the first run those in which parameter β was assigned too high values, and preserved only variants I.4, I.5, I.6, I.7, I.8 and I.10. We mention that, in the case of variants I.7, I.8 and I.10 also, we obtained economically unlikely high values for the growth rate and the final leve of capital.

*RUN I*

$\gamma = 0$; $\delta = 1$; $K_0 = 1{,}000$; $\mu = 0.052$

| Variant | α | β | Behaviour of the optimal control | $K_{30}$ |
|---|---|---|---|---|
| I.4 | 0.945 | 0.215 | Decreases from $u_0 = 0.999$ to $u_{26} = 0.931$; $u_{27} = 0.888$; $u_{28} = 0.763$; $u_{29} = 0$. | 67,070 |
| I.5 | 0.800 | 0.215 | Decreases from $u_0 = 0.999$ to $u_{23} = 0.91$; $u_{24} = 0.88$: $u_{25} = 0.83$ $u_{26} = 0.76$; $u_{27} = 0.6$; $u_{28} = 0.08$; $u_{29} = 0$. | 48,860 |
| I.6 | 0.715 | 0.215 | Decreases from $u_0 = 0.999$ to $u_{23} = 0.88$; $u_{24} = 0.84$; $u_{25} = 0{,}77$; $u_{26} = 0{,}66$; $u_{27} = 0{,}41$; $u_{28} = u_{29} = 0$. | 43,570 |
| I.7 | 0.643 | 0.643 | Decreases from $u_0 = 0.999$ to $u_{26} = 0.944$; $u_{27} = 0{,}86$; $u_{28} = 0{,}59$; $u_{29} = 0$. | 509,033,750 |
| I.8 | 0.487 | 0.503 | Decreases from $u_0 = 0{,}999$ to $u_{25} = 0.954$; $u_{26} = 0.896$; $u_{27} = 0.735$; $u_{28} = u_{29} = 0$. | 25,703,740 |
| I.10 | 0.500 | 0.500 | Decreases from $u_0 = 0.999$ to $u_{25} = 0.953$; $u_{26} = 0.894$; $u_{27} = 0.736$; $u_{28} = u_{29} = 0$. | 24,227,600 |

*RUN II*

$\gamma = 0.1$; $\delta = 0.4$; $K_0 = 1{,}000$; $\mu = 0.05$; $\beta = 0.33$

| Variant | $\alpha$ | Optimal control | $C_0$ | $C_{29}$ | $K_{30}$ | Performance |
|---|---|---|---|---|---|---|
| II.1 | 0.9 | $u_n = 0.4$ $n = 0, ..., 28$; $u_{29} = 0.1$ | 198 | 2,919 | 9,663 | 100,297 |
| II.2 | 0.8 | $u_n = 0.4$ $n = 0, ..., 28$; $u_{29} = 0.1$ | 198 | 2,919 | 9,663 | 85,607 |
| II.3 | 0.7 | $u_n = 0.4$ $n = 0, ..., 27$; $u_{28} = u_{29} = 0.1$ | 198 | 2,653 | 8,779 | 73,177 |
| II.4 | 0.6 | $u_n = 0.4$ $n = 0, ..., 27$; $u_{28} = u_{29} = 0.1$ | 198 | 2,653 | 8,779 | 62,676 |
| II.5 | 0.5 | $u_n = 0.4$ $n = 0, ..., 26$; $u_{27} = 0.351$; $u_{28} = u_{29} = 0.1$ | 198 | 2,613 | 8,649 | 53,698 |
| II.6 | 0.4 | $u_n = 0.4$ $n = 0, ..., 26$; $u_{27} = u_{28} = u_{29} = 0.1$ | 198 | 2,410 | 7,976 | 46,092 |
| II.7 | 0.3 | $u_n = 0.4$ $n = 0, ..., 26$; $u_{27} = u_{28} = u_{29} = 0.1$ | 198 | 2,410 | 7,976 | 39,621 |
| II.8 | 0.2 | $u_n = 0.4$ $n = 0, ..., 26$; $u_{27} = u_{28} = u_{29} = 0.1$ | 198 | 2,410 | 7,976 | 34,069 |
| II.9 | 0.1 | $u_n = 0.4$ $n = 0, ..., 25$; $u_n = 0.1$ $n = 26, 27, 28, 29$ | 198 | 2,189 | 7,246 | 29,341 |
| II.10 | 0.0 | $u_n = 0.4$ $n = 0, ..., 25$ $u_n = 0.1$ $n = 26, 27, 28, 29$. | 198 | 2,189 | 7,246 | 25,311 |

We remark that, at RUN I, where the range of admissible controls is maximal, namely the interval [0,1], the optimal control takes its values, at the majority of moments, in the interval (0,1); besides, we mentioned in sections 3 and 4 that $u = 1$ may not be optimal. At RUN II, where the range of admissible controls is much narrower, we meet the opposite situation: except for variant II.5, we have optimal control of the "bang-bang" type in all the other variants. As concerns variant II.10, this result is theoretically justified, since we are in the limit-case when the utility function reduces to the linear function: $\mathcal{U}(C) = C$. As concerns the other variants, we consider that the "bang-bang" solution is not the only theoretically possible one, but it is a

consequence of the way we have chosen the parameters of the model and especially of the narrow range of the control parameter.

As concerns the dynamics of capital, we note that, though we have not imposed any constraint on its final level, it has a harmonious and mainly ascendent evolution (except for a few moments at the end of the prognosis period). In the tables below we give the average rates of growth of capital for all the variants considered.

| Variant | I.4 | I.5 | I.6 | I.7 | I.8 | I.10 |
|---|---|---|---|---|---|---|
| Average growth rate of capital | 15.5% | 13.8% | 13.4% | 55% | 40.2% | 40% |
| | | | | | | |
| Variant | II.1 | II.2 | II.3 | II.4 | II.5 | |
| Average growth rate of capital | 7.6% | 7.6% | 7.3% | 7.3% | 7.2% | |
| | | | | | | |
| Variant | II.6 | II.7 | II.8 | II.9 | II.10 | |
| Average growth rate of capital | 7% | 7% | 7% | 6.8% | 6,8% | |

In the following, we shall refer only to RUN II, where we have a more detailed information, since consumption at every moment, as well as the optimal value of the performance for every variant are edited.

Moreover, since the ten variants differ from each other only in the values of parameter $\alpha$, we can draw conclusions concerning the sensitivity of the optimal trajectory with respect to this parameter.

In the following table we give the average growth rates of consumption for variants II.1—II.10.

| Variant | II.1 | II.2 | II.3 | II.4 | II.5 |
|---|---|---|---|---|---|
| Average growth rate of consumption | 9.7% | 9.7% | 9,4% | 9.4% | 9.3% |
| Variant | II.6 | II.7 | II.8 | II.9 | II.10 |
| Average growth rate of consumption | 9% | 9% | 9% | 8.6% | 8.6% |

In conclusion, we mention that the sensitivity of the concrete model with respect to $\alpha$ is small. One can observe that variants II.1—II.10, although they correspond to different values of $\alpha$, group together in pairs or even triplets of variants having identical behaviours of the optimal trajectories. However, one can note that all the elements specified in table RUN II (the final value of consumption, the final value of capital, as well as the optimal value of the objective function) are increasing functions of $\alpha$.

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## UN MODELE DE CROISSANCE ECONOMIQUE DONT LA FONCTION DE BUT DEPEND EXPLICITEMENT DU NIVEAU DES FONDS PRODUCTIFS


L'auteur traite d'un problème du taux optimal des investissements utilisant un modèle de croissance économique dont la fonction de production est linéaire et la fonction d'utilité dépend non seulement du niveau de la consommation, mais aussi du niveau des fonds productifs. On déduit les conditions nécessaires et suffisantes d'optimalité à l'aide desquelles on construit un algorithme pour le calcul de la trajectoire optimale. Le dernier paragraphe présente quelques conclusions imposées par la solution de plusieurs exemples numériques.


## МОДЕЛЬ ЭКОНОМИЧЕСКОГО РОСТА С ЦЕЛЕВОЙ ФУНКЦИЕЙ, ЯВНО ЗАВИСЯЩЕЙ ОТ УРОВНЯ ПРОДУКТИВНЫХ ФОНДОВ


В статье рассматривается задача оптимального темна накопления, на основе модели экономического роста с линейной производственной функцией и с функцией полезности, зависящей не только от уровня потребления, но текжа от уровня продуктивных фондов. Используютса необходимые и достаточные условия оптимальности и на основе этих условий строится алгорифм для расчёта оптимальной траектории.

Последний абзац содержит рассуждения относительно решении некоторых конкретиых примеров.



16.02.1976.