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# An Algorithm for Solving the Transportation Tridimensional Problem

*The paper presents an algorithm for solving the transportation tridimensional problem — a double sum model — by using the Dantzig-Wolfe decomposition method. The paper describes the advantages provided by the algorithm-programmed in FORTRAN — and the possibilities of using the computer for solving the transportation tridimensional problem.*

## 1. DESCRIBING THE PROBLEM

The practice of economic planning has imposed the solution of problems of the kind: a certain product necessary in $n$ consuming centres is made in $m$ production centres. Knowing in each production centre $i$ the available quantity $a_i (i = 1, ..., m)$, the necessary quantity $b_j (j = 1, ..., n)$ in each consuming centre $j$ and the cost of transportation $c_{ij}$, per unit of product from the producer to the consumer, a distribution of quantities $x_{ij}$ is required, which are to be transported with minimum total expenses so that the demand to be satisfied. The mathematical formulation gives the model of the linear programming problem called of "transportation" [8]:

$$min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij};$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, ..., m;$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, ..., n;$$

$$x_{ij} \geqslant 0; \quad i = 1, ..., m, \quad j = 1, ..., n;$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

To solve the problem in this way — equality constraints — the specialist literature gives many methods some of them for obtaining the optimum by improving an initial solution which is found out others, approximate, which give directly the solution close to the optimum one.

Yet, problems are occuring whose solution by the classical methods of linear programming is difficult. For them, multidimensional programming — the space of the unknown values is no longer bidimensional — is advantageous. For example, if in case of the above ennunciated problem, the transport of the product is made by $p$ means of conveyance and we know quantity $c_k$ which has to be transported by means of conveyance $k(k = 1, \ldots p)$ and $c_{ijk}$ is the transportation cost of the unit of that product from producer $i$ to consumer $j$ by the means of conveyance $k$, to establish a transportation plan which should be made with minimum total expenses is a transportation tridimensional problem. According to the classification of the transportation tridimensional problems [4], [5], the problem presented is a transportation triplane problem with 3 categories of constraints expressed by double sums with the following mathematical model [2]:

$$min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} \tag{1}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijk} = a_i, \quad i = 1, \ldots, m \tag{2}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = b_j \quad j = 1, \ldots, n \tag{3}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = c_k \quad k = 1, \ldots, p \tag{4}$$

$$x_{ijk} \geqslant 0 \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \quad k = 1, \ldots, p \tag{5}$$

$$a_i > 0, \quad b_j > 0, \quad c_k > 0, \quad c_{ijk} \geqslant 0 \tag{6}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} c_k = T \tag{7}$$

According to the interpretation given to the known elements the model may have multiple applications: for example, if in the previous model the quantities are replaced by a number of transports, the costs by time, the optimal solution will give the minimum motorvehicles per hour. The same mathematical model can be adapted to other distribution, selection and affecting problems.

The model can be easily inferred if the matrix of the unknown values and the data of the problem in a parallelepiped are written and the sum of the unknown values $x_{ijk}$ on each of the $m$, $n$, $p$ planes of the parallelepiped is made.

Matrix $X = \{x_{ijk} | \, i = 1, \ldots, m, \, j = 1, \ldots, n, \, k = 1, \ldots, p\}$ which satisfies ratios (2), (3), (4), (5) is called a transportation plan and plan $X$ is called optimum if it satisfies (1).

The results are known [1], [2]: condition (7) is the necessary and sufficient condition for the existence of the solution; the level of the matrix of the constraint system is $m + m + p - 2$ showing that a non-degenerated transportation plan of problem (1—7) contains at least $m + n + p - 2$ non-nul components; the problem admits endless solutions and at least one optimal.

The big volume of calculus generated in the solution by the methods known [1], [2], [4], [5] leaves tridimensional transportation problems unsolved.

The Dantzig-Wolfe decomposition method-known in linear programming for solving problems of big dimensions allowing a decomposition into smaller problems — for the transportation problem having a particular structure, creates the possibility of a simple algorithmization for solving on an electronic computer and the possibility of solving big dimension problems.

The advantage of the decomposition method for the usual transportation problem described in [9] is obvious if for a problem of the dimension $15 \times 3\,000$ the solution was obtained with the computer in 50 minutes through the algorithm resulting from the decomposition method and 80 minutes with the stepping-stone method.

By using the decomposition method for the tridimensional problem (1—7) described in [3] the author formulates the algorithm which is inferred for solving problem (1—7) with the computer and presents its advantages.

## 2. THE DECOMPOSITION ALGORITHM FOR THE TRANSPORTATION TRIDIMENSIONAL PROBLEM

The Dantzig-Wolfe method pursues the turning of the given problem into one with less constraints and more unknown values, implying a more simple and rapid solution, a transformation justified by the "reduction theorem":

Consider sets $X$ and Y, the functional defined on $X$ and functional $\psi$ defined on Y. Suppose that $\varphi$ has a minimal point for $\psi$ on Y. If $M$ is a function with field Y and co-field $S$ and

1. $M$ is an application from Y to $X$,
2. $\varphi M(y^\circ) = \psi(y^\circ)$
3. any minimal point for $\psi$ in $X$ has a preimage through $M$ in Y,

in this case $x^\circ = M(y^\circ)$ is a minimal point for $\varphi$ on $X$.

In problem (1—7) with $m + n + p$ constrains and $mnp$ variables we consider the set of constraints:

$$\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijk} = a_i \quad i = 1, ..., m$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p} x_{ijk} = b_j \quad j = 1, ... n \tag{8}$$

If $X_\nu$ with $\nu = 1, ... n$ are all the extreme solutions of (8), in this case for any solution $X$ of (8) there is a set of numbers $z_\nu \geqslant 0$ so that

$$\sum_{\nu=1}^{N} z_\nu = 1 \quad X = \sum_{\nu=1}^{N} X_\nu z_\nu \quad x_{ijk} = \sum_{\nu=1}^{N} x_{ijk}^{\nu} z_\nu \tag{9}$$

and for each set of nonnegative numbers $z_\nu$ so that $\sum_{\nu=1}^{N} z_\nu = 1$ sum $X$ in (9) is the solution of (8).

The set of solutions (plans) to problem (1—7) is the subset of solutions to (8) for which the constraints (4) are also satisfied. In this case $z_\nu$ will be the set of nonnegative numbers satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\nu=1}^{N} x_{ijk}^{\nu} z_\nu = c_k \quad k = 1, \ldots p$$

with $x_{ijk}$ the solution to (8).

If we note

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{\nu} = \rho_\nu^{(k)} \tag{10}$$

we have

$$\sum_{\nu=1}^{N} \rho_\nu^{(k)} z_\nu = c_k \quad k = 1, \ldots, p \tag{11}$$

We associate with each extreme solution of (8) a cost

$$\gamma_\nu = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk} x_{ijk}^{\nu} \tag{12}$$

and, in this case the given functional $L(X)$

$$L(X) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk} x_{ijk}$$

becomes

$$\varphi(z) = \sum_{\nu=1}^{N} \gamma_\nu z_\nu \tag{13}$$

We are in the conditions of the theorem of reducing by $X$ the set $\{(x_{ijk}).\ i = 1, \ldots, m,\ j = 1, \ldots, n,\ k = 1, \ldots p\}$ nonnegative numbers verifying the constraints in (1—7) $L$ functionals in (1) $Z$ the set $\{z_\nu,\ \nu = 1, \ldots, N\}$ verifying (11), $\varphi$ the functional (13), $M$ is defined on $Z$ by (9) and is an application from $Z$ to $X$ by $\varphi = LM$. In this case, to find the solution to problem (1—7) consists in finding the solution to the following linear programming problem:

$$min \sum_{\nu=1}^{N} \gamma_\nu z_\nu$$

$$\sum_{\nu=1}^{N} \rho_\nu^{(k)} z_v = c_k \quad k = 1, \ldots, p \qquad \text{II}$$

$$z_\nu \geqslant 0 \quad \nu = 1, \ldots N$$

The initial problem (1—7) with $m + n + p$ constraints and $mnp$ unknown values was reduced to a problem (II) with $p$ constraints and $N$ unknown values so that the optimum solution to the transportation tridimensional problem given may be calculated through (9).

The decomposition algorithm results by solving problem (II) by the inverse matrix method which is a variant of the simplex method by adding the estimation generalized operation.

In the set of $N$ columns $\rho_{\nu}^{(k)}$ only $p$ sufficient are explicitely necessary for forming a base. The algorithm starts by constructing $p$ columns $\rho_{\nu}^{(k)}$, $\nu = 1, ..., p$ of the initial matrix of constraints which will form the first base, so that only the column necessary to formulate the new base should be explicitely constructed at each iteration. One works only with the basic matrix inverse. The updating of the basic matrix inverse, of the solution vector is done according to the rectangle rule.

To construct the initial base $B$ for problem (II) $p$ common transportation problems are solved:

$$min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijt} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, ..., m \tag{14}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, ..., n$$

$$x_{ij} \geqslant 0.$$

With the solutions of problems (14) $x_{ij}^t$ $t = 1, ..., p$ are considered $p$ transportation plans of (8) so that:

$$x_{ijk}^{\nu t} = \begin{cases} x_{ij}^t & k = t \\ 0 & k \neq t \end{cases} \quad t = 1, ..., p.$$

In this case, owing to (10) the initial base for problem (II) will be constructed out of vectors

$$\rho_{\nu t}^{(k)} = \begin{cases} T & k = t \\ 0 & k \neq t \end{cases} \quad t = 1, ..., p$$

**Initial step.** We have the initial base $B$ of size $p \times p$ given by the matrix of components $T\delta_{kt}$ and the inverse of the basic initial matrix will have elements $(1/T)\ \delta_{kt}$. Obviously, the initial basic plan is:

$$z_{\nu t} = \frac{c_{\nu t}}{T} \quad t = 1, ..., p \tag{15}$$

(according to the general theory equal to the product between the basic matrix inverse and the free vector of constraints). The cost coefficients corresponding to the basic variables in the objective function will be according to (12):

$$\gamma_{\nu_t} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijt} x_{ij}^t \quad t = 1, ..., p. \tag{16}$$

**Iterative step.** For a certain iteration step we note the elements of the basic matrix inverse with $\bar{\rho}_{tk}$. The "estimation" vector (given by the product between the vector of the cost coefficients corresponding to the basic variables and the basic current matrix inverse) by components $\pi_k$ is given by

$$\pi_k = \sum_{t=1}^{p} \gamma_t \bar{\rho}_{kt} \quad k = 1, ..., p. \tag{17}$$

The simplex input criterion makes differences intervene between the cost coefficients and the product between the estimation vector and the columns of the initial matrix (the differences are nul on the set of basic variables), that is:

$$L_\nu = \gamma_\nu - \sum_{k=1}^{p} \pi_k \rho_\nu^{(k)}$$

which can also be written

$$L_\nu = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}x_{ijk} - \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n} \pi_k x_{ijk} =$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} (c_{ijk} - \pi_k)\, x_{ijk}$$

The estimation generalized operation consists in finding that column vector $\rho_\nu^{(k)}$ (of components $\rho_\nu^1, \ldots \rho_\nu^p$) which corresponds to $x_{ijk}^\nu$ for which quantity $L_\nu$ is minimum. More explicitely, the selection of the vector included into the base is made by solving the following transportation problem: Find the extreme solution of (8) which gives the minimum of $L_\nu$ on the set of the basic solutions of (8), that is

$$min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} (c_{ijk} - \pi_k)\, x_{ijk}.$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijk} = a_i \qquad i = 1, \ldots m \tag{18}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p} x_{ijk} = b_j \quad j = 1, \ldots n$$

$$x_{ijk} \geqslant 0.$$

We put:

$$c_{ij} = c_{ijs(i,j)} - \pi_{s(i,j)} = \min_k (c_{ijk} - \pi_k). \tag{19}$$

Problem (18 ) is reduced to problem

$$min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} y_{ij}$$

$$\sum_{j=1}^{n} y_{ij} = a_i \qquad i = 1, \ldots m \tag{20}$$

$$\sum_{i=1}^{m} y_{ij} = b_j \qquad j = 1, \ldots n.$$

It is clear that if $y_{ij}^{\nu_0}$ is the solution to this common transportation problem, the solution of (18) will be:

$$x_{ijk}^{\nu_0} = \begin{cases} y_{ij}^{\nu_0} & k = s(i, j) \\ 0 & k \neq s\ (i,\ j) \end{cases} \qquad k = 1, ..., p. \tag{21}$$

Having this solution due to (10) we also have vector $\rho_{\nu_0}^{(k)}$ which will enter the new base. With the components of this vector we shall calculate the elements of the only column of interest in the current matrix of constraints, as is known:

$$u_r = \sum_{k=1}^{p} \overline{\rho}_{rk} \rho_{\nu_0}^{(k)} \quad r = 1, ... p. \tag{22}$$

Further, we use column $u_r$ in the output criterion. If the current basic solution is $z_t$ we find out a $r_0$ for which:

$$\frac{z_{r_0}}{u_{r_0}} = \min_r \left\{ \frac{z_r}{u_r} \mid u_r > 0 \right\} \tag{23}$$

In this case vector $\rho_{\nu_0}^{(k)}$ will replace $\rho_{r_0}^{(k)}$ in the new base. The new solution is obtained as is known

$$\begin{aligned} z'_{\nu_0} &= \frac{z_{r_0}}{u_{r_0}} \\ z'_t &= z_t - z_{r_0} \frac{u_t}{u_{r_0}} \qquad t \neq \nu_0 \end{aligned} \tag{24}$$

and the elements of the new inverse

$$\begin{aligned} \overline{\rho}'_{tk} &= \overline{\rho}_{tk} - \overline{\rho}_{r_0 k} \frac{u_t}{u_{r_0}} \qquad t \neq \nu_0 \\ \overline{\rho}'_{\nu_0 k} &= \frac{\overline{\rho}_{r_0 k}}{u_{r_0}} \end{aligned} \tag{25}$$

We note

$$\Delta_{\nu_0} = -L_{\nu_0}(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij}^{\nu_0}$$

The simplex input criterion shows the resuming of the process of improving the solution in case $\Delta_{\nu_0} > 0$. If $\Delta_{\nu_0} \leqslant 0$ we have:

**Final step.** The algorithm stops because the base for problem (II) is optimum and $z_\nu$ provides through (9) the best solution for problem (1—7)

One comes to the optimum solution in a finite number of steps because the number of bases is finite.

Schematically, the decomposition algorithm which provides those nonnegative numbers $z_\nu$ which together with the extreme solutions of (8) form the optimum solution of (1—7) is presented in the follwing stages:

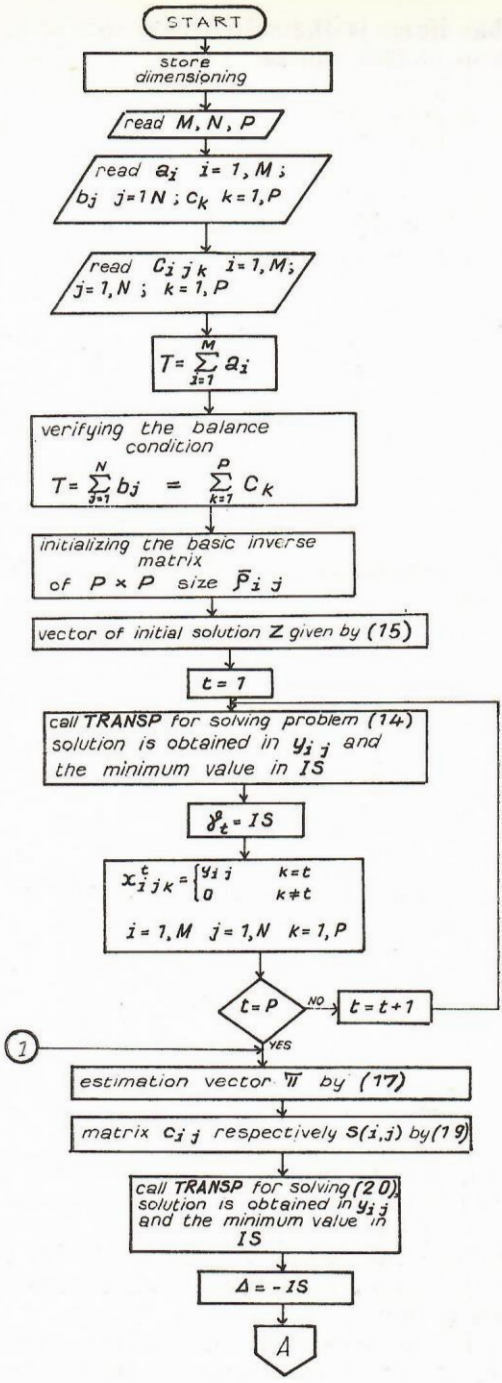

START
store dimensioning
read M, N, P
read $a_i$ $i = 1, M$; $b_j$ $j = 1 N$ ; $c_k$ $k = 1, P$
read $C_{ijk}$ $i = 1, M$; $j = 1, N$ ; $k = 1, P$
$T = \sum_{i=1}^{M} a_i$
verifying the balance condition
$T = \sum_{j=1}^{N} b_j = \sum_{k=1}^{P} c_k$
initializing the basic inverse matrix of $P \times P$ size $\bar{P}_{ij}$
vector of initial solution Z given by (15)
$t = 1$
call TRANSP for solving problem (14) solution is obtained in $y_{ij}$ and the minimum value in IS
$\delta_t^0 = IS$
$x_{ijk}^t = \begin{cases} y_{ij} & k = t \\ 0 & k \neq t \end{cases}$
$i = 1, M$ $j = 1, N$ $k = 1, P$
$t = P$
NO
$t = t + 1$
YES
1
estimation vector $\bar{\pi}$ by (17)
matrix $c_{ij}$ respectively $s(i,j)$ by (19)
call TRANSP for solving (20) solution is obtained in $y_{ij}$ and the minimum value in IS
$\Delta = -IS$
A

Fig. 1 a.

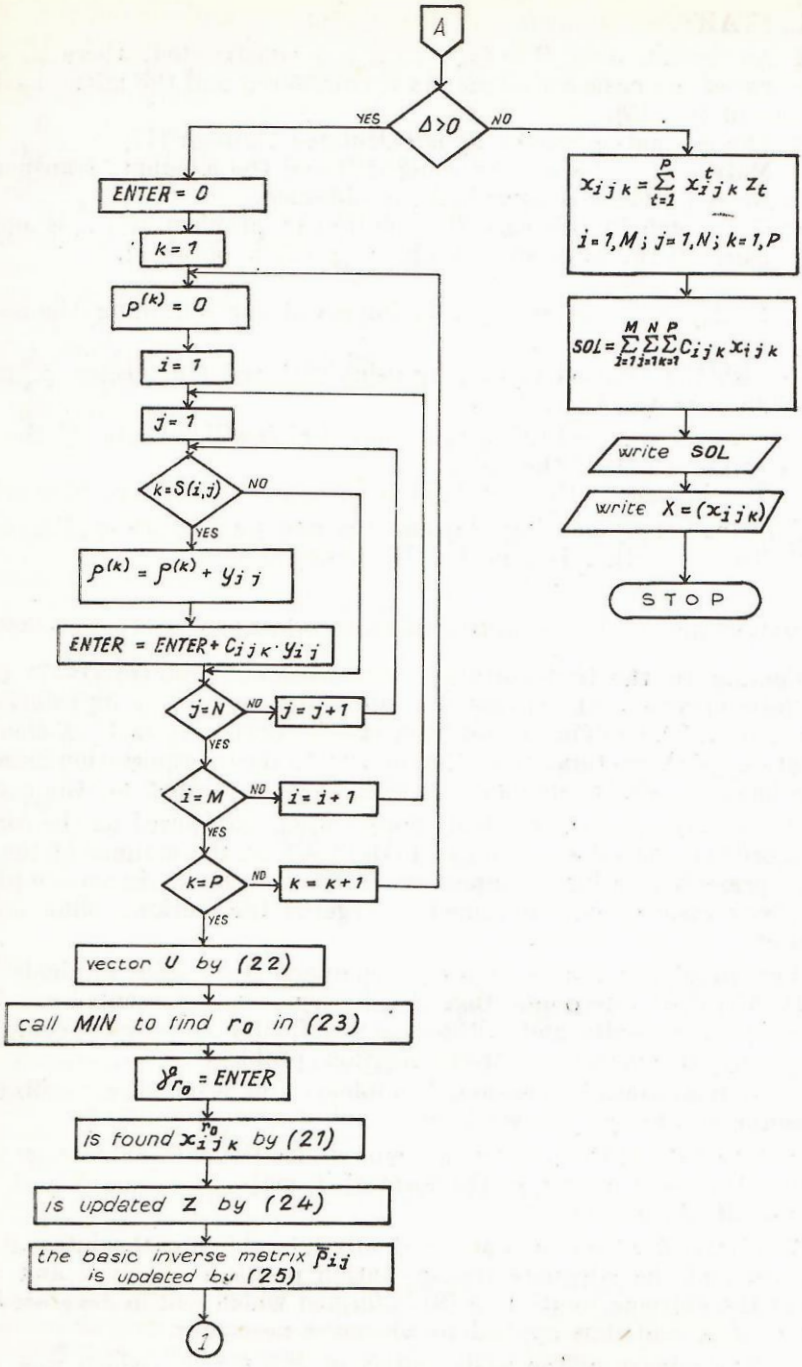

A
Δ>0
YES
NO
ENTER = 0
k = 1
p(k) = 0
i = 1
j = 1
k=S(i,j)
NO
YES
p(k) = p(k) + yij
ENTER = ENTER + Cijk · yij
j=N
NO
j = j+1
YES
i=M
NO
i = i+1
YES
K=P
NO
k = k+1
YES
vector U by (22)
call MIN to find r0 in (23)
y r0 = ENTER
is found x r0 ijk by (21)
is updated Z by (24)
the basic inverse matrix P̄ij is updated by (25)
1
xijk = Σ t=1 P x t ijk zt
i = 1,M; j = 1,N; k = 1,P
SOL = Σ i=1 M Σ j=1 N Σ k=1 P Cijk xijk
write SOL
write X = (xijk)
STOP

Fig. 1 b.

1. START.

2. An initial base $B = (\rho_{v_1}^{(k)}, ..., \rho_{v_p}^{(k)})$ is constructed, where $\rho_{v_t}^{k} = T\delta_{kt}$. The inverse of the basic initial matrix is considered and the initial basic plan is calculated by (15).

3. The estimation vector $\pi_k$ is calculated through (17).

4. Matrix $c_{ij}$ is formed by using (19) and the adequate transportation bidimensional problem is solved. $\Delta_{v_0}$ is obtained.

a) If $\Delta_{v_0} < 0$ in this case the solution to problem (II) $z_v$ is optimum and by using (9) the solution to problem (1—7) is obtained.
STOP.

b) If $\Delta_{v_0} > 0$, the solution is improved for obtaining the optimum one:

— with the solution to (20) by using (21) and (10) vector $\rho_{v_0}^{(k)}$ is found which will enter the base.

— by using (22) and (23) a $r_0$ is found which will indicate $\rho_{r_0}^{(k)}$ the vector which will come out of the base.

5. The inverse of the new base is calculated — the new base will contain $\rho_{v_0}^{(k)}$ instead of $\rho_{r_0}^{(k)}$ — using (25) and the new plan by using (24).

6. The algorithm is repeated from step 3.

## 3. ADVANTAGES OF THE ALGORITHM AS REGARDS COMPUTER UTILIZATION

Coming to the transportation tridimensional problem (1—7) we note from those presented above that the difficulties concerning its solution have been reduced. The solving of problem (1—7) consists in fact of elementary calculations with vectors and of the solution to the transportation bidimensional problem for which efficient algorithms exist, tested on the computer.

The efficiency of the method appears when transposed on the computer. The algorithm was programmed in FORTRAN in the manner of the logical scheme presented in Fig. 1, upon its testing on small dimension problems satisfactory results being obtained as regards the working time with the computer.

For simplicity and rapidity in comparison to the methods known [1], [2], [4], [5], the decomposition algorithm which is presented can be used with very good results and without any difficulty in using the computer for solving any tridimensional transportation problem.

In solving multidimensional problems the economical utilization of the computer storage is necessary:

1. The data of the problem are stored into the external storage (magnetic disc): the costs matrix in the form of $P$ matrix of $M \times N$ size, vectors $a$, $b$, $c$ of $M$, $N$, $P$ size.

2. Matrix $k$ vectors $a$, $b$ are gradually brought into the internal storage ($P$ times) and the adequate transportation problem is solved and with its solution the extreme solution of (8) is formed which will be be stored on the magnetic disc and it is applied to whenever necessary.

3. The inverse of the basic matrix of $P \times P$ size (which was initially non-nul only on the diagonal) is stored.

4. The basic solution of problem (II) is calculated which is stored into a vector of $P$ size.

5. The vector of cost coefficients corresponding to the basic variables of $P$ size, the estimation vector of $P$ size are determined.

6. By applying to the data on the disc the costs matrix is formed for which the transportation bidimensional problem is solved by using the same store space from Step 2.

7. With the solution to this transportation problem, we determine the size vector $P$ in the initial matrix of constraints which will enter the new base. The column vector which is of interest in the current matrix of constraints is calculated .

8. The optimum condition is verified and it is repeated from Step 3.

The efficiency of the algorithm for multidimensional problems is being analysed as regards the working time on the computer and the organization of the internal storage.

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## UN ALGORITHME POUR RESOUDRE UN PROBLEME DE TRANSPORT TRIDIMENSIONNEL


On présente un algorithme pour résoudre un problème tridimensionnel de transport — le modèle à sommes doubles — en utilisant la méthode de décomposition Dantzig-Wolfe. L'auteur présente les avantages offerts par cet algorithme — programmé en FORTRAN —, ainsi que les possibilités d'utilisation de l'ordinateur.


## АЛГОРИТМ РЕШЕНИЯ ТРЕХРАЗМЕРНОГО ВОПРОСА ТРАНСПОРТА


В работе даётся алгорифм решения трёхразмерного вопроса транспорта — модель с двойными суммами — путём использования метода разложения Дантих-Вольф. Описаны преимущества алгорифма — запрограммированный в ФОРТРАНЕ — а также возможности использования вычислителя для разрешения трёразмерного вопроса транспорта.



17.07.1976.